A Test for Non-stationarity of Time-series

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Summary

We consider the problem of testing a given time-series for stationarity. The approach is based on evolutionary spectral analysis, and the proposed method consists essentially in testing the "homogeneity" of a set of evolutionary spectra evaluated at different instants of time.

Using a logarithmic transformation, we show that the mechanics of the test are formally equivalent to a two-factor analysis of variance procedure when the residual variance is known, *a priori.*

In addition to testing stationarity, the analysis provides also a method for testing whether the observed series fits a "uniformly modulated" model, and a test for "randomness" (constancy of spectra).

1. Introduction

Several authors have proposed methods for testing whether or not a given time-series may be regarded as stationary. Some of these methods are designed to detect non-stationary "trends" in a particular characteristic of the series, such as the mean or the variance (Grenander and Rosenblatt, 1957; Parthasarathy, 1961; Granger and Hatanaka, 1964; Sen, 1965; Subba Rao, 1968), whilst others are designed to test whether the correlation or spectral properties of two sections of a time-series are "compatible" (Quenouille, 1958; Jenkins, 1961). In the later cases the two sections of the series have to be specified, *a priori*, and it is assumed that within each section the series is stationary. The method to be described in this paper differs from those mentioned above in that it may be used to test the *overall* stationarity of the complete second-order properties of a time-series. This test makes use of the concept of the "evolutionary (that is, time-dependent) spectrum" of a non-stationary process, and the basis of the method consists essentially in testing the "uniformity" of a set of evolutionary spectra evaluated at different instants of time. The mechanics of the test are almost identical to those of a two-factor analysis of variance procedure, and the basic approach follows the lines suggested by Priestley (1965).

A further advantage of this method is that it enables one to test not only the overall stationarity of the series, but also to examine the character of the non-stationarity (when it exists).

As a preliminary step, we summarize below the main ideas and results of evolutionary spectral theory. (For a more detailed discussion see Priestley, 1965, 1967.)

2. Evolutionary Spectra

We consider the class of non-stationary processes, $\{X(t)\}$, with $E\{X(t)\} = 0$, $E\{X^2(t)\} < \infty$, all t, which admit spectral representations of the form

$$X(t) = \int_\Lambda e^{i\omega t} A_t(\omega)\, dZ(\omega), \tag{2.1}$$

where $Z(\omega)$ is an orthogonal process with $E|dZ(\omega)|^2 = \mu(d\omega)$; for each fixed ω, $A_t(\omega)$ (considered as a function of t) has a Fourier transform whose absolute maximum occurs at the origin, and $\Lambda = (-\infty, \infty)$ or $(-\pi, \pi)$ according as to whether the parameter t is continuous or integral valued. Thus, we are assuming that the mean of the series has been removed, either by regression analysis or by filtering techniques—cf. Granger and Hatanaka (1964)—or by incorporating "mean detrending" in the procedure for estimating spectra, as is normally done when dealing with stationary processes.) The *evolutionary spectrum at time t* with respect to the family

$$\mathscr{F} \equiv \{e^{i\omega t} A_t(\omega)\}$$

is defined by

$$dF_t(\omega) = |A_t(\omega)|^2 \mu(d\omega), \quad \omega \in \Lambda. \tag{2.2}$$

Assuming that the measure μ is absolutely continuous with respect to $d\omega$ (as will be assumed throughout the rest of the paper), we define the *evolutionary spectral density function at time t* by

$$f_t(\omega) = F_t'(\omega) = |A_t(\omega)|^2 \frac{d\mu}{d\omega}. \tag{2.3}$$

Given a sample from a continuous parameter process, $X(t)$, $0 \leqslant t \leqslant T$, we may estimate $f_t(\omega)$ $(0 \leqslant t \leqslant T)$ by using the "double-window" technique (Priestley, 1965, 1966), as follows. Choose a "filter" (or "window") $\{g(u)\}$ which is square integrable and normalized, so that

$$2\pi \int_{-\infty}^{\infty} |g(u)|^2 \, du = \int_{-\infty}^{\infty} |\Gamma(\omega)|^2 \, d\omega = 1. \tag{2.4}$$

