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STATISTICAL TESTS FOR STATIONARITY WITHIN THE FRAMEWORK OF HARMONIZABLE PROCESSES

by

N. R. Goodman

RESEARCH REPORT

2 August 1965

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STATISTICAL TESTS FOR STATIONARITY WITHIN THE FRAMEWORK OF HARMONIZABLE PROCESSES

Research Report 65-28

2 August 1965

Prepared By:

N. R. Goodman
Special Consultant
Mathematics and Statistics
Research

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Technical Review and Approval:

J. M. Zimmerman
Group Scientist
Mathematics and Statistics
Research

J. E. Witherspoon
Chief
Physics, Mathematics and Measurements
Research

Release Approval:

R. J. Thompson, Jr.
Research Director

TABLE OF CONTENTS

INTRODUCTION AND SUMMARY

Real harmonizable time series $X(t)$ which possesses the (real) spectral representation

$$X(t) = \int_0^\infty [\cos \omega t dU(\omega) + \sin \omega t dV(\omega)]$$

are considered. Essentially all weakly stationary and many nonstationary time series fall in the above category. Properties of the "increments" $dU(\omega)$, $dV(\omega)$ are reviewed under the null hypothesis that $X(t)$ is weakly stationary. From a sample of $X(t)$ taken over $-T \leq t \leq T$ finite counterparts (sample increments) $\Delta U(\omega)$, $\Delta V(\omega)$ of the increments $dU(\omega)$, $dV(\omega)$ are defined and (under certain smoothness conditions) are shown to possess properties analogous to those of the increments $dU(\omega)$, $dV(\omega)$. Various tests for stationarity of $X(t)$ based on statistics that are functions of these sample increments are considered as suggested by the properties of the $dU(\omega)$, $dV(\omega)$ that hold only on the null hypothesis of weak stationarity. These statistics are obtained by partitioning the entire frequency band of interest into, say, J frequency sub-bands (indexed by $j = 1, \ldots, J$) and considering a collection of sample increments $[\Delta U(\omega_{jk}), \Delta V(\omega_{jk})]$, $k = 1, \ldots, K$, associated with each frequency sub-band. Tests involving both a single frequency band and two or more are considered (within the framework of harmonizable times series).

The distribution of each test statistic is given in closed form or in such a manner that critical values required in statistical tests may be readily obtained.

The selection of test statistics was to a certain extent motivated by the desire to characterize the type of nonstationarity occurring when the hypothesis of stationarity was rejected. Thus, in each case a description of the type of nonstationarity that possibly prevails (when that particular test statistic rejects the null hypothesis) is also developed.

A. STATIONARY TIME SERIES CONSIDERED WITHIN THE FRAMEWORK OF HARMONIZABLE TIME SERIES

Consider a real harmonizable time series $X(t)$ with the spectral representation written in real form:

$$X(t) = \int_0^\infty [\cos \omega t dU(\omega) + \sin \omega t dV(\omega)] \quad (0 < \omega < \infty) . \tag{1}$$

When weak (second order) stationarity prevails the increments $dU(\omega)$, $dV(\omega)$, $(0 < \omega < \infty)$ of the processes $U(\omega)$ and $V(\omega)$ have special properties not enjoyed if $X(t)$ is nonstationary. These properties for the case when the spectrum $S(\omega)$ of $X(t)$ is absolutely continuous are

$$EdU(\omega) = 0 = EdV(\omega) , \quad 0 < \omega < \infty; \tag{2}$$

$$E(dU(\omega))^2 = E(dV(\omega))^2 = dS(\omega) = s(\omega)d\omega \tag{3a}$$

$$EdU(\omega)dV(\omega) = 0 ; \quad 0 < \omega < \infty ;$$

and

$$\begin{aligned} &EdU(\omega)dU(\omega') = 0 \\ &EdU(\omega)dV(\omega') = 0 \\ &EdV(\omega)dV(\omega') = 0 , \quad \text{for} \quad \omega \neq \omega' , \quad 0 < \omega , \omega' < \infty . \end{aligned} \tag{3b}$$

In the above equations E denotes the expectation operator, and $s(\omega)$ the spectral density of the time series $X(t)$. (Absolute continuity of

$S(\omega)$ permits writing $dS(\omega) = s(\omega)d\omega$.) In general, for a real harmonizable time series $X(t)$ not necessarily weakly stationary, the increments $dU(\omega)$, $dV(\omega)$, $(0 < \omega < \infty)$ of the processes $U(\omega)$ and $V(\omega)$ possess first and second moment spectra defined by the expressions

$$EdU(\omega) = dM_U(\omega)$$
$$EdV(\omega) = dM_V(\omega) \; ; \quad 0 < \omega < \infty \; ; \tag{4}$$

and

$$EdU(\omega)dU(\omega') = dC_{UU}(\omega,\omega')$$
$$EdU(\omega)dV(\omega') = dC_{UV}(\omega,\omega') \tag{5a}$$
$$EdV(\omega)dV(\omega') = dC_{VV}(\omega,\omega') \; ,$$

with the symmetry conditions

$$dC_{UU}(\omega,\omega') = dC_{UU}(\omega',\omega) \; ,$$
$$dC_{VV}(\omega,\omega') = dC_{VV}(\omega',\omega) \; ; \; 0 < \omega,\omega' < \infty \; . \tag{5b}$$

Here the functions $M_U(\omega)$ and $M_V(\omega)$ in (4) may be regarded to be first moment spectra corresponding to $U(\omega)$ and $V(\omega)$, respectively. Similarly, one may regard the functions $C_{UU}(\omega,\omega')$, $C_{UV}(\omega,\omega')$, and $C_{VV}(\omega,\omega')$ in (5) to be second moment spectra of $U(\omega)$ and $V(\omega)$. Together, $M_U(\omega)$, $M_V(\omega)$ and $C_{UU}(\omega,\omega')$, $C_{UV}(\omega,\omega')$, $C_{VV}(\omega,\omega')$ are defined, respectively, to be the first moment spectra and the second moment spectra of the harmonizable time series $X(t)$. Upon comparing equations (2), (3) with (4), (5) it is seen that in the framework of real harmonizable time series, weak stationarity prevails if and only if the spectra of the harmonizable time series satisfy

$$M_U(\omega) = 0, \quad M_V(\omega) = 0; \quad 0 < \omega < \infty ; \tag{6}$$

$$dC_{UU}(\omega,\omega') = \delta(\omega,\omega')dC(\omega,\omega') = dC_{VV}(\omega,\omega') \tag{7a}$$

$$\text{with} \quad C(\omega,\omega) = S(\omega) ;$$

and

$$C_{UV}(\omega,\omega') = 0; \quad 0 < \omega , \quad \omega' < \infty . \tag{7b}$$

Equation (7a) defined (by the formalism of the Dirac delta function) $C(\omega,\omega')$ such that a Stieltjes integral $\int_R\int f(\omega,\omega')dC_{UU}(\omega,\omega')$ of a function $f(\omega,\omega')$ over a region R in $0 < \omega \le \omega' < \infty$ reduces to a Stieltjes line integral $\int_L f(\omega,\omega)dC(\omega,\omega)$ where L is that part of the line $\omega' = \omega$ contained in R. A similar result holds for integrals with respect to $dC_{VV}(\omega,\omega')$.

