STAT 415 — Midterm 2 (Spring 2021)

Name: ______________________________

Exam rules:

- You have 90 minutes to complete the exam.
- There are **4** Questions.
- You may use the provided formula sheet (this is a closed book exam).
- You are only allowed to use a pen and pencil (or ipad for writing). You cannot use the internet to find anwers and you cannot confer with class mates.
- Explain your arguments carefully, show all your work, and write out all your steps. Credit will be given for clarity. No credit will be given for a solution without working.
- If you are caught cheating or helping someone to cheat on this exam, you will both receive a grade of zero on the exam and you will be turned in.

1. Define the bivariate random vector

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \right).$$

Let $Y_1 = X_1 - X_2$ and $Y_2 = X_1 + X_2$.

Obtain (necessary and sufficient) conditions that ensure that Y_1 and Y_2 are independent random variables. [2]

2. Suppose that $\{X_t\}_{t=1}^n$ are iid (independent, identically distributed) random variables with $\mathrm{E}[X_t] = \mu$ and $\mathrm{var}[X_t] = \sigma^2$. Suppose that U and $\{X_t\}_{t=1}^n$ are independent of each other. At the time points $t = 2^k$ $(k = 1, 2, 3 \ldots)$ X_t is corrupted with the noise U where $\mathrm{E}[U] = 0$ and $\mathrm{var}[U] = \sigma_U^2$. We observe the random variables $\{Y_t\}$ where

$$Y_t = \begin{cases} X_t & \text{if } t \neq 2^k \text{ where } k = 1, 2, 3, \ldots \\ X_t + U & \text{if } t = 2^k. \end{cases}$$

Let $\bar{Y} = n^{-1} \sum_{t=1}^n Y_t$.

(i) Show that $\bar{Y}$ is an unbiased estimator of μ.

(ii) Obtain the Mean Squared error of $\bar{Y}$, $\mathrm{E}[(\bar{Y} - \mu)^2]$.

(iii) Is $\bar{Y}$ an asymptotically consistent estimator of μ? Prove your claim (yes or no is not enough). [4]

3. Suppose that $\{X_i\}_{i=1}^n$ are iid random variables with the exponential distribution

$$f(x;\theta) = \frac{1}{\theta}\exp(-x/\theta), \quad x \geq 0.$$

You can use that $\mathrm{E}[X_i] = \theta$.

(i) Obtain the maximum likelihood estimator of θ and the Fisher information $I(\theta)$.

(ii) For a fixed sample size n, make a sketch of the log-likelihood function corresponding to a large Fisher information and a small Fisher information (and the corresponding θ). Indicate on your plots which log-likelihood will have an MLE with a small variance and which will have a large variance. [4]

4. Suppose $\{X_i\}_{i=1}^{n}$ are iid geometrically distributed random variables with probability mass function

$$P(X = k; \pi) = \pi(1-\pi)^{k-1} \qquad k = 1, 2, \ldots$$

You can use that $\mathrm{E}[X_i] = 1/\pi$ and $\mathrm{var}[X_i] = (1-\pi)/\pi^2$.

(i) Define the new random variable

$$Y_i = \begin{cases} 1 & \text{if } X_i = 1 \\ 0 & \text{if } X_i > 1 \end{cases}$$

Obtain $\mathrm{E}[Y_i]$ and derive the method of moments estimator of p based on $\{Y_i\}$. Call this estimator $\widehat{\pi}_{MoM}$

(ii) Derive the asymptotic distribution of $\widehat{\pi}_{MoM}$. [3]

(iii) Using $\{X_i\}$ derive the maximum likelihood estimator of π. Call this $\widehat{\pi}_{MLE}$

(iv) Derive the asymptotic distribution of $\widehat{\pi}_{MLE}$ (using any of the results we derived in class). [3]

(v) Construct asymptotic 95% confidence intervals for π using $\widehat{\theta}_{MoM}$ and $\widehat{\theta}_{MLE}$ (this should be two intervals).

(vi) Compare the asymptotic variance of $\widehat{\pi}_{MoM}$ and asymptotic variance of $\widehat{\pi}_{MLE}$. In terms of variance which estimator is better and how does this impact the confidence interval in part (v)?

(vii) Which of the random variables $\{Y_i\}$ and $\{X_i\}$ contain more "information" about the parameter π. [4]

Only a heuristic explanation is required you do not need to calculate the Fisher information of both estimators.