Here

$$\Gamma(\omega) = \int_{-\infty}^{\infty} g(u)\, e^{-iu\omega} \, du \tag{2.5}$$

denotes the frequency response function of $\{g(u)\}$. Now write, for any frequency ω,

$$U(t, \omega) = \int_{t-T}^{t} g(u)\, X(t-u)\, e^{-i\omega(t-u)} \, du. \tag{2.6}$$

In practice, the time-domain "width" of the filter $\{g(u)\}$ will be small compared with T so that, for $t \gg 0$, the limits of the integral in (2.6) may be replaced by $(-\infty, \infty)$. (In other words, we are supposing that t is sufficiently large for the effects of "transients" in the filter output to have decayed.)

Now choose a second "window", $w_{T'}(t)$, depending on the parameter T', which satisfies

(a) $w_{T'}(t) \geqslant 0$, for all t, T',

(b) $w_{T'}(t)$ decays to zero as $|t| \to \infty$, for all T',

(c) $\int_{-\infty}^{\infty} w_{T'}(t)\, dt = 1$, for all T',

(d) $\int_{-\infty}^{\infty} \{w_{T'}(t)\}^2 \, dt < \infty$, for all T'.

Write

$$W_{T'}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda t} w_{T'}(t)\, dt; \tag{2.7}$$

assume that

(e) there exists a constant C such that

$$\lim_{T' \to \infty} \left\{ T' \int_{-\infty}^{\infty} |W_{T'}(\lambda)|^2 \, d\lambda \right\} = C. \tag{2.8}$$

Then we may estimate $f_t(\omega)$ by

$$\hat{f}_t(\omega) = \int_{t-T}^{t} w_{T'}(u)\, |U(t-u, \omega)|^2 \, du. \tag{2.9}$$

The remarks noted above apply also to the limits of the integral in (2.9), which may also be replaced by $(-\infty, \infty)$. With the obvious modifications, a similar procedure may be used to estimate the evolutionary spectra of discrete-parameter processes.

It has been shown by Priestley (1965) that the mean and variance of $\hat{f}_t(\omega)$ are approximately given by

$$E\{\hat{f}_t(\omega)\} \sim \int_{-\infty}^{\infty} \bar{f}_t(\omega+\theta)\, |\Gamma(\theta)|^2 \, d\theta, \tag{2.10}$$

$$\operatorname{var}\{\hat{f}_t(\omega)\} \sim (C/T')\bar{f}_t^2(\omega) \left\{ \int_{-\infty}^{\infty} |\Gamma(\theta)|^4 \, d\theta \right\}, \quad \omega \neq 0, \tag{2.11}$$

where

$$\bar{f}_t(\omega+\theta) = \int_{-\infty}^{\infty} W_{T'}(u) f_{t-u}(\omega+\theta)\, du$$

and

$$\bar{f}_t^2(\omega) = \frac{\int_{-\infty}^{\infty} f_{t-u}^2 \{W_{T'}(u)\}^2 \, du}{\int_{-\infty}^{\infty} \{W_{T'}(u)\}^2 \, du}.$$

(When $\omega = 0$,—or π, in the discrete case—the above expression for the variance should be doubled.)

As a further approximation, it may be shown that (i) if the "bandwidth" of $|\Gamma(\theta)|^2$ is small compared with the "frequency domain bandwidth" of $f_t(\omega)$, and (ii) if the "bandwidth" of $W_{T'}(u)$ is small compared with the "time-domain bandwidth" of $f_t(\omega)$, then

$$E\{\hat{f}_t(\omega)\} \sim f_t(\omega) \tag{2.10}$$

(that is, $\hat{f}_t(\omega)$ is approximately an unbiased estimate of $f_t(\omega)$) and

$$\operatorname{var}\{\hat{f}_t(\omega)\} \sim (C/T') f_t^2(\omega) \left\{ \int_{-\infty}^{\infty} |\Gamma(\theta)|^4 \, d\theta \right\}. \tag{2.11}$$

(Note that $\bar{f}_t(\omega)$ and $\bar{f}_t^2(\omega)$ may be interpreted as "smoothed" forms of $f_t(\omega)$ and $f_t^2(\omega)$ respectively.)

