Big and little oh

Big and little oh in mathematics

(a) **Big oh** If we write $S_n = O(a_n)$, then this implies there exists a finite constant C such that for all n, $|\frac{S_n}{a_n}| \leq C$.

(b) **Little Oh** If we write $S_n = o(a_n)$, then this implies the sequence $|\frac{S_n}{a_n}| \to 0$ as $n \to \infty$.

Example: Often we write $S_n = \mu + b_n + o(a_n)$. This implies that the reminder $S_n - \mu - b_n$ is such that $|S_n - \mu - b_n|/a_n \to 0$ as $n \to \infty$. Thus a_n dominates the remainder $S_n - \mu - b_n$.

Big and little oh in probability

Often we are given an estimator, S_n of μ. Sometimes we can evaluate $\mathbb{E}[S_n - \mu]^2$. Typically we can show $\mathbb{E}[S_n - \mu]^2 = O(n^{-\gamma})$ (usually $\gamma = 1/2$).

(a) **Big oh** If we write $S_n = O_p(a_n)$. This means for every $\varepsilon > 0$ there exists a finite constant C_ε such that for all n

$$P\left(\left|\frac{S_n}{a_n}\right| > C_\varepsilon\right) \leq \varepsilon.$$

Example: Suppose $\mathbb{E}|S_n - \mu|^2 \leq \frac{K}{n}$, then we can say that $S_n = O_p(n^{-1/2})$. This is due to Chebyshev's inequality. That is

$$P\left(\left|\frac{S_n}{n^{-1/2}}\right| > C_\varepsilon\right) \leq \frac{n\mathbb{E}|S_n|^2}{C_\varepsilon^2} \leq \frac{K}{C_\varepsilon^2},$$

Thus, for a given ε if we choose $C_\varepsilon = (K/\varepsilon)^{1/2}$, we have that

$$P(|\frac{S_n}{n^{-1/2}}| > C_\varepsilon) \leq \varepsilon.$$

This implies $S_n = O_p(n^{-1/2})$.

(b) **Little Oh** If we write $S_n = \mu + X_n + o_p(a_n)$, then this implies the sequence $|\frac{S_n - \mu - X_n}{a_n}| \stackrel{\mathcal{P}}{\to} 0$ as $n \to \infty$. In other words, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(\left|\frac{S_n - \mu - X_n}{a_n}\right| > \varepsilon\right) = 0$$

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(c) Suppose $a_n \to 0$ as $n \to \infty$. If $|S_n - \mu| = O_p(a_n)$, then $|S_n - \mu|^2 = O_p(a_n^2) = o_p(a_n)$. If $|S_{1,n} - \mu_1| = O_p(a_n)$ and $|S_{2,n} - \mu_2| = O_p(a_n)$, then $|S_{1,n} - \mu_1||S_{2,n} - \mu_2| = O_p(a_n^2) = o_p(a_n)$.