B. FINITE ATTAINABLE COUNTERPARTS OF THE INCREMENTS OF A HARMONIZABLE TIME SERIES

Consider a finite realization (sample) of a real harmonizable time series $X(t)$ $-T \le t \le T$ with its spectral representation given by (1). Let $\omega_k > 0$ denote a fixed frequency, and define

$$\Delta U(\omega_k) \equiv \int_0^T \tfrac{1}{2} K_{c\omega_k}(t)[X(t) + X(-t)]dt$$

$$\Delta V(\omega_k) \equiv \int_0^T \tfrac{1}{2} K_{s\omega_k}(t)[X(t) - X(-t)]dt \;. \qquad (8)$$

We suppose that $K_{c\omega_k}(t)$, $K_{s\omega_k}(t)$ have been chosen so that $\Delta U(\omega_k)$, $\Delta V(\omega_k)$ approximate $dU(\omega_k)$ and $dV(\omega_k)$, respectively. From (1)

$$\tfrac{1}{2}[X(t) + X(-t)] = \int_0^\infty \cos \omega t dU(\omega)$$

$$\tfrac{1}{2}[X(t) - X(-t)] = \int_0^\infty \sin \omega t dV(\omega) \;, \qquad (9)$$

and thus from (8) and (9) one obtains

$$\Delta U(\omega_k) = \int_0^\infty F_{c\omega_k}(\omega) dU(\omega)$$

$$\Delta V(\omega_k) = \int_0^\infty F_{s\omega_k}(\omega) dV(\omega) \;, \qquad (10)$$

where

$$F_{c\omega_k}(\omega) \equiv \int_0^T K_{c\omega_k}(t) \cos \omega t dt ,$$
$$F_{s\omega_k}(\omega) \equiv \int_0^T K_{s\omega_k}(t) \sin \omega t dt . \tag{11}$$

The functions $F_{c\omega_k}(\omega)$ and $F_{s\omega_k}(\omega)$ are called filters. Note that if the filters $F_{c\omega_k}(\omega)$, $F_{s\omega_k}(\omega)$ were Dirac delta functions centered at $\omega = \omega_k$, one would have from (10) the formal results $\Delta U(\omega_k) = dU(\omega_k)$, and $\Delta V(\omega_k) = dV(\omega_k)$. It can be shown that filters $F_{c\omega_k}$, $F_{s\omega_k}(\omega)$ which best "approximate" Dirac delta functions centered at $\omega = \omega_k$ for finite T in the frequency range $\omega \geq 0$, have $F_{c\omega_k}(\omega)$ and $F_{s\omega_k}(\omega)$ very nearly equal and the smallest attainable bandwidth B of each is approximately $B = 4\pi/T$. Formulas for such attainable filters are contained in [2]; for present purposes, it will be sufficient to state certain properties of such filters as they are needed.

The first and second moments of $\Delta U(\omega_k)$, $\Delta V(\omega_k)$, $\Delta U(\omega_{k'})$, $\Delta V(\omega_{k'})$ corresponding to any pair of frequencies ω_k, $\omega_{k'}$, may be expressed in terms of the filters $F_{c\omega_k}(\omega)$, $F_{s\omega_k}(\omega)$, $F_{c\omega_{k'}}(\omega)$ $F_{s\omega_{k'}}(\omega)$ and the spectra $M_U(\omega)$, $M_V(\omega)$, $C_{UU}(\omega,\omega')$, $C_{UV}(\omega,\omega')$, $C_{VV}(\omega,\omega')$. for example,

$$E\Delta V(\omega_k) = E \int_0^\infty F_{s\omega_k}(\omega)dV(\omega) = \int_0^\infty F_{s\omega_k}(\omega)EdV(\omega) = \int_0^\infty F_{s\omega_k} dM_V(\omega) , \tag{12}$$

and

$$E\,\Delta U(\omega_k)\,\Delta V(\omega_k) = E\left(\int_0^\infty F_{c\omega_k}(\omega)dU(\omega)\right)\left(\int_0^\infty F_{s\omega_k}(\omega')dV(\omega')\right)$$

$$= E\int_0^\infty\int_0^\infty F_{c\omega_k}(\omega)F_{s\omega_{k'}}(\omega')dU(\omega)dV(\omega') \tag{13}$$

$$= \int_0^\infty\int_0^\infty F_{c\omega_k}(\omega)F_{s\omega_{k'}}(\omega')EdU(\omega)dV(\omega') = \int_0^\infty\int_0^\infty F_{c\omega_k}(\omega)F_{s\omega_{k'}}(\omega')dC_{UV}(\omega,\omega') .$$

The other first and second moments of $\Delta U(\omega_k)$, $\Delta V(\omega_k)$, $\Delta U(\omega_{k'})$, $\Delta V(\omega_{k'})$ are given by analogous formulas. From these expressions, the formulas for first and second moments of linear combinations

$L \equiv \sum_k [a_k \Delta U(\omega_k) + b_k \Delta V(\omega_k)]$ and $L' \equiv \sum_k [a_{k'} \Delta U(\omega_{k'}) + b_{k'} \Delta V(\omega_{k'})]$

are readily determined. For example,

$$E\left(\sum_k a_k \Delta U(\omega_k)\right) \equiv \int_0^\infty \left(\sum_k a_k F_{c\omega_k}(\omega)\right) dM_U(\omega) , \tag{14}$$

and

$$E\left(\sum_k a_k \Delta U(\omega_k)\right)\left(\sum_{k'} b_{k'} \Delta V(\omega_{k'})\right) = \int_0^\infty\int_0^\infty \left(\sum_{k,k'} a_k b_{k'} F_{c\omega_k}(\omega)F_{s\omega_{k'}}(\omega')\right) dC_{UV}(\omega,\omega')$$

$$= \int_0^\infty\int_0^\infty \left(\sum_k a_k F_{c\omega_k}(\omega)\right)\left(\sum_{k'} b_{k'} F_{s\omega_{k'}}(\omega')\right) dC_{UV}(\omega,\omega') .$$

When the filters $F_{c\omega_k}(\omega)$, $F_{s\omega_k}(\omega)$ are selected so that they approximate Dirac delta functions centered at $\omega = \omega_k$ in the sense described above the corresponding $\Delta U(\omega_k)$, $\Delta V(\omega_k)$ will be referred to as finite attainable counterparts of the corresponding increments $dU(\omega_k)$, $dV(\omega_k)$. Henceforth, when $\Delta U(\omega_k)$, $\Delta V(\omega_k)$ of a finite $-T \leq t \leq T$ sample of a real harmonizable time series $X(t)$ are considered, it is persumed that the corresponding filters $F_{c\omega_k}(\omega)$, $F_{s\omega_k}(\omega)$ are chosen so that they possess the properties of the smallest attainable bandwidth filters described earlier.

C. PROPERTIES OF THE FINITE ATTAINABLE COUNTERPARTS OF THE INCREMENTS OF WEAKLY (SECOND ORDER) STATIONARY TIME SERIES

When weak (second order) stationarity of $X(t)$ prevails, then (under suitable regularity conditions that will be stated subsequently) the finite attainable counterparts $\Delta U(\omega_k)$, $\Delta V(\omega_k)$ corresponding to $dU(\omega_k)$, $dV(\omega_k)$ possess properties analogous to those of $dU(\omega_k)$, $dV(\omega_k)$ expressed by (2) and (3). One has

$$E\Delta U(\omega_k) = 0 = E\Delta V(\omega_k). \tag{16}$$

Also,

$$\begin{aligned} E(\Delta U(\omega_k))^2 &= \int_0^\infty F_{c\omega_k}(\omega)dS(\omega) \doteq \int_{\omega_k-\frac{1}{2}B}^{\omega_k+\frac{1}{2}B} F^2_{c\omega_k}(\omega)dS(\omega) , \\ E(\Delta V(\omega_k))^2 &= \int_0^\infty F_{s\omega_k}(\omega)dS(\omega) \doteq \int_{\omega_k-\frac{1}{2}B}^{\omega_k+\frac{1}{2}B} F^2_{s\omega_k}(\omega)dS(\omega) , \end{aligned} \tag{17}$$

so that

$$E(\Delta U(\omega_k))^2 \doteq E(\Delta V(\omega_k))^2 = \int_{\omega_k-\frac{1}{2}B}^{\omega_k+\frac{1}{2}B} F^2_{\omega_k}(\omega)dS(\omega) \tag{18}$$

where the filters $F_{c\omega_k}(\omega)$, $F_{s\omega_k}(\omega)$ have been designed such that

$$F_{c\omega_k}(\omega) \doteq F_{\omega_k}(\omega) \doteq F_{s\omega_k}(\omega) \quad \text{for} \quad \omega_k-\tfrac{1}{2}B \leq \omega \leq \omega_k+\tfrac{1}{2}B . \tag{19}$$