To illustrate the use of (2.11) suppose, for example, that we choose $\{g(u)\}$ to be of the form

$$g\{u) = \begin{cases} 1/\{2\sqrt{(h\pi)}\}, & |u| \leqslant h, \\ 0, & |u| > h. \end{cases} \tag{2.12}$$

Then

$$|\Gamma(\omega)|^2 = \frac{1}{\pi}\frac{\sin^2 h\omega}{h\omega^2},$$

corresponding to the Bartlett window, and we find that

$$\int_{-\infty}^{\infty} |\Gamma(\theta)|^4 \, d\theta = 2h/3\pi.$$

Further, if we choose $W_{T'}(t)$ to be of the form

$$W_{T'}(t) = \begin{cases} 1/T', & -\tfrac{1}{2}T' \leqslant t \leqslant \tfrac{1}{2}T', \\ 0, & \text{otherwise}, \end{cases} \tag{2.13}$$

corresponding to the Daniell window, then it is not difficult to show that

$$C = \lim_{T' \to \infty} \left\{ T' \int_{-\infty}^{\infty} |W_{T'}(\lambda)|^2 \, d\lambda \right\} = 2\pi.$$

Using these results in (2.11), we now obtain

$$\operatorname{var} \hat{f}_t(\omega) \sim (4h/3T') f_t^2(\omega). \tag{2.14}$$

The expression for the covariance between $\hat{f}_{t_2}(\omega_2)$ and $\hat{f}_{t_1}(\omega_1)$ is somewhat complicated, but is given in Priestley (1966). It is sufficient here to quote the result that this covariance will be effectively zero if either

(i) $\qquad |\omega_1 \pm \omega_2| \gg$ bandwidth of $|\Gamma(\theta)|^2$ or

(ii) $\qquad |t_1 - t_2| \gg$ "width" of the function $\{W_{T'}(u)\}$.

(For a fuller discussion of these estimation procedures, together with numerical illustrations and a treatment of the "design relations" governing the choice of the parameters of the estimates, see the references quoted above.)

3. THE BASIS OF THE TEST

It is well known that, in the case of stationary processes, a logarithmic transformation will stabilize (approximately) the variance of the estimated spectral density function, and this device has been suggested by Jenkins and Priestley (1957) and Grenander and Rosenblatt (1957) in connection with goodness of fit tests. It follows from (2.10) and (2.11) that this transformation will produce the same effect when applied to evolutionary spectral estimates. With this in mind, let us write

$$Y(t, \omega) = \log_e \hat{f}_t(\omega). \tag{3.1}$$

Then we have, approximately,

$$E\{Y(t,\omega)\} = \log_e f_t(\omega) \tag{3.2}$$

and

$$\operatorname{var}\{Y(t,\omega)\} = \sigma^2 \quad (\omega \neq 0, \pi), \tag{3.3}$$

where

$$\sigma^2 = (C/T')\left\{\int_{-\infty}^{\infty} |\Gamma(\theta)|^4 \, d\theta\right\} \tag{3.4}$$

is now independent of ω and t. Alternatively, we may write

$$Y(t,\omega) = \log_e f_t(\omega) + \epsilon(t,\omega), \tag{3.5}$$

where approximately,

$$E[\epsilon(t,\omega)] = 0, \quad \text{all } t, \omega,$$

$$\operatorname{var}[\epsilon(t,\omega)] = \sigma^2, \quad \text{all } t, \text{ all } \omega \neq 0, \pi.$$

