Solutions (Final 2016) 613

Q1) $\mathcal{L}(\mu, \sigma^2, \theta)$

$$= -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2 - \frac{n}{2}\log\sigma^2 - \underbrace{\frac{n}{2}\log 2\pi}_{\text{does not matter}}$$

$$- m\log\theta - \sum_{j=1}^{m}\theta^{-1} y_j$$

(b) The sufficient statistics are $\sum x_i^2$, $\sum x_i$ and $\sum y_j$

(c) In the case $\mu = \gamma$, $\sigma^2 = \gamma^2$ and $\theta = \gamma$ (note $\gamma > 0$)

the likelihood becomes

$$\mathcal{L}(\gamma) = -\frac{1}{2\gamma^2}\sum_{i=1}^{n}(x_i^2 - 2x_i\gamma + \gamma^2) - \frac{n}{2}\log\gamma^2$$

$$- m\log\gamma - \gamma^{-1}\sum y_j$$

$$= -\frac{1}{2\gamma^2}S_{xx} + \frac{1}{\gamma}S_x - \frac{n}{2} - n\log\gamma - m\log\gamma - \gamma^{-1}S_y$$

$$\frac{\partial \mathcal{L}}{\partial \gamma} = \frac{S_{xx}}{\gamma^3} - \frac{S_x}{\gamma^2} - \frac{(n+m)}{\gamma} + \frac{S_y}{\gamma^2} = 0$$

$$\Rightarrow -(n+m)\gamma^2 - (S_x - S_y)\gamma + S_{xx} = 0$$

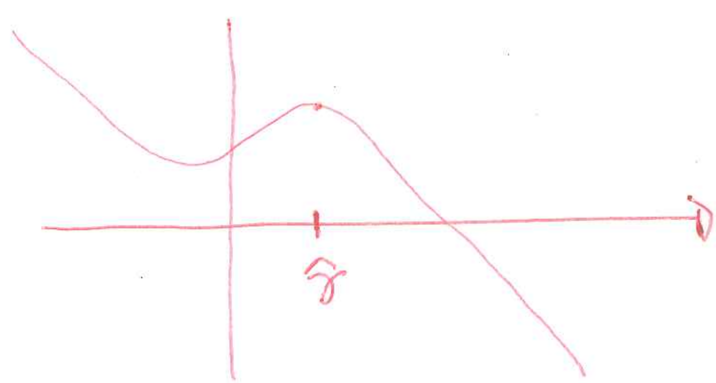
$$\Rightarrow (n+m)\gamma^2 + (S_x - S_y)\gamma - S_{xx} = 0$$

leading to a solution

$$\hat{\gamma} = \frac{-(S_x - S_y) \pm \sqrt{(S_x - S_y)^2 + 4 S_{xx}(n+m)}}{2(n+m)}$$

Since $\hat{\gamma}$ must be positive (in order to be a variable solution) this leads to the unique solution

$$\hat{\gamma} = \frac{-(S_x - S_y) + \sqrt{(S_x - S_y)^2 + 4 S_{xx}(n+m)}}{2(n+m)}.$$



② $H_0: \theta = \theta_0$ vs. $H_A: \theta \neq \theta_0$

$$T = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{Y_i - \mu(\theta_0)}{\sqrt{V(\theta_0)}}.$$

(i) Under null $T \xrightarrow{D} N(0, 1)$

(ii) $$T = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{[Y_i - \mu(\theta_1)]}{\sqrt{V(\theta_0)}} + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{[\mu(\theta_1) - \mu(\theta_0)]}{\sqrt{V(\theta_0)}}$$

Consider the local alternative $\theta_1 = \theta_0 + \phi/\sqrt{n}$

$\simeq$ $\mu(\theta_1) \simeq \mu(\theta_0) + \frac{\phi}{\sqrt{n}} \mu'(\theta_0)$

then

$$T \sim Z + \boxed{\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{[\mu(\theta_1) - \mu(\theta_0)]}{\sqrt{V(\theta_0)}}} \to = \frac{\phi}{\sqrt{V(\theta_0)}}$$

assume that $\phi > 0$, then

$$P\{T > z_{1-\alpha/2}\} = P\left\{Z > z_{1-\alpha/2} - \frac{\phi}{\sqrt{V(\theta_0)}}\right\}$$

