**STAT 415 — Midterm 1 (Spring 2021)**

Name: Solutions

**Exam rules:**

- You have 90 minutes to complete the exam.
- There are **4** Questions.
- You may use the provided formula sheet.
- You are only allowed to use a pen and pencil (or ipad for writing). You cannot use the internet to find anwers and you cannot confer with class mates.
- Explain your arguments carefully, show all your work, and write out all your steps. Credit will be given for clarity. No credit will be given for a solution without working.
- If you are caught cheating or helping someone to cheat on this exam, you will both receive a grade of zero on the exam and you will be turned in.

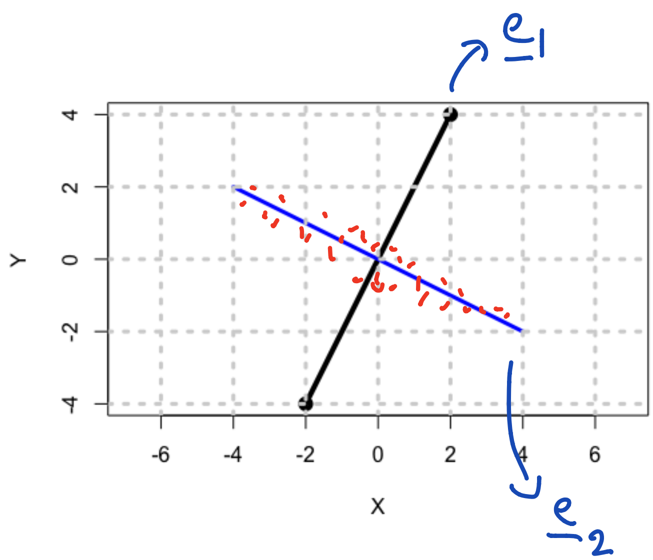
1. Define the bivariate random vector

$$\underline{U} = \begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{0.2} Z_1 \underline{e}_1 + \mathbf{2} Z_2 \underline{e}_2$$
$$= 0.2 \begin{pmatrix} Z_1/2 \\ Z_1 \end{pmatrix} + 2 \begin{pmatrix} Z_2 \\ -Z_2/2 \end{pmatrix}$$

where $Z_1, Z_2$ are iid normal standard random variables, $\underline{e}_1' = (1/2, 1)$ and $\underline{e}_2' = (1, -1/2)$.

(i) Suppose that $\{\underline{U}\}_{i=1}^{n}$ are iid random variables where $\underline{U}_i \sim \underline{X}$. Draw a plot (a plot as a guide is given) of 30 or so typical realisations of $\{\underline{U}\}_{i=1}^{n}$ (plotting $X$ against $Y$); enough to get the general behaviour of $\underline{U}$. [2]

* Variation on $\underline{e}_1$ very small.
* Variation on $\underline{e}_2$ is large




(ii) Derive the distribution of $(X, Y)$. [2]

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0.1 & 2 \\ 0.2 & -1 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$$

$$\Rightarrow E\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0.1 & 2 \\ 0.2 & -1 \end{pmatrix} E\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 0.1 & 2 \\ 0.2 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{var}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0.1 & 2 \\ 0.2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0.1 & 0.2 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 4.01 & -1.98 \\ -1.98 & 1.04 \end{pmatrix}$$

Negative correlation matches the plot.

Sine $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \sim N\left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$

$\Rightarrow \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0.1 & 2 \\ 0.2 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ is normal

(linear combination of normal is normal)

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \sim N\left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4.01 & -1.98 \\ -1.98 & 1.04 \end{pmatrix} \right)$$

2. Suppose that $\{Z_i\}_{i=1}^{3}$ are iid random variables with mean 0 and variance 1. Let

$$Y_1 = Z_1 + Z_2, \quad Y_2 = Z_1 - Z_2, \quad Y_3 = Z_1 + Z_2 Z_3.$$

Obtain the covariance matrix of $(Y_1, Y_2, Y_3,$ [illegible]$)$ and make a network plot based on its covariance structure. [3]

$\mathrm{Var}(Y_1) = 2$ , $\mathrm{Var}(Y_2) = 2$ ,

$\mathrm{Var}(Y_3) = \mathrm{Var}(Z_1) + \mathrm{Var}(Z_2 Z_3) = \mathrm{Var}(Z_1) + \mathrm{Var}(Z_2)\mathrm{Var}(Z_3) = 2$

$\mathrm{Cov}(Y_1, Y_2) = \mathrm{Var}(Z_1) + \mathrm{Var}(Z_2) = 1 - 1 = 0$

$\mathrm{Cov}(Y_1, Y_3) = \mathrm{Var}(Z_1) + \mathrm{Cov}(Z_2, Z_2 Z_3)$

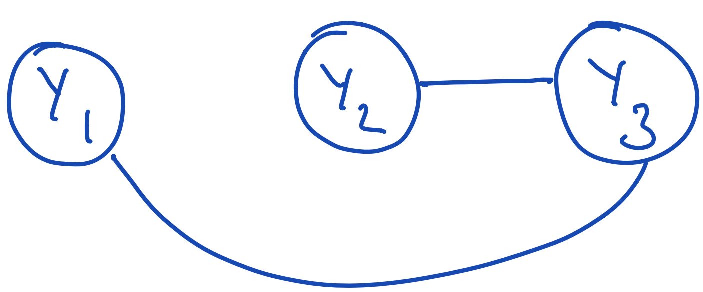
$= 1 + E(Z_2^2 Z_3) = 1 + E(Z_2^2)E(Z_3)$

$= 1 + 0 = 1$

$\mathrm{Cov}(Y_2, Y_3) = \mathrm{Var}(Z_2) - \mathrm{Cov}(Z_2, Z_2 Z_3) = 1$

$$\mathrm{Var}\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

The network



3. Suppose the estimator $\widehat{\theta}_n$ is an *unbiased* estimator of $\theta$ (i.e. $\mathrm{E}[\widehat{\theta}_n] = \theta$), $\mathrm{var}[\widehat{\theta}_n] = \mathrm{E}[(\widehat{\theta}_n - \theta)^2] = \sigma^2/n$ and $\mathrm{E}[(\widehat{\theta}_n - \theta)^3] = \kappa_3/n^{3/2}$.

(i) Show that $\widehat{\theta}_n^3$ is a biased estimator of $\theta^3$, and obtain the bias. [3]

(ii) Shows that asymptotically $\widehat{\theta}_n^3$ is an unbiased estimator of $\theta^3$. [2]

(iii) Using that $\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow{\mathcal{D}} N(0, \sigma^2)$ (or $\widehat{\theta}_n \xrightarrow{\mathcal{D}} N(\theta, \sigma^2/n)$). Obtain the asymptotic distribution of $\widehat{\theta}^3$.
(State the main condition under which it will hold). [3]

(i) Use $E[(\hat{\theta}_n - \theta)^3]$ and expand.

$$E[(\hat{\theta}_n - \theta)^3] = E[\hat{\theta}_n^3] - 3E[\hat{\theta}_n^2 \theta] + 3E[\hat{\theta}_n \theta^2] - \theta^3$$

$$= E[\hat{\theta}_n^3] - 3\theta E[\hat{\theta}_n^2] + 3\theta^2 \overset{\theta}{\overset{\shortparallel}{E[\hat{\theta}_n]}} - \theta^3$$

$$= E[\hat{\theta}_n^3] - 3\theta\left[\mathrm{var}(\hat{\theta}_n) + (E[\hat{\theta}_n])^2\right] + 2\theta^3$$