In (17) a regularity condition is imposed on $S(\omega)$, namely that $S(\omega)$ outside the range $\omega_k-\frac{1}{2}B \leq \omega < \omega_k+\frac{1}{2}B$ is such that in relation to the attainable filters $F_{c\omega_k}(\omega)$, $F_{s\omega_k}(\omega)$ the contributions to the infinite integrals from outside the range $\omega_k-\frac{1}{2}B \leq \omega \leq \omega_k+\frac{1}{2}B$ are negligible. Equations (16) and (18) are the finite analogs of (2), and the first equation of (3a), respectively. The finite analog of the second equation of (3a)

$$E(\Delta U(\omega_k))(\Delta V(\omega_k)) = 0 \tag{20}$$

follows from (7b) and (15) since

$$E(\Delta U(\omega_k))(\Delta V(\omega_k)) = \int_0^\infty\int_0^\infty F_{c\omega_k}(\omega)F_{s\omega_k}(\omega')dC_{UV}(\omega,\omega') = 0 \ . \tag{21}$$

For a pair of frequencies $0 < \omega_k,\ \omega_{k'}$ such that $|\omega_k-\omega_{k'}| \geq B$ one also has the following finite analog of (3b):

$$\begin{aligned} &E(\Delta U(\omega_k))(\Delta U(\omega_{k'})) \doteq 0 \ , \\ &E(\Delta U(\omega_k))(\Delta V(\omega_{k'})) = 0 \ , \\ &E(\Delta V(\omega_k))(\Delta V(\omega_{k'})) \doteq 0, \quad \text{for} \quad 0 < \omega_k, \quad \omega_{k'} < 0, \ |\omega_k-\omega_{k'}| \geq B. \end{aligned} \tag{22}$$

The equations of (22) follow from (6), (7a), (7b), and the properties of the filters $F_{c\omega_k}(\omega)$, $F_{s\omega_k}(\omega)$, $F_{c\omega_{k'}}(\omega)$, $F_{s\omega_{k'}}(\omega)$. For example,

$$
\begin{aligned}
E(\Delta U(\omega_k))(\Delta U(\omega_{k'})) &= E\left(\int_0^\infty F_{c\omega_k}(\omega)dU(\omega)\right)\left(\int_0^\infty F_{c\omega_{k'}}(\omega)dU(\omega)\right) \\
&= \int_0^\infty\int_0^\infty F_{c\omega_k}(\omega)F_{c\omega_{k'}}(\omega')dU(\omega)dU(\omega') \\
&= \int_0^\infty\int_0^\infty F_{c\omega_k}(\omega)F_{c\omega_{k'}}(\omega')EdU(\omega)dU(\omega') \\
&= \int_0^\infty\int_0^\infty F_{c\omega_k}(\omega)F_{c\omega_{k'}}(\omega')dC_{UU}(\omega,\omega') \\
&= \int_0^\infty\int_0^\infty F_{c\omega_k}(\omega)F_{c\omega_{k'}}(\omega')\delta(\omega-\omega')dC(\omega,\omega') \\
&= \int_0^\infty F_{c\omega_k}(\omega)F_{c\omega_{k'}}(\omega)dS(\omega) \doteq 0 \;.
\end{aligned}
\tag{23}
$$

The final step in (23) is achieved by imposing the regularity condition on $S(\omega)$ that with the filters $F_{c\omega_k}(\omega)$, $F_{c\omega_{k'}}(\omega)$ "spaced so that their main lobes are non-overlapping" the final integral in (23) essentially vanishes. Similar derivations yield the other equations of (22). Keeping in mind how the attainable filters $F_{c\omega_k}(\omega)$, $F_{s\omega_k}(\omega)$ act as weighting functions in integrals, one may view the regularity conditions imposed on $S(\omega)$ to be effectively equivalent to regarding $F_{c\omega_k}(\omega)$, $F_{s\omega_k}(\omega)$ as function on $0 < \omega < \infty$ that are equal on and vanish outside the frequency band $\omega_k-\frac{1}{2}B \leq \omega \leq \omega_k+\frac{1}{2}B$. The filters $F_{c\omega_k}(\omega)$, $F_{s\omega_k}(\omega)$ are non-negative within this frequency band and furthermore by design, for every pair of frequencies $\omega_{k'}$, $\omega_k > \frac{1}{2}B$, $F_{c\omega_{k'}}(\omega)$, $F_{s\omega_{k'}}(\omega)$, $F_{c\omega_k}(\omega)$, $F_{s\omega_k}(\omega)$ are such that

$$F_{c\omega_{k'}}(\omega) \doteq F_{c\omega_k}(\omega + \omega_k - \omega_{k'})$$
$$F_{s\omega_{k'}}(\omega) \doteq F_{s\omega_k}(\omega + \omega_k - \omega_{k'}) \text{ , } 0 < \omega < \infty \text{ .} \qquad (24)$$

Thus the filters $F_{c\omega_{k'}}(\omega)$, $F_{s\omega_{k'}}(\omega)$ centered at $\omega = \omega_{k'}$, are translates of the filters $F_{c\omega_k}(\omega)$, $F_{s\omega_k}(\omega)$ centered at $\omega = \omega_k$, respectively, for $0 < \omega < \infty$.

Consider now the case where the weak (second order) stationary time series $X(t)$ possesses an absolutely continuous spectrum $S(\omega)$ so that the spectral density $s(\omega)$ exists. We impose the additional regularity conditions that (a) $s(\omega)$ effectively vanishes beyond some suitably high frequency Ω, i.e., $s(\omega) \doteq 0$ when $\omega > \Omega$; and that (b) $s(\omega)$ is sufficiently "smooth" within the frequency band $0 < \omega \le \Omega$ so that if this frequency band is partitioned into say J sub-bands $0 = \omega^{(0)} < \omega \le \omega^{(1)}$, $\omega^{(1)} < \omega \le \omega^{(2)}$, ... , $\omega^{(J-1)} < \omega \le \omega^{(J)} = \Omega$ one may regard $s(\omega)$ to be approximately constant within each such sub-band. Thus, one has

$$s(\omega) \doteq s_j \quad \text{for} \quad \omega^{(j-1)} < \omega \le \omega^{(j)}, \; j = 1,2,\ldots,J. \qquad (25)$$

In the present report it is presumed that the frequency sub-bands

$\omega^{(j-1)} < \omega < \omega^{(j)}$, $j = 1, 2, \ldots, J$ are of equal bandwith Ω/J:

$$\omega^{(j)} = \frac{j}{J}\Omega, \; j = 1, \ldots, J, \tag{26}$$

and further that the bandwidth Ω/J satisfies

$$\Omega/J = KB. \tag{27}$$

That is, the frequency bands over which the spectral density is essentially constant may be further partitioned into an integral number K of frequency bands of fundamental bandwidth B. Consider now the frequencies

$$\omega_{jk} \equiv (j-1)\frac{\Omega}{J} + \tfrac{1}{2}(2k-1)B; \quad j = 1, \ldots, J; \quad k = 1, \ldots, K \tag{28}$$

which are equally spaced a bandwidth B apart; the frequencies $\omega_{jk}(k = 1, \ldots, K)$ are in the j-th frequency sub-band $\omega^{(j-1)} < \omega \leq \omega^{(j)}$, $(j = 1, \ldots, J)$. Corresponding to each ω_{jk} we consider $\Delta U(\omega_{jk})$, $\Delta V(\omega_{jk})$, $(j = 1, \ldots, J; \; k = 1, \ldots, K)$. From the regularity conditions introduced above and (17) one obtains for each $j = 1, \ldots, J$