Hence the power function is

$$= \Phi\left(z_{1-\alpha/2} - \frac{\phi}{\sqrt{V(\theta_0)}}\right)$$

(Q3) (i) $P\{U - V \le y\} = \int P\{U - V \le y \mid V = v\} f_v(v)\,dv$

$$= \int P_u(U \le y + v) f_v(v)\,dv$$

$$= \int F_u(y+v) f_v(v)\,dv.$$

(ii) $X_i = \min(U_i, V_i)$

$\Rightarrow P\{X_i \ge x\} = P\{U_i \ge x\} P\{V_i \ge x\}$ (by independence)

$$= \exp\{-x(\theta_1 + \theta_2)\}$$

$$\Rightarrow f_x(x) = (\theta_1 + \theta_2)\exp\{-x(\theta_1 + \theta_2)\}$$

(*) This immediately shows that we cannot identify the parameters (θ_1 and θ_2) individually! No matter what algorithm we use. Below we try to estimate θ_1 and θ_2 separately via the EM, but we will see in the limit a unique solution cannot exist.

(iii) Define the dummy variable

$$\delta_i = \begin{cases} 1 & \text{if } U_i < V \\ 0 & \text{if } U_i > V \end{cases}$$

$$f[X_i = x \text{ and } \delta_i = 1\} = f\{X_i = x \text{ and } u_i - v_i < 0\}$$

$$= f(u_i = x \text{ and } v_i \geq x)$$

$$= f(u_i = x) P\{v_i \geq x\} = \theta_1 \exp\{-x(\theta_1 + \theta_2)\}$$

similarly

$$f[X_i = x \text{ and } \delta_i = 0] = \theta_2 \exp\{-x(\theta_1 + \theta_2)\}$$

Thus the full likelihood if δ_i were observed is

$$\mathcal{L}(y, \underline{u}; \theta) = \sum_1^n \delta_i \{\log \theta_1 - X_i(\theta_1 + \theta_2)\}$$
$$+ \sum_1^n (1-\delta_i)\{\log \theta_2 - X_i(\theta_1 + \theta_2)\}$$

we also observe that

$$E\{\delta_i \mid X_i\} = \frac{P\{\delta_i = 1, \min(u, v) = X_i\}}{f(\min(u, v) = X_i)}$$

$$= \frac{P\{u = X_i \text{ and } v \geq X_i\}}{f[\min(u, v) = X_i\}} = \frac{\theta_1 \exp[-X_i(\theta_1 + \theta_2)]}{(\theta_1 + \theta_2)\exp\{-X_i(\theta_1 + \theta_2)}$$

$$= \frac{\theta_1}{\theta_1 + \theta_2}$$

so $E\{1 - \delta_i \mid X_i\} = \dfrac{\theta_2}{\theta_1 + \theta_2}$

Thus

$$Q(\theta^*, \theta) = E\{\ell(\underline{y}, \underline{u}; \theta) \mid \theta^*\}$$

$$= \frac{\theta_1^*}{\theta_1^* + \theta_2^*} \sum_i' \{\log \theta_1 - x_i(\theta_1 + \theta_2)\}$$

$$+ \frac{\theta_2^*}{\theta_1^* + \theta_2} \sum_i' \{\log \theta_2 - x_i(\theta_1 + \theta_2)\}$$

Let $\pi_1^* = \frac{\theta_1^*}{\theta_1^* + \theta_2^*}$ and $\cancel{(1-\pi_2)}$

Then the above reduces to

$$n\pi_1^* \log \theta_1 - \pi_1^* n\bar{x}(\theta_1 + \theta_2)$$

$$+ n(1-\pi_1^*) \log \theta_2 - (1-\pi^*) n\bar{x}(\theta_1 + \theta_2)$$

$$= n[\pi^* \log \theta_1 + (1-\pi^*) \log \theta_2] - n\bar{x}(\theta_1 + \theta_2)$$

Diff. wrt θ_1 and θ_2 gives

$$\frac{\cancel{n}\pi^*}{\theta_1} = \cancel{n}\bar{x} \quad \text{hence} \quad \theta_1 = \frac{\pi^*}{\bar{x}}$$

and $\theta_2 = \dfrac{1-\pi^*}{\bar{x}}$

This is the $(n+1)$th iteration of the algorithm

Thus

$$\theta_{1,k+1} = \frac{\theta_{1,k}}{\theta_{1,k}+\theta_{2,k}} \cdot \frac{1}{\bar{X}} \quad \text{and}$$

$$\theta_{2,k+1} = \frac{\theta_{2,k}}{\theta_{1,k}+\theta_{2,k}} \cdot \frac{1}{\bar{X}} . \quad \text{let } k \to \infty \text{ means}$$

$$\theta_k \to \theta_k$$

Then $\cancel{\theta}_1 = \frac{\cancel{\theta}_1}{\theta_1+\theta_2} \cdot \frac{1}{\bar{X}}$ and $\cancel{\theta}_2 = \frac{\cancel{\theta}_2}{\theta_1+\theta_2} \cdot \frac{1}{\bar{X}}$.

