STAT 415 — Midterm 2 (Spring 2020)

Name: ______________________________

Exam rules:

- You have 2hours 20 minutes to complete the exam, scan it and upload onto Ecampus.
- There are **4** Questions.
- This exam is open book. You are free to use your class notes and HW solutions to solve the problems.

  Do not do a brain dump. Answers which are irrelevant will be penalized.

  Do not blindly copy answers from the HW solutions.
- State precisely in the derivations all the results that you use.
- Explain your arguments carefully, show all your work, and write out all your steps. Credit will be given for clarity. No credit will be given for a solution without working.
- If you are caught cheating or helping someone to cheat on this exam, you will both receive a grade of zero on the exam and you will be turned in.
- If a problem requires a numerical answer, you may express answer in terms of elementary functions or cdfs (e.g. $\Gamma(r+1)$ or $r!$).

(1) Suppose that $\underline{X} = (X_1, X_2)'$ are jointly normal where

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\right).$$

Let $Y = (X_1 - 3^{-1}X_2)^2$ and $W = (X_1 + X_2)^2$.

Show that Y and W are independent random variables (stating all the results that you use).

(2) Suppose that $\{X_i\}_{i=1}^n$ are iid Poisson distributed random variables with probability mass function $f(k;\lambda) = \lambda^k \exp(-\lambda)/k!$.

(i) Obtain the maximum likelihood estimator of λ. Denote this estimator as $\widehat{\lambda}$

(ii) Obtain the Mean Squared error: $\mathrm{E}[(\widehat{\lambda} - \lambda)^2]$.
Hint: You can use that

$$\mathrm{var}[X] = \lambda.$$

(iii) Obtain the asymptotic sampling distribution of the MLE.

(iv) Construct the approximate 99% confidence interval for λ based on the MLE.
The confidence interval should be "feasible" (something that a practitioner can immediately evaluate given the MLE).

(v) Obtain the Fisher information of λ.

(vi) For a fixed sample size n, make a sketch of the **log-likelihood** function corresponding to a large Fisher information and a small Fisher information. Indicate on your plots which **log-likelihood** will have an MLE with a small variance and which will have a large variance.

(3) Suppose that $\{X_i\}$ are iid random variables with density

$$f(x;\beta) = \beta(1-x)^{\beta-1} \qquad x \in [0,1] \qquad \beta > 0.$$

(i) Evaluate the expectation $\mathrm{E}[X_i]$.

Hint: You may use the two identities

$$\int_0^1 x^a(1-x)^b dx = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} \text{ and } \frac{\Gamma(a+1)}{\Gamma(a)} = a,$$

where $\Gamma(a)$ is the Gamma function. If a is an integer, then $\Gamma(a) = (a-1)!$ and $\Gamma(1) = 1$.

(ii) Based on your answer in (i) obtain the method of moments estimator of β. Call this estimator $\widehat{\beta}_{MoM}$.

(iii) Obtain the asymptotic sampling properties of $\widehat{\beta}_{MoM}$.

Hint: You can use that

$$\mathrm{var}[X] = \frac{\beta}{(1+\beta)^2(2+\beta)}$$

(iii) Obtain the maximum likelihood estimator of β. Call this estimator $\widehat{\beta}_{MLE}$.

(iv) Obtain the asymptotic sampling properties of $\widehat{\beta}_{MLE}$.

(v) By comparing the asymptotic variance of $\widehat{\beta}_{MoM}$ and $\widehat{\beta}_{MLE}$ explain for what values of β the two estimators have a variance which is very close.

(4) Suppose that $\{X_i\}_{i=1}^n$ are iid random variables with density

$$f(x;\alpha) = \begin{cases} \frac{1}{10-\alpha} & x \in [10-\alpha, 10] \\ 0 & x \notin [10-\alpha, 10] \end{cases}$$

Obtain the maximum likelihood estimator of α.