$$\begin{aligned} E(\Delta U(\omega_{jk}))^2 &\doteq \int_{\omega_{jk}-\frac{1}{2}B}^{\omega_{jk}+\frac{1}{2}B} F^2_{c\omega_{jk}}\, dS(\omega) = s_j \int_{\omega_{jk}-\frac{1}{2}B}^{\omega_{jk}+\frac{1}{2}B} F^2_{c\omega_{jk}}(\omega)d\omega, \\ E(\Delta V(\omega_{jk}))^2 &\doteq \int_{\omega_{jk}-\frac{1}{2}B}^{\omega_{jk}+\frac{1}{2}B} F^2_{s\omega_{jk}}(\omega)dS(\omega) = s_j \int_{\omega_{jk}-\frac{1}{2}B}^{\omega_{jk}+\frac{1}{2}B} F^2_{s\omega_{jk}}(\omega)d\omega, \end{aligned} \tag{29}$$

$$(k = 1, \ldots, K).$$

The translation property (24) and the other properties of the filters $F_{c\omega_{jk}}(\omega)$, $F_{s\omega_{jk}}(\omega)$ described above, yield

$$\int_{\omega_{jk}-\frac{1}{2}B}^{\omega_{jk}+\frac{1}{2}B} F^2_{c\omega_{jk}}(\omega)d\omega \doteq \int_{\omega_{jk}-\frac{1}{2}B}^{\omega_{jk}+\frac{1}{2}B} F^2_{s\omega_{jk}}(\omega)d\omega \doteq \text{Constant} \tag{30}$$

$$(j = 1, \ldots, J;\ k = 1, \ldots, K).$$

By suitable normalization of the filters $F_{c\omega_{jk}}(\omega)$, $F_{s\omega_{jk}}(\omega)$ one may take the constant in (30) to be unity, so that (29) then becomes

$$E(\Delta U(\omega_{jk}))^2 \doteq s_j,$$
$$E(\Delta V(\omega_{jk}))^2 \doteq s_j, \quad (j = 1, \ldots, J;\ k = 1, \ldots, K). \tag{31}$$

For the frequencies ω_{jk} given by (28) one has (22) prevailing, so that for $j, j' = 1, \ldots, J;\ k, k' = 1, \ldots, K$

$$E(\Delta U(\omega_{jk}))(\Delta U(\omega_{j'k'})) \doteq 0\ ,$$
$$E(\Delta U(\omega_{jk}))(\Delta V(\omega_{j'k'})) = 0, \tag{32}$$
$$E(\Delta V(\omega_{jk}))(\Delta V(\omega_{j'k'})) \doteq 0; \quad \text{unless } j = j' \text{ and } k = k'\ .$$

From (20) it follows (for all j and k) that

$$E(\Delta U(\omega_{jk}))(\Delta V(\omega_{jk})) = 0\ , \tag{33}$$

and from (16)

$$E\Delta U(\omega_{jk}) = 0 = E\Delta V(\omega_{jk}) \; . \qquad (34)$$

Under the smoothness conditions described above, equations (34), (31), (32), and (33) give the means, variances, and covariances respectively of all the $\Delta U(\omega_{jk})$, $\Delta V(\omega_{jk})$ for $j = 1, \ldots, J$; $k = 1, \ldots, K$. In Section A, the first moment and second moment properties of the increments $dU(\omega)$, $dV(\omega)$, $(0 < \omega < \infty)$ of a real harmonizable time series $X(t)$ were described in terms of first moment and second moment spectra, respectively. Furthermore, it was pointed out that weak (second order) stationarity of $X(t)$ prevailed if and only if the first moment and second moment spectra satisfied certain relations. From equations (31) through (34) we have that the finite attainable counterparts $\Delta U(\omega_{jk})$, $\Delta V(\omega_{jk})$, $(j = 1, \ldots, J;\; k = 1, \ldots, k)$ of the increments $dU(\omega)$, $dV(\omega)$, $(0 < \omega < \infty)$ satisfy analogous equations. If one considers equations (31) through (34) in the case when $J = 1$, the analogy is especially clear. Generally, however, J will be taken to be greater than unity in order to handle the statistical tests for stationarity discussed in the following section.

D. STATISTICAL TESTS FOR STATIONARITY OF A TIME SERIES

Suppose a single finite $-T \le t \le T$ realization (sample) of a real harmonizable time series $X(t)$ is observed. When the time series $X(t)$ is weakly (second order) stationary, it is presumed that its spectrum $S(\omega)$ is absolutely continuous, and that the spectral density $s(\omega)$ of $X(t)$ satisfies the "smoothness" conditions described in Section C. The finite attainable counterparts $\Delta U(\omega_{jk})$, $\Delta V(\omega_{jk})$, $(j = 1, \ldots, J; k = 1, \ldots, K)$ of the increments $dU(\omega)$, $dV(\omega)$, $(0 < \omega < \infty)$ are then presumed to satisfy equation (31) through (34) and statistical tests for stationarity of $X(t)$ will be developed based on the <u>null hypothesis</u> that the $\Delta U(\omega_{jk})$, $\Delta V(\omega_{jk})$, $(j = 1, \ldots, J; k = 1, \ldots, K)$ possess the special properties stated by these equations. In this regard, J and K are presumed known, however, the s_j, $(j=1, \ldots, J)$ are presumed unknown.

If $X(t)$ is a stationary Gaussian time series, then $\Delta U(\omega_{jk})$, $\Delta V(\omega_{jk})$, are JK-variate Gaussian distributed with means, variances, and covariances given by equations (31) through (34). The $\Delta U(\omega_{jk})$, $\Delta V(\omega_{jk})$, are obtained from $X(t)$, $-T \le t \le T$ by the process of "narrow band-pass" filtering. Many stationary <u>non-Gaussian</u> time series become nearly Gaussian when "passed through" sufficiently narrow band filters (see [4]). The bandwith B of the filters $F_{c\omega_{jk}}(\omega)$, $F_{s\omega_{jk}}(\omega)$ will be small when

T is moderately large. In order to determine the distribution of test statistics on the null hypothesis, we will assume further the time series $X(t)$ is such that the procedure of "narrow band-pass filtering" $X(t)$, $-T \leq t \leq T$ employed to obtain the $\Delta U(\omega_{jk})$, $\Delta V(\omega_{jk})$, makes these quantities Gaussian. Thus, in addition to the properties expressed by equations (31) through (34), the $\Delta U(\omega_{jk})$, $\Delta V(\omega_{jk})$, $(j = 1, \ldots , J; k = 1, \ldots , K)$ are JK-variate Gaussian distributed on the null hypothesis.

Test statistics will now be considered which, in a certain sense, detect or measure types of nonstationarity implicitly on rejecting the null hypothesis of stationarity. Since the test statistics for stationarity will be functions of the frequency domain random variables $\Delta U(\omega_{jk})$, $\Delta V(\omega_{jk})$, $(j = 1, \ldots , J; k = 1, \ldots , K)$ such descriptions will generally occur in the frequency domain.

For ease of notation, the random variables $\Delta U(\omega_{jk})$, $\Delta V(\omega_{jk})$ will now be denoted by U_{jk}, V_{jk}, respectively, $(j = 1, \ldots, J; k = 1, \ldots , K)$. As a consequence of the null hypothesis described above one is able to introduce the test statistics given below, and either obtain from existing statistical theory or derive the sampling distribution of each such test statistic.

TEST STATISTICS BASED ON REAL MULTIVARIATE GAUSSIAN STATISTICAL ANALYSIS

Tests Within a Single Frequency Interval

The random variables (U_{jk}, V_{jk}) $(k = 1, \ldots, K)$ obtained from the j-th frequency sub-band comprise a random sample of size K from a bivariate zero mean Gaussian distribution with correlation coefficient zero and equal variances s_j.