Thus it is impossible to individually identify θ_1 and θ_2.

However we see the algorithm does (?!) achieve a limit we have $\widehat{(\theta_1+\theta_2)} = \bar{X}$ (which is what we would expect!

(Q4) Let

$$L_n(\beta) = \sum_i (Y_i - g(\beta x_i))^2 \qquad \beta_0 = \text{true value where}$$

$$E[Y_i] = g(\beta_0 x_i)$$

$\hat{\beta}_n = \arg\min L_n(\beta)$ and

solves the estimating equation

$$\nabla L_n(\beta) = 0 \quad \text{where}$$

$$\nabla L_n(\beta) = -2\sum_{i=1}^{n}[Y_i - g(\beta x_i)]\, g'(\beta x_i) x_i .$$

To answer the question define

$$V_n(\beta_0) = \text{var}[\nabla L_n(\beta_0)]$$

$$= 4\sum_i v(\beta x_i)[g'(\beta x_i) x_i]^2$$

and

$$H_n(\beta_0) = E[\nabla^2 L_n(\beta_0)]$$

$$= E\Big[2\sum_{i=1}^{n}[g'(\beta_0 x_i) x_i]^2 - 2\sum_{i=1}^{n}(Y - g(\beta_0 x_i))\, g''(\beta x_i) \times x_i^2\Big]$$

$$= 2\sum_{i=1}^{n}[g'(\beta_0 x_i) x_i]^2$$

Note, neither $V_n(\beta_0)$ or $H_n(\beta_0) \to 0$ as $n \to \infty$, else $\hat{\beta}_n$ will have a non-standard limiting distribution.

We assume that

$$\frac{1}{\sqrt{V_n(\beta_0)}} \nabla L_n(\beta_0) \xrightarrow{D} N(0,1) \qquad (1).$$

By using Taylor expansion we have

$$(\hat{\beta}_n - \beta_0) = [\nabla^2 L_n(\bar{\beta})]^{-1} \nabla L_n(\beta_0) \qquad (2)$$

where $\bar{\beta} = \alpha \beta_0 + (1-\alpha)\hat{\beta}_n$ for some α.

We replace $\bar{\beta}$ with β_0 and assume the error in (2) is negligible;

$$(\hat{\beta}_n - \beta_0) = [\nabla^2 L_n(\beta_0)]^{-1} \nabla L_n(\beta_0) + \cancel{o_p(1)} \; o_p\left(\frac{1}{\sqrt{V_n(\beta_0)}}\right)$$

multiply the above by our standardiser $\sqrt{V_n(\beta_0)}$

$$\sqrt{V_n(\beta_0)}(\hat{\beta}_n - \beta_0) = \Bigg[\underbrace{\frac{1}{V_n(\beta_0)} \nabla^2 L_n(\beta_0)}\Bigg]^{-1} \frac{1}{\sqrt{V_n(\beta_0)}} \nabla L_n(\beta_0) + o_p(1) \qquad (3)$$

next assumption

$$\frac{1}{V_n(\beta_0)} \nabla^2 L_n(\beta_0) = \frac{1}{V_n(\beta_0)} E[\nabla^2 L_n(\beta_0)] + o_p(1)$$

$$= \frac{H_n(\beta_0)}{\quad} + o_p(1) \qquad (4)$$

Replace (4) into (3) to give

$$\sqrt{V_n(\beta_0)}\,(\hat{\beta}_n - \beta_0) = \left[\frac{H_n(\beta_0)}{V_n(\beta_0)}\right]^{-1} \underbrace{\frac{1}{\sqrt{V_n(\beta_0)}}\,\nabla L_n(\beta_0)}_{\xrightarrow{D} N(0,1)} + o_p(1)$$

Hence

$$\frac{H_n(\beta_0)}{V_n(\beta_0)^{1/2}}\,(\hat{\beta}_n - \beta_0) \xrightarrow{D} N(0,1)$$

under the assumption that $\dfrac{H_n(\beta_0)}{V_n(\beta_0)^{1/2}} \to \infty$ as $n \to \infty$.