Suppose now that we have evaluated the estimated evolutionary spectra $\hat{f}_t(\omega)$, over the interval $(0, T)$. We now choose a set of times $t_1, t_2, \ldots, t_I$ (say) and a set of frequencies $\omega_1, \omega_2, \ldots, \omega_J$ (say) which cover the range of times and frequencies of interest, and are such that conditions (i) and (ii) of Section 2 are both satisfied. If now we write

$$\left.\begin{aligned} Y_{ij} &= Y(t_i, \omega_j), \\ f_{ij} &= f_{t_i}(\omega_j), \\ \epsilon_{ij} &= \epsilon(t_i, \omega_j), \end{aligned}\right\} \quad \begin{aligned} i &= 1, \ldots, I, \\ j &= 1, \ldots, J, \end{aligned}$$

then we have the model

$$Y_{ij} = f_{ij} + \epsilon_{ij}, \tag{3.6}$$

and if the $\{t_i\}$ and $\{\omega_j\}$ are spaced "sufficiently wide apart" the $\{\epsilon_{ij}\}$ will be approximately uncorrelated. As yet, only the first two moments of the $\{\hat{f}_t(\omega)\}$ have been investigated, but Jenkins (1961) suggested that, in the case of stationary processes, the logarithmic transformation would bring the distribution of the spectral estimates closer to normality. Although this statement was not substantiated, its validity is rendered highly plausible by the analogous result given in Kendall and Stuart (1966, Vol. 3, p. 93) relating to the distribution of the sample variance of a set of independent normal observations. It is shown by these authors that the variance-stabilized logarithmically transformed sample variance tends to normality more rapidly than the untransformed sample variance. The relevance of this result follows from the fact that a spectral estimate is distributed as a weighted sum of χ^2 variables, and consequently may be treated approximately as a χ^2 variable—as suggested by Blackman and Tukey (1958) and Jenkins (1961). It seems reasonable to suppose, therefore, that (at least approximately) we may treat the $\{\epsilon_{ij}\}$ as independent $N(0, \sigma^2)$. (This result requires rather more than the approximate normality of $\log \hat{f}_t(\omega)$; it requires, in fact, that the joint distribution of $\{\log \hat{f}_{t_i}(\omega_j)\}$, $i = 1, \ldots, I$, $j = 1, \ldots, J$, be approximately normal. A rigorous proof of this would certainly require a separate paper, but there seems little doubt that a result of this type could be proved under sufficiently strong conditions.)

It should be recalled that the variances of $\hat{f}_t(0)$, and $\hat{f}_t(\pi)$ (in the discrete case) are equal to $2\sigma^2$. Accordingly, these frequencies should either be omitted from the set $(\omega_1, \ldots, \omega_J)$, or alternatively for each t, $\hat{f}_t(0)$ and $\hat{f}_t(\pi)$ should be replaced by the single "entry", $\frac{1}{2}\{\hat{f}_t(0)+\hat{f}_t(\pi)\}$. This device will not affect the test for stationarity, but will certainly affect a test for "randomness"—see below. With the above assumption, (3.6) becomes the usual "*two-factor analysis of variance*" model, and may be rewritten in the conventional form

$$H : Y_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ij} \quad (i = 1, \ldots, I; j = 1, \ldots, J). \tag{3.7}$$

If $\{X(t)\}$ is a stationary process, then

$$E\{\hat{f}_t(\omega)\} \sim f(\omega) \quad \text{(independent of } t\text{)},$$

where $f(\omega)$ is the usual (non-time-dependent) spectral density function. Consequently, we may test the stationarity of $\{X(t)\}$ by using standard techniques to test the model

$$H_1 : Y_{ij} = \mu + \beta_j + \epsilon_{ij} \tag{3.8}$$

against the general model H given by (3.7). Note that we may test for the presence of the "interaction" term, γ_{ij}, even with one observation per "cell", since in this situation we know the value of

$$\sigma^2 = \text{var}\{\epsilon_{ij}\}$$

a priori. In fact, it turns out that the interaction term has a rather interesting interpretation, and we discuss this point in the next section.

4. Interpretation of the Parameters

It is fairly obvious that the parameters $\{\alpha_i\}$, $\{\beta_j\}$ may be interpreted as the "main-effects" of the time and frequency "factors" (respectively), and that the $\{\gamma_{ij}\}$ represent an "interaction" between these two factors. However, it is interesting to inquire under what circumstances we would expect the $\{\gamma_{ij}\}$ to be all zero. Now if all the $\{\gamma_{ij}\}$ are in fact zero, then $\log f_t(\omega)$ is additive in terms of time and frequency, so that $f_t(\omega)$ is "multiplicative", that is, may be written in the form,

$$f_t(\omega) = c^2(t) f(\omega), \tag{4.1}$$

for some functions $c(t)$ and $f(\omega)$.