The test statistics

$$t_{Uj} = \frac{\sqrt{K-1}\left(\sum_{k=1}^{K} U_{jk}\right)}{\sqrt{K}\left(\sum_{k=1}^{K}\left[U_{jk}-\left(\frac{1}{K}\sum_{k=1}^{K} U_{jk}\right)\right]^2\right)^{1/2}}$$

$$t_{Vj} = \frac{\sqrt{K-1}\left(\sum_{k=1}^{K} V_{jk}\right)}{\sqrt{K}\left(\sum_{k=1}^{K}\left[V_{jk}-\left(\frac{1}{K}\sum_{k=1}^{K} V_{jk}\right)\right]^2\right)^{1/2}} \qquad (35)$$

are therefore independently student's t distributed with $K-1$ degrees of freedom. If only the zero mean condition of the null hypothesis may possibly be violated, the statistics given by (35) then serve as tests to detect such violations. With reference to the discussion on harmonizable time series of Section A, such violations indicate that the first moment spectra do not vanish in the j-th frequency sub-band, so that the harmonizable time series $X(t)$ then possesses a non-constant "trend" component.

The test statistic

$$F_j = \frac{\sum_{k=1}^{K} U_{jk}^2}{\sum_{k=1}^{K} V_{jk}^2} \tag{36}$$

possesses the standard F distribution with K degrees of freedom in both numerator and denominator. With reference to (31) let now

$$\begin{aligned} EU_{jk}^2 &= s_{Uj} , \\ EV_{jk}^2 &= s_{Vj} , \quad (k = 1, \dots , K) . \end{aligned} \tag{37}$$

If only the condition $s_{Uj} = s_{Vj}$ of the null hypothesis may possibly be violated, the test statistic F_j then serves to detect such a violation of variance equality. Within the framework of harmonizable time series, such a violation indicates that from the j-th frequency sub-band more variance or "power" is contributed to $X(t)$ from the cosine components than from the sine components (or vice versa).

The test statistic

$$F_j' = \frac{\sum_{k=1}^{K} \left[U_{jk} - \left(\frac{1}{K}\sum_{k=1}^{K} U_{jk}\right)\right]^2}{\sum_{k=1}^{K} \left[V_{jk} - \left(\frac{1}{K}\sum_{k=1}^{K} V_{jk}\right)\right]^2} \tag{38}$$

possesses the standard F distribution with $K-1$ degrees of freedom in both numerator and denominator. In testing for the variance equality $s_{Uj} = s_{Vj}$, the statistic F_j' is preferable to F_j if one suspects that the U_{jk}, V_{jk} do not have zero mean.

The test statistic

$$\hat{\rho}_{UVj} = \frac{\sum_{k=1}^{K} U_{jk}V_{jk}}{\left(\sum_{k=1}^{K} U_{jk}^2\right)^{1/2}\left(\sum_{k=1}^{K} V_{jk}^2\right)^{1/2}} \tag{39}$$

is the sample correlation coefficient between U_{jk} and V_{jk}, $(k = 1, \ldots, K)$, and possesses the usual sample correlation coefficient distribution with parameters $\rho = 0$ and degrees of freedom K. The probability density function of that distribution is given by

$$\frac{\Gamma\left[\frac{1}{2}K\right]}{\Gamma\left[\frac{1}{2}(K-1)\right]\sqrt{\pi}} (1-r^2)^{\frac{1}{2}(K-3)} \qquad (-1 < r < 1) . \tag{40}$$

The statistic $\hat{\rho}_{UVj}$ can be used to test that the correlation coefficient ρ_{UVj} between U_{jk} and V_{jk} in the j-th frequency sub-band is zero. It is to be noted that the distribution (40) holds even when the condition of equal variance $(s_{Uj} = s_{Vj})$ of the null hypothesis is violated. Within the framework of harmonizable time series, when only the condition $\rho_{UVj} = 0$ is violated it is an indication that the cosine and sine components of $X(t)$ from the j-th frequency sub-band are correlated.

The test statistic

$$\hat{\rho}'_{UVj} = \frac{\sum_{k=1}^{K} \left[U_{jk} - \left(\frac{1}{K}\sum_{k=1}^{K} U_{jk}\right)\right]\left[V_{jk} - \left(\frac{1}{K}\sum_{k=1}^{K} V_{jk}\right)\right]}{\left(\sum_{k=1}^{K}\left[U_{jk} - \left(\frac{1}{K}\sum_{k=1}^{K} U_{jk}\right)\right]^2\right)^{1/2}\left(\sum_{k=1}^{K}\left[V_{jk} - \left(\frac{1}{K}\sum_{k=1}^{K} V_{jk}\right)\right]^2\right)^{1/2}} \tag{41}$$

is preferable to $\hat{\rho}_{UVj}$ in testing for zero correlation if the U_{jk}, V_{jk} $(k = 1, \ldots , K)$ violate the null hypothesis by not possessing zero means (or one suspects that to be so on other grounds). The probability density of the distribution of $\hat{\rho}'_{UVj}$ is given by (40) with the parameter K replaced by $K-1$.

Tests Involving More Than One Frequency Interval

Consider the random variables $(U_{jk}, V_{jk}, U_{j'k}, V_{j'k})$ $(k = 1, \ldots , K)$ obtained from the j-th and the j'-th frequency sub-bands $(j \neq j')$.

These random variables comprise a random sample of size K from a four-variate zero mean Gaussian distribution with a diagonal covaraince given by

$$\sum_{Rjj'} = \begin{bmatrix} s_j & & & 0 \\ & s_j & & \\ & & s_{j'} & \\ 0 & & & s_{j'} \end{bmatrix} . \tag{42}$$

The test statistics

$$\hat{\rho}_{U_j U_{j'}} = \frac{\sum_{k=1}^{K} U_{jk} U_{j'k}}{\left(\sum_{k=1}^{K} U_{jk}^2\right)^{1/2} \left(\sum_{k=1}^{K} U_{j'k}^2\right)^{1/2}} , \tag{43a}$$

$$\hat{\rho}_{V_j V_{j'}} = \frac{\sum_{k=1}^{K} V_{jk} V_{j'k}}{\left(\sum_{k=1}^{K} V_{jk}^2\right)^{1/2} \left(\sum_{k=1}^{K} V_{j'k}^2\right)^{1/2}} , \tag{43b}$$

$$\hat{\rho}_{U_j V_{j'}} = \frac{\sum_{k=1}^{K} U_{jk} V_{j'k}}{\left(\sum_{k=1}^{K} U_{jk}^2\right)^{1/2} \left(\sum_{k=1}^{K} V_{j'k}^2\right)^{1/2}} , \tag{43c}$$

$$\hat{\rho}_{U_{j'},V_j} = \frac{\sum_{k=1}^{K} U_{j'k}V_{jk}}{\left(\sum_{k=1}^{K} U^2_{j'k}\right)^{1/2}\left(\sum_{k=1}^{K} V^2_{jk}\right)^{1/2}} , \qquad (43d)$$

are sample correlation coefficients which estimate the correlations $\rho_{U_jU_{j'}}$, $\rho_{V_jV_{j'}}$, $\rho_{U_jV_{j'}}$, $\rho_{U_{j'}V_j}$, respectively. These are the various correlations between the random variables U_{jk}, V_{jk} in the j-th frequency sub-band and the random variables $U_{j'k}$, $V_{j'k}$ in the j'-th frequency sub-band. Each of the statistics in (43) is distributed with the usual sample correlation coefficient distribution (with parameters $\rho = 0$ and degrees of freedom K) having the probability density function given by (40) and can be used to test that the correlation coefficients $\rho_{U_jU_{j'}}$, $\rho_{V_jV_{j'}}$, $\rho_{U_jV_{j'}}$, $\rho_{U_{j'},V_j}$ are zero. The variance equalities of the null hypothesis indicated by the diagonal elements of the matrix $\Sigma_{Rjj'}$ given by (42) may be violated without affecting the distribution of these test statistics. Within the framework of harmonizable time series, violations of the condition that the $\rho_{U_jU_{j'}}$, $\rho_{V_jV_{j'}}$, $\rho_{U_jV_{j'}}$, $\rho_{U_{j'},V_j}$ are zero indicate, respectively, that the second moment spectra $C_{UU}(\omega,\omega')$, $C_{VV}(\omega,\omega')$, $C_{UV}(\omega,\omega')$ are not zero in a region of the (ω,ω') plane off the diagonal $\omega = \omega'$. Specifically, this region is defined by the inequalities $\omega^{(j-1)} < \omega \leq \omega^{(j)}$, $\omega^{(j'-1)} < \omega' \leq \omega^{(j')}$, $(j \neq j')$.

