STAT 613 Midterm (1 hour) **April 11th, 2011**

NAME: **Total number of Marks: /25**

Answer all the questions (questions are on both sides of the paper).

Marks will be given for clarity of the solution.

Write your solutions in the question paper.

Good Luck!

(1) Suppose that Z is a Weibull random variable with density $f(x;\phi,\alpha) = (\frac{\alpha}{\phi})(\frac{x}{\phi})^{\alpha-1}\exp(-(x/\phi)^{\alpha})$. Show that

$$\mathbb{E}(Z^r) = \phi^r\Gamma\left(1+\frac{r}{\alpha}\right).$$

[3]

Hint: Use

$$\int x^a \exp(-x^b)dx = \frac{1}{b}\Gamma(\frac{a}{b}+\frac{1}{b}) \qquad a,b>0.$$

This result may be useful in the questions below.

(2) Let us suppose that the random variable X is a mixture of Weibull distributions

$$f(x;\theta) = p(\frac{\alpha_1}{\phi_1})(\frac{x}{\phi_1})^{\alpha_1 - 1}\exp(-(x/\phi_1)^{\alpha_1}) + (1-p)(\frac{\alpha_2}{\phi_2})(\frac{x}{\phi_2})^{\alpha_2 - 1}\exp(-(x/\phi_2)^{\alpha_2}).$$

(i) Derive the mean and variance of X. [3]

(ii) Obtain the exponential distribution which best fits the above mixture Weibull according to the Kullbach-Lieber criterion (recall that the exponential is $g(x;\lambda) = \frac{1}{\lambda}\exp(-x/\lambda)$). [3]

(3) Let us suppose that $\{T_i\}_i$ are the survival times of lightbulbs. We will assume that $\{T_i\}$ are iid random variables with the density $f(\cdot;\theta_0)$ and survival function $\mathcal{F}(\cdot;\theta_0)$, where θ_0 is unknown. The survival times are censored, and $Y_i = \min(T_i, c)$ and δ_i are observed ($c > 0$), where $\delta_i = 1$ if $Y_i = T_i$ and is zero otherwise.

(a) (i) State the log-likelihood of $\{(Y_i, \delta_i)\}_i$. [1]

(ii) We denote the above log-likelihood as $\mathcal{L}_T(\theta)$. Show that

$$-\mathbb{E}\left(\frac{\partial^2 \mathcal{L}_T(\theta)}{\partial \theta^2}\rfloor_{\theta=\theta_0}\right) = \mathbb{E}\left(\frac{\partial \mathcal{L}_T(\theta)}{\partial \theta}\rfloor_{\theta=\theta_0}\right)^2,$$

stating any important assumptions that you may use. [3]

(b) Let us suppose that the above survival times satisfy a Weibull distribution $f(x;\phi,\alpha) = (\frac{\alpha}{\phi})(\frac{x}{\phi})^{\alpha-1}\exp(-(x/\phi)^{\alpha})$ and as in part (a) we observe and $Y_i = \min(T_i, c)$ and δ_i, where $c > 0$.

(i) Using your answer in part 2a(i), give the log-likelihood of $\{(Y_i,\delta_i)\}_i$ for this particular distribution (we denote this as $\mathcal{L}_T(\alpha,\phi)$) and derive the profile likelihood of α (profile out the nusiance parameter ϕ).
Suppose you wish to test $H_0 : \alpha = 1$ against $H_A : \alpha \neq 1$ using the log-likelihood ratio test, what is the limiting distribution of the test statistic under the null? [3]

(ii) Let $\hat{\phi}_T, \hat{\alpha}_T = \arg\max \mathcal{L}_T(\alpha, \phi)$ (maximum likelihood estimators involving the censored likelihood). Do the estimators $\hat{\phi}_T$ and $\hat{\alpha}_T$ converge to the true parameters ϕ and α (you can assume that $\hat{\phi}_T$ and $\hat{\alpha}_T$ converge to some parameters, and your objective is to find whether these parameters are ϕ and α). [3]

(iii) Obtain the (expected) Fisher information matrix of maximum likelihood estimators. [3]

(iv) Using your answer in part 2b(iii) derive the limiting variance of the maximum likelihood estimator of $\hat{\alpha}_T$. [3]