If $f_t(\omega)$ is of the form (4.1) it is not difficult to show that $\{X(t)\}$ must be of the form

$$X(t) = c(t)\, X_0(t), \tag{4.2}$$

where $\{X_0(t)\}$ is a *stationary* process with spectral density function, $f(\omega)$. Processes of the form (4.2) have been discussed by Herbst (1963) and Priestley (1965), who described them as *Uniformly Modulated Processes. Thus, a test for the presence of interaction is equivalent to testing whether or not $\{X(t)\}$ is a Uniformly Modulated Process.*

We would point out that, in addition to the test for stationarity, this approach provides also a test for "*complete randomness*" (in other words, constancy of spectra over frequency). This is achieved simply by testing the model

$$H_2 : Y_{ij} = \mu + \alpha_i + \epsilon_{ij} \tag{4.3}$$

against the general model H.

5. THE TEST PROCEDURE

Given the computed values of $Y_{ij} \equiv \log \hat{f}_{t_i}(\omega_j)$, we first construct the standard analysis of variance table for a two-factor design, which, with the usual notation, is set out below.

Item	*Degrees of freedom*	*Sum of squares*
Between times	$I-1$	$S_T = J \sum_{i=1}^{I} (Y_{i.} - Y_{..})^2$
Between frequencies	$J-1$	$S_F = I \sum_{j=1}^{J} (Y_{.j} - Y_{..})^2$
Interaction + residual	$(I-1)(J-1)$	$S_{I+R} = \sum_{i=1}^{I} \sum_{j=1}^{J} (Y_{ij} - Y_{i.} - Y_{.j} + Y_{..})^2$
Total	$IJ-1$	$S_o = \sum_{i=1}^{I} \sum_{j=1}^{J} (Y_{ij} - Y_{..})^2$

(1) In testing for stationarity, the first step is to test the interaction sum of squares, using the result $S_{I+R}/\sigma^2 = \chi^2_{(I-1)(J-1)}$. (Recall that, since σ^2 is known, all comparisons are based on χ^2 rather than F-tests.)

(2) If the interaction is not significant, we conclude that $\{X(t)\}$ is a uniformly modulated process, and proceed to test for stationarity by testing S_T, using

$$S_T/\sigma^2 = \chi^2_{(I-1)}.$$

(3) If, however, the interaction turns out to be significant, we conclude that $\{X(t)\}$ is *non-stationary*, and non-uniformly modulated. As is usually the case, there is now little point in testing the "main-effect" S_T, but we may well wish to examine whether the non-stationarity of $\{X(t)\}$ is restricted only to *some* frequency components. (For example, we may wish to test whether the "high" frequencies are stationary and only the "low" frequencies non-stationary.) To test this type of hypothesis we select those frequencies of interest, say $\{\omega_{j_1}, \omega_{j_2}, \dots \omega_{j_k}\}$, and test for stationarity *at these frequencies* by using the statistic

$$\sum_{j \in K} \sum_{i=1}^{I} (Y_{ij} - Y_{.j})^2 = \sigma^2 \chi^2_{k(I-1)}. \tag{5.1}$$

(Here K denotes the set of integers $\{j_1, j_2, \dots, j_k\}$.) In particular, this type of test may be used to examine whether any one particular frequency component is stationary.

(4) Reversing the roles of "times" and "frequencies", the above procedure may be used in exactly the same way to test for "complete randomness", either at all times (using S_F when S_{I+R} is not significant), or at a particular subset of times (using a statistic analogous to (5.1)).

6. EXAMPLES

We now apply our test to two examples.