In testing for the vanishing of the correlations $\rho_{U_j U_{j'}}$, $\rho_{V_j V_{j'}}$, $\rho_{U_j V_{j'}}$, $\rho_{U_{j'} V_j}$ test statistics $\hat{\rho}_{U_j U_{j'}}$, $\hat{\rho}_{V_j V_{j'}}$, $\hat{\rho}_{U_j V_{j'}}$, $\hat{\rho}_{U_{j'} V_j}$ given by

$$\hat{\rho}'_{U_j U_{j'}} = \frac{\sum_{k=1}^{K} \left[U_{jk} - \left(\frac{1}{K} \sum_{k=1}^{K} U_{jk} \right) \right] \left[U_{j'k} - \left(\frac{1}{K} \sum_{k=1}^{K} U_{j'k} \right) \right]}{\left(\sum_{k=1}^{K} \left[U_{jk} - \left(\frac{1}{K} \sum_{k=1}^{K} U_{jk} \right) \right]^2 \right)^{1/2} \left(\sum_{k=1}^{K} \left[U_{j'k} - \left(\frac{1}{K} \sum_{k=1}^{K} U_{j'k} \right) \right]^2 \right)^{1/2}} \tag{44}$$

and analogous expressions for $\hat{\rho}_{V_j V_{j'}}$, $\hat{\rho}_{U_j V_{j'}}$, $\hat{\rho}_{U_{j'} V_j}$ are preferable to $\hat{\rho}_{U_j U_{j'}}$, $\hat{\rho}_{V_j V_{j'}}$, $\hat{\rho}_{U_j V_{j'}}$, $\hat{\rho}_{U_{j'} V_j}$ if the zero mean conditions of the null hypothesis are in doubt. On the null hypothesis each of the test statistics of (44) is distributed with the sample correlation coefficient distribution with parameters $\rho = 0$ and degrees of freedom K-1.

These tests over two frequency bands can be generalized. Consider the random variables $\eta'_k = (U_{j_1 k}, V_{j_1 k}, \ldots, U_{j_p k}, V_{j_p k})$, $(k = 1, \ldots, K)$ obtained from p different frequency sub-bands indexed by $j_1, j_2, \ldots, j_p$, $(j_1 < j_2 < \ldots < j_p)$. These random variables comprise a random sample of size K from a 2p-variate zero mean Gaussian distribution with a diagonal covariance matrix given by

$$\sum_{Rj_1, j_2, \ldots, j_p} = \begin{bmatrix} s_{j_1} & & & & & 0 \\ & s_{j_1} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ 0 & & & & s_{j_p} & \\ & & & & & s_{j_p} \end{bmatrix} . \quad (45)$$

The multiple correlation coefficient between any component of η_k, say for example $U_{j_1 k}$, and the remaining 2p-1 components $(V_{j_1 k}, U_{j_2 k}, V_{j_2 k}, \ldots, U_{j_p k}, V_{j_p k})$ is denoted by $R_{U_{j_1} \cdot V_{j_1}, U_{j_2}, V_{j_2}, \ldots, U_{j_p}, V_{j_p}}$. On the null hypothesis such multiple correlation coefficients are zero. As before we do not invoke the variance equalities of the null hypothesis indicated by the diagonal elements of the matrix $\Sigma_{Rj_1, j_2, \ldots, j_p}$ given by (45). From the η_k, $(k = 1, \ldots, K)$ one obtains then the $2p \times 2p$ sample covariance matrix

$$\hat{\sum}_{Rj_1, j_2, \ldots, j_p} = \frac{1}{K} \sum_{k=1}^{K} \eta_k \eta_k' . \quad (46)$$

from whose elements the test statistic (sample multiple correlation coefficient) $\hat{R}_{U_{j_1} \cdot V_{j_1}, U_{j_2}, V_{j_2}, \ldots, U_{j_p}, V_{j_p}}$ is determined. Let such test statistics be denoted simply by $\hat{R}$. On the null hypothesis, the quantity

$$\left(\frac{K+1-2p}{2p-1}\right)\left[\frac{\hat{R}^2}{1-\hat{R}^2}\right] \tag{47}$$

has the F distribution with 2p-1 and K+1-2p degrees of freedom. (It is presumed that $K > 2p$.) The statistics $\hat{R}$ are used to test whether multiple correlation coefficients such as

$R_{U_{j_1} \cdot V_{j_1}, U_{j_2}, V_{j_2}, \ldots, U_{j_p}, V_{j_p}}$ are indeed zero.

One may consider other multiple correlation coefficients, for example, $R_{U_{j_1} \cdot U_{j_2}, U_{j_3}, \ldots, U_{j_p}}$, that are zero on the null hypothesis, and use the corresponding sample multiple correlation coefficients ($\hat{R}_{U_{j_1} \cdot U_{j_2}, U_{j_3}, \ldots, U_{j_p}}$ in this case), to test that such multiple correlation coefficients are zero. Letting $\hat{R}$ denote $\hat{R}_{U_{j_1} \cdot U_{j_2}, U_{j_3}, \ldots, U_{j_p}}$ then on the null hypothesis the quality

$$\left(\frac{K+1-p}{p-1}\right)\left[\frac{\hat{R}^2}{1-\hat{R}^2}\right] \tag{48}$$

is F distributed with p-1 and K+1-p degrees of freedom.

Within the framework of harmonizable time series, the detection of non-vanishing multiple correlation coefficients of the type introduced above indicates more subtle or complex relations between the frequency components of the time series then could be indicated by the criteria discussed previously.

If the zero mean conditions of the null hypothesis are in doubt, it is preferable to determine sample multiple correlation coefficients from the sample covariance matrix

$$\hat{\sum}'_{R_j, j_2, \ldots, j_p} = \frac{1}{K-1} \sum_{k=1}^{K} (\eta_k - \bar{\eta})(\eta_k - \bar{\eta})' \qquad (49)$$

$$\left(\text{where} \quad \bar{\eta} = \frac{1}{K} \sum_{k=1}^{K} \eta_k\right) .$$

On the null hypothesis, the distributional statements made above are then modified by changing K to $K-1$.

TEST STATISTICS BASED ON MULTIVARIATE COMPLEX GAUSSIAN STATISTICAL ANALYSIS

In this section we present tests for stationarity which are based on complex Gaussian statistical analysis (for an introduction to the latter topic see [3]).

Coherence Tests

Consider the complex random variables $W_{jk} = U_{jk} - iV_{jk}$, $W_{j'k} = U_{j'k} - iV_{j'k}$, $(k = 1, \ldots, K)$ obtained from the j-th and j'-th frequency sub-bands $(j \neq j')$. The complex random variables $(W_{jk}, W_{j'k})$

comprise a random sample of size K from a bivariate zero mean complex Gaussian distribution with a diagonal covariance matrix given by

$$\sum_{cjj'} = \begin{bmatrix} 2s_j & 0 \\ 0 & 2s_{j'} \end{bmatrix} . \tag{50}$$

[The complex random variables $(W_{jk}, \bar{W}_{j'k})$, $(k = 1, \ldots , K)$ also comprise a random sample of size K with the same properties as those stated for $(W_{jk}, W_{j'k})$, $(k = 1, \ldots , K)$. The results described below also hold if $(W_{jk}, \bar{W}_{j'k})$ is substituted for $(W_{jk}, W_{j'k})$.]