Example 1. Here we consider the uniformly modulated process (in discrete time),

$$X(t) = \{e^{-(t-500)^2/2(200)^2}\} X_0(t), \quad (t = 0, 1, 2, \dots),$$

where $X_0(t)$ is the (stationary) second-order autoregressive process

$$X_0(t+2) - 0{\cdot}8X_0(t+1) + 0{\cdot}4X_0(t) = Z(t),$$

in which the $\{Z(t)\}$ as independent variables, each having the distribution $N(0, 100^2)$. Artificial realizations of this process were constructed, and the evolutionary spectra estimated for $t = 108\,(150)\,558$. The estimates, $\hat{f}_t(\omega)$, were obtained by using the discrete time analogue of (2.9) in which $W_{T'}(u)$ is given by (2.13), with $T' = 200$, and $g(u)$ has the form (2.12) with $h = 7$. (For further details see Priestley, 1965).) For this analysis we find that

$$\sigma^2 = 7/150 \quad \text{(cf. equation (2.14))}.$$

Also, the window $|\Gamma(\omega)|^2$ has a bandwidth of approximately $\pi/h = \pi/7$, and the window $\{W_{T'}(t)\}$ has width $T' = 200$. Thus, in order to obtain approximately uncorrelated estimates, the points $\{\omega_j\}$, $\{t_i\}$ should be chosen so that the spacings between the $\{\omega_j\}$ are at least $\pi/7$ and the spacings between the $\{t_i\}$ are at least 200. In fact, as will be shown below, the test works quite satisfactorily even with a (uniform) $\{t_i\}$ spacing as low as 150. (This smaller spacing was used as it allowed us to include an extra value of t in Tables 1 and 3.) The $\{\omega_j\}$ were chosen as follows:

$$\omega_j = \pi j/20, \quad j = 1\,(3)\,19,$$

corresponding to a uniform spacing of $3\pi/20$ (which just exceeds $\pi/7$). The values of $\log_e \hat{f}_t(\omega)$ are shown in Table 1.

TABLE 1

t \ ω	$\pi/20$	$4\pi/20$	$7\pi/20$	$10\pi/20$	$13\pi/20$	$16\pi/20$	$19\pi/20$
108	2·3597	2·3245	2·1499	1·9856	1·6258	1·4274	1·2494
258	3·1849	3·2967	3·3749	2·8425	2·3800	2·0380	2·0579
408	3·7692	3·8002	3·6135	3·1199	2·8137	2·5727	2·4673
558	3·7253	3·6672	3·5288	3·1247	2·7545	2·7050	2·4871

TABLE 2

Item	*Degrees of freedom*	*Sum of squares*	$\chi^2 =$ (*sum of squares*/σ^2)
Between times	3	7·6353	163·61
Between frequencies	6	6·4716	138·68
Interaction + residual	18	0·1848	3·96
Total	27	14·2917	306·25

The analysis of variance is shown in Table 2.

As expected, the interaction is extremely small (confirming the uniformly modulated model) and both the "between times" and "between frequencies" sums of squares are highly significant, confirming that the process is non-stationary and that the spectra are non-uniform.

Example 2. Here we consider a non-stationary, *non*-uniformly modulated process which was generated by taking the same stationary process, $\{X_0(t)\}$, of Example 1 and passing it through each of the three (approximately) "band-pass" filters with ranges $(0, \pi/3)$, $(\pi/3, 2\pi/3)$, $(2\pi/3, \pi)$ respectively. The outputs of the filters were then multiplied by three different functions of time and recombined to form the process $\{X(t)\}$.

The evolutionary spectra of $\{X(t)\}$ are as follows:

$$f_t(\omega) = \begin{cases} |C_1(t)|^2 f(\omega), & 0 \leqslant \omega \leqslant \pi/3, \\ |C_2(t)|^2 f(\omega), & \pi/3 < \omega \leqslant 2\pi/3, \\ |C_3(t)|^2 f(\omega), & 2\pi/3 < \omega \leqslant \pi, \end{cases}$$

where

$$C_1(t) = e^{-(t-100)^2/2(200)^2},$$

$$C_2(t) = [1+(t-300)^2/275]^{-1},$$

$$C_3(t) = \frac{\pi}{2(300)^2} t^2 e^{-t/300}.$$

(For further details see Abdrabbo, 1966).

The spectra were estimated from artificial realizations using the same formulae and parameters as in Example 1 (so that again $\sigma^2 = 7/150$), and evaluated at the same values of t and ω. The values of $\log_e \hat{f}_t(\omega)$ are as shown in Table 3.