The test statistic

$$\hat{R}^2_{cjj'} \equiv \hat{Coh}_{jj'} = \frac{\left| \sum_{k=1}^{K} W_{jk}\bar{W}_{j'k} \right|^2}{\left(\sum_{k=1}^{K} \left| W_{jk} \right|^2 \right)\left(\sum_{k=1}^{K} \left| W_{j'k} \right|^2 \right)} \tag{51}$$

is the sample coherence between W_{jk} and $W_{j'k}$, $(k = 1, \ldots , K)$, and is distributed with the sample coherence distribution with parameters population coherence = $Coh_{jj'} \equiv R^2_{cjj'} = 0$ and degrees of freedom K (see [3]). The probability density function of that distribution is given by

$$p(\hat{R}^2_c) = (K-1)(1-\hat{R}^2_c)^{K-2} , \quad (0 \leq \hat{R}^2_c < 1). \tag{52}$$

and the statistic $\hat{R}^2_{cjj'}$ can be used to test that the coherence $R^2_{cjj'}$ between the random variables W_{jk} and $W_{j'k}$, $(k = 1, \ldots, K)$, is zero. Unlike the real case discussed in the previous section the distribution of the test statistic $\hat{R}^2_{cjj'}$ utilizes the variance equalities of the null hypothesis indicated by the diagonal elements of the matrix $\Sigma_{Rjj'}$ given by (42). Within the framework of harmonizable time series, a violation solely of the condition that the coherence $R^2_{cjj'}$ is zero indicates that the second moment spectra $C_{UU}(\omega,\omega')$, $C_{UV}(\omega,\omega')$, $C_{VV}(\omega,\omega')$ are not all zero off the diagonal in a region of the (ω,ω') plane defined by $\omega^{(j-1)} < \omega \leq \omega^{(j)}$, $\omega^{(j'-1)} < \omega' \leq \omega^{(j')}$, $(j \neq j')$. Even more can be said; let the j-th filtered component of the time series $X(t)$ be defined by

$$X_{jj}(t) = \int_{\omega^{(j-1)}}^{\omega^{(j)}} \left[\cos \omega t dU(\omega) + \sin \omega t dV(\omega)\right] \tag{53}$$

and the j'-th filtered component of the time series $X(t)$ "heterodyned" from the frequency band $\omega^{(j'-1)} < \omega \leq \omega^{(j')}$ to the frequency band $\omega^{(j-1)} < \omega \leq \omega^{(j)}$ be defined by

$$X_{j'j}(t) \equiv \int_{\omega^{(j-1)}}^{\omega^{(j)}} \left[\cos \omega t dU(\omega + \omega^{(j')} - \omega^{(j)}) + \sin \omega t dV(\omega + \omega^{(j')} - \omega^{(j)})\right]. \tag{54}$$

A violation solely of the condition that the coherence $R^2_{cjj'}$ is zero indicates that the time series $X_{jj}(t)$ is to some degree related to the time series $X_{j'j}(t)$ by means of a linear time invariant operator L, i.e., to some degree the relation

$$X_{jj}(t) = LX_{j'j}(t) \tag{55}$$

prevails.

If the zero mean conditions of the null hypothesis are in doubt, it is recommended that the test statistic

$$\hat{R}'^2_{cjj'} = \frac{\left| \sum_{k=1}^{K} (W_{jk}-\tilde{W}_j)\, \overline{(W_{j'k}-\tilde{W}_{j'})} \right|^2}{\left(\sum_{k=1}^{K} |W_{jk}-\tilde{W}_j|^2 \right) \left(\sum_{k=1}^{K} |W_{j'k}-\tilde{W}_{j'}|^2 \right)} \tag{56}$$

where

$$\tilde{W}_j \equiv \frac{1}{K} \sum_{k=1}^{K} W_{jk},\quad \tilde{W}_{j'k} = \frac{1}{K} \sum_{k=1}^{K} W_{j'k} \tag{57}$$

be used instead of (51). If the complex random variables $(W_{jk}, W_{j'k})$, $(k = 1, \ldots, K)$, possess a constant mean, i.e., if $E(W_{jk}, W_{j'k}) = (\nu_j, \nu_{j'})$, $(k = 1, \ldots, K)$, then the distribution of $\hat{R}'^2_{cjj'}$ of (56) is given by (52) with K changed to K-1. Also, in that case, the above discussion concerning the relationship between $X_{jj}(t)$ and $X_{j'j}(t)$ still holds provided the latter are redefined so as to have no trend component.

Tests Involving the Complex Second Moment Spectra

Next recalling the discussion of Section A, we let

$$dW(\omega) \equiv dU(\omega) - idV(\omega), \quad (0 < \omega < \infty) \tag{58}$$

and for a real harmonizable time series $X(t)$ define the complex second moment spectra $C(\omega,\omega')$ and $C^*(\omega,\omega')$ by means of the expressions

$$\begin{aligned} C(\omega,\omega') &\equiv E dW(\omega)\overline{dW(\omega')} \\ C^*(\omega,\omega') &\equiv E dW(\omega) dW(\omega'), \quad 0 < \omega,\ \omega' < \infty . \end{aligned} \tag{59}$$

Next we consider harmonizable time series in a manner analogous to the discussion of Section C. From the condition that the complex second moment spectra are essentially constant, $C(\omega,\omega') \doteq C_{jj'}$, $C^*(\omega,\omega') \doteq C^*_{jj'}$ in the (ω,ω') regions defined by $\omega^{(j-1)} < \omega < \omega^{(j)}$, $\omega^{(j'-1)} < \omega' \leq \omega^{(j')}$, $(j \neq j')$, it is natural to introduce

$$\begin{aligned} \hat{C}_{jj'} &= \frac{1}{K^2} \sum_{k,k'=1}^{K} W_{jk}\overline{W}_{j'k} , \\ \hat{C}^*_{jj'} &= \frac{1}{K^2} \sum_{k,k'=1}^{K} W_{jk}W_{j'k} , \quad (j \neq j') \end{aligned} \tag{60}$$

as estimators for the respective $C_{jj'}$, $C^*_{jj'}$. We can then find statistics for testing the null hypothesis that the $C_{jj'}$, $C^*_{jj'}$ vanish. (Results will be given for $C_{jj'}$; in any equation, however, if $W_{j'k'}$ is replaced by $\bar{W}_{j'k'}$ one then obtains the corresponding equation for $C^*_{jj'}$. The statistical results pertaining for the $C_{jj'}$ pertain identically for the $C^*_{jj'}$.)

The test statistic (a normalized form of $|\hat{C}_{jj'}|^2$) given by

$$\hat{T}_{\beta jj'} = \frac{\left| \sum_{k,k'=1}^{K} W_{jk}\bar{W}_{j'k'} \right|^2}{K^2\left(\sum_{k=1}^{K} |W_{jk}|^2\right)\left(\sum_{k'=1}^{K} |W_{j'k'}|^2\right)} \tag{61}$$

can be used to test that a particular $C_{jj'}$ is zero. On the null hypothesis $\hat{T}_{\beta jj'}$ is distributed as is the product of two independent and identically distributed beta random variables with parameters $\alpha = 0$ and $\beta = K-2$ each of which has the density function

$$f_\beta(x) = (K-1)(1-x)^{K-2}, \quad (0 \le x \le 1). \tag{62}$$

The Laplace transform of the probability density function of a random variable $-\ell nX$ with X having the probability density (62) is

$$E \exp(t\ell nX) = \int_0^1 (K-1)x^t(1-x)^{K-2}dx = \Gamma(K)\,\frac{\Gamma(1+t)}{\Gamma(K+t)}, \tag{63}$$

$$(0 < t < \infty).$$

Thus, if X_1 and X_2 denote the two independent beta random variables the Laplace transform of the probability density function of the random variable $-\ell n(X_1 X_2) = -\ell n\, X_1 - \ell n X_2$ is given by

$$E \exp[t \ell n(X_1 X_2)] = \Gamma^2(K) \frac{\Gamma^2(1+t)}{\Gamma^2(K+t)}, \quad (0 < t < \infty). \tag{64}$$

Since K is an integer, the Laplace transform in (64) simplifies to the reciprocal of a polynomial in t with zeros of order 2 at $t = -1, -2, \ldots, -(K-1)$. The inverse Laplace transform of such a function is readily obtained by partial fraction decomposition and yields the probability density function $g(y)$ of the random variable $Y = -\ell n(X_1 X_2)$. The function $g(y)$ is of the form

$$g(y) = \sum_{k=1}^{K-1} (a_k + b_k y) e^{-ky}, \quad (0 \le y < \infty), \tag{65}$$

where a_k, b_k, $(k = 1, \ldots, K-1)$ are appropriate constants. The cumulative distribution function of the random variable Y (or equivalently the random varaible $\hat{T}_{\beta jj'}$) may be obtained in closed form by integrating $g(y)$ between suitable limits.