TABLE 3

t \ ω	$\pi/20$	$4\pi/20$	$7\pi/20$	$10\pi/20$	$13\pi/20$	$16\pi/20$	$19\pi/20$
108	4·5786	4·6293	5·0356	4·6723	4·0922	3·8433	3·7419
258	4·1970	4·1694	4·9950	4·4738	4·2272	4·1818	4·0763
408	3·1430	3·3359	4·0464	3·9416	4·0465	4·3829	4·1709
558	2·7110	2·8195	3·7248	3·7301	4·1195	4·5665	4·1467

TABLE 4

Item	*Degrees of freedom*	*Sum of squares*	$\chi^2 =$ (*sum of squares*/σ^2)
Between times	3	2·3187	49·69
Between frequencies	6	1·9071	40·87
Interaction + residual	18	4·3055	92·26
Total	27	8·5313	182·82

The analysis of variance is shown in Table 4.

In this case it will be noticed that the interaction term is highly significant (0·1 per cent), confirming that $\{X(t)\}$ is non-uniformly modulated. The "between times" and "between frequencies" sums of squares are also highly significant, but it is instructive here to decompose the total χ^2 for (between times+interaction) into its various frequency components, as suggested in Section 5. We obtain the results set out in Table 5.

TABLE 5

j	1	4	7	10	13	16	19	Total
χ^2	49·27	42·54	28·46	12·60	0·38	6·16	2·54	141·95

To test each component for stationarity we refer each entry to χ^2 on three degrees of freedom. Thus, on the basis of these results we might conclude that, whilst the low frequencies were certainly non-stationary, there was no evidence to suggest non-stationarity in the high frequencies (although χ^2 for $j = 16$ is reasonably large compared with $\chi^2_3(0{\cdot}05) = 7{\cdot}81$). However, since the estimated spectra at different times appear quite close at the upper frequencies, this result is hardly surprising.

REFERENCES

ABDRABBO, N. A. (1966). Prediction and filtering of non-stationary processes. Ph.D. Thesis, University of Manchester (unpublished).

BLACKMAN, R. B. and TUKEY, J. W. (1958). The measurement of power spectra from the point of view of communications engineering. *Bell Systems Tech. J.*, **37**, 1, 2.

GRANGER, C. W. J. and HATANAKA, M. (1964). *Spectral Analysis of Economic Time Series.* Princeton: University Press.

GRENANDER, U. and ROSENBLATT, M. (1957). *Statistical Analysis of Stationary Time Series.* New York: Wiley.

HERBST, L. J. (1963). Periodogram analysis and variance fluctuations. *J. R. Statist. Soc.* B, **25**, 442–450.

JENKINS, G. M. (1961). General considerations in the estimation of spectra. *Technometrics*, **3**, 133–166.

JENKINS, G. M. and PRIESTLEY, M. B. (1957). The spectral analysis of time-series. *J. R. Statist. Soc.* B, **19**, 1–12.

KENDALL, M. G. and STUART, A. (1966). *The Advanced Theory of Statistics.* London: Griffin.

PARTHASARATHY, K. R. (1961). On testing the mean of a discrete linear process. *Sankhyā*, **23**, 221–224.

PRIESTLEY, M. B. (1965). Evolutionary spectra and non-stationary processes. *J. R. Statist. Soc.* B, **27**, 204–237.

—— (1966). Design relations for non-stationary processes. *J. R. Statist. Soc.* B, **28**, 228–240.

—— (1967). Power spectral analysis of non-stationary random processes. *J. Sound Vib.*, **6**, 86–97.

QUENOUILLE, M. H. (1958). The comparison of correlations in time-series. *J. R. Statist. Soc.* B, **20**, 158–164.

SEN, P. K. (1965). Some non-parametric tests for *m*-dependent time-series. *J. Amer. Statist. Ass.*, **60**, 134–147.

SUBBA RAO, T. (1968). A note on the asymptotic relative efficiencies of Cox and Stuart's tests for testing trend in dispersion of a *p*-dependent time series. *Biometrika*, **55**, 381–385.