When the zero mean conditions of the null hypothesis are suspect the test statistic (which is an alternative normalized form of $|\hat{C}_{jj'}|^2$) given by

$$\hat{T}_{Fjj'} = \frac{(K-1)^2 \left| \sum_{k,k'=1}^{K} W_{jk} \bar{W}_{j'k'} \right|^2}{K^2 \left(\sum_{k=1}^{K} |W_{jk} - \tilde{W}_j|^2 \right) \left(\sum_{k=1}^{K} |W_{j'k'} - \tilde{W}_{j'}|^2 \right)} \tag{66}$$

(where $\tilde{W}_j$ and $\tilde{W}_{j'}$ are given by (57)) may be used to test that a particular $C_{jj'}$, $(j \neq j')$, is zero. On the null hypothesis $\hat{T}_{Fjj'}$ is distributed as is the product of two independent and identically distributed $F_{2,2(K-1)}$ random variables each of which possess a probability density function given by

$$f_F(x) = \left[1 + \frac{x}{K-1}\right]^{-K}, \quad (0 \leq x < \infty). \tag{67}$$

Thus

$$\text{Prob}\left[\hat{T}_{Fjj'} \geq c_o\right] = \int_{o^+}^{\infty} \left(1 + \frac{x_1}{K-1}\right)^{-K} \left[\int_{c_o/x_1}^{\infty} \left(1 + \frac{x_2}{K-1}\right)^{-K} dx_2 \right] dx_1$$

$$= (K-1) \int_o^{\infty} \frac{u^{K-1} du}{[c_o(K-1)^{-2} + u]^{K-1} [1+u]^K} . \tag{68}$$

The numerical evaluation of the final integral of (68) to determine critical values c_o should present no difficulty.

The two test statistics $\hat{T}_{\beta jj'}$ and $\hat{T}_{Fjj'}$ are closely related to estimates of the complex second moment spectrum $C(\omega, \omega')$ of a real harmonizable time series $X(t)$ in regions (ω, ω') described previously. These regions do not intersect the diagonal $\omega = \omega'$ and if the null hypothesis is rejected, it is an indication that the complex second moment spectrum $C(\omega, \omega')$ fails to vanish in such regions. Suppose that the tests based on $\hat{T}_{\beta jj'}$ and $\hat{T}_{Fjj'}$ are matched so that both have the same probability of rejecting the null hypothesis when that hypothesis is true. The tests based on $\hat{T}_{Fjj'}$ will then have a higher probability of rejecting the null hypothesis due to violations of the zero mean conditions than will the tests based on $\hat{T}_{\beta jj'}$ since in this case the former will have a smaller denominator.

Tests Over More Than Two Frequency Bands

Consider the random variables $\xi_k' = (W_{j_1,k}, W_{j_2,k}, \ldots, W_{j_p,k})$, $(k = 1, \ldots, K)$ obtained from p different frequency sub-bands indexed by $j_1, j_2, \ldots, j_p$, $(j_1 < j_2 < \ldots < j_p)$. These random variables comprise a random sample of size K from a p-variate zero mean complex Gaussian distribution with a diagonal covariance matrix given by

$$\sum\nolimits_{Cj_1,j_2,\ldots,j_p} = \begin{bmatrix} 2s_{j_1} & & & 0 \\ & 2s_{j_2} & & \\ & & \ddots & \\ 0 & & & 2s_{j_p} \end{bmatrix} . \qquad (69)$$

(Complex random variables such as $(W_{j_1,k}, \bar{W}_{j_2,k}, W_{j_3,k}, \ldots, \bar{W}_{j_p k})$, $(k = 1, \ldots, K)$ also comprise a random sample of size K with the same properties as those stated for the ξ_k'. Results analogous to those now described for the ξ_k' therefore hold for such random variables.)

The multiple coherence between any component of ξ_k, say for example $W_{j_1,k}$, and the remaining $p=1$ components $(W_{j_2 k}, \ldots, W_{j_p k})$ is denoted by $R^2_{cj_1 \cdot j_2, j_3, \ldots, j_p}$. Under the null hypothesis such multiple coherences are zero. From the ξ_k, $(k = 1, \ldots, K)$ one obtains then the $p \times p$ sample Hermitian covariance matrix

$$\hat{\Sigma}_{Cj_1, j_2, \ldots, j_p} = \frac{1}{K}\sum_{k=1}^{K} \xi_k \bar{\xi}_k' . \tag{70}$$

From the elements of the matrix $\hat{\Sigma}_{Cj_1, j_2, \ldots, j_p}$ one determines (see [3]) the test statistic corresponding sample multiple coherence $\hat{R}^2_{cj_1 \cdot j_2, j_3, \ldots, j_p}$, which we denote simply by $\hat{R}^2_c$. On the null hypothesis (employing equality of variances) the distribution of $\hat{R}^2_c$ has the probability density function

$$p(\hat{R}^2_c) = \frac{\Gamma(K)}{\Gamma(p-1)\Gamma(K-p+1)} (\hat{R}^2_c)^{p-2}(1-\hat{R}^2_c)^{K-p}, \quad (0 \le \hat{R}^2_c < 1) . \tag{71}$$

The statistic $\hat{R}^2_{cj_1 \cdot j_2, j_3, \ldots, j_p}$ is used to test that the coherence $R^2_{cj_1 \cdot j_2, j_3, \ldots, j_p}$ is zero.

Finally consider the j_1-th filtered component of the time series $X(t)$ given by

$$X_{j_1 j_1}(t) = \int_{\omega^{(j_1-1)}}^{\omega^{(j_1)}} \left[\cos \omega t dU(\omega) + \sin \omega t dV(\omega)\right] \quad (72)$$

and also the j_2-th, j_3-th, ... , j_p-th filtered components of the time series $X(t)$ heterodyned to the frequency band $\omega^{(j_1-1)} < \omega \leq \omega^{(j_1)}$:

$$X_{j_s j_1}(t) = \int_{\omega^{(j_1-1)}}^{\omega^{(j_1)}} \left[\cos \omega t dU(\omega + \omega^{(j_s)} - \omega^{(j_1)}) + \sin \omega t dV(\omega + \omega^{(j_s)} - \omega^{(j_1)})\right],$$

$$(s = 2, 3, \ldots, p) . \quad (73)$$

A violation solely of the condition that the multiple coherence $R^2_{cj_1 \cdot j_2, j_3, \ldots, j_p}$ is zero indicates that the time series $X_{jj}(t)$ is to some degree determinable from the time series $X_{j_2 j_1}(t), X_{j_3 j_1}(t), \ldots, X_{j_p j_1}(t)$ by means of linear time invariant operators $L_{j_2 j_1}, \ldots, L_{j_p j_1}$, and to some degree

$$X_{j_1 j_1}(t) = L_{j_2 j_1} X_{j_2 j_1}(t) + L_{j_3 j_1} X_{j_3 j_1}(t) + \ldots + L_{j_p j_1} X_{j_p j_1}(t) \quad (74)$$

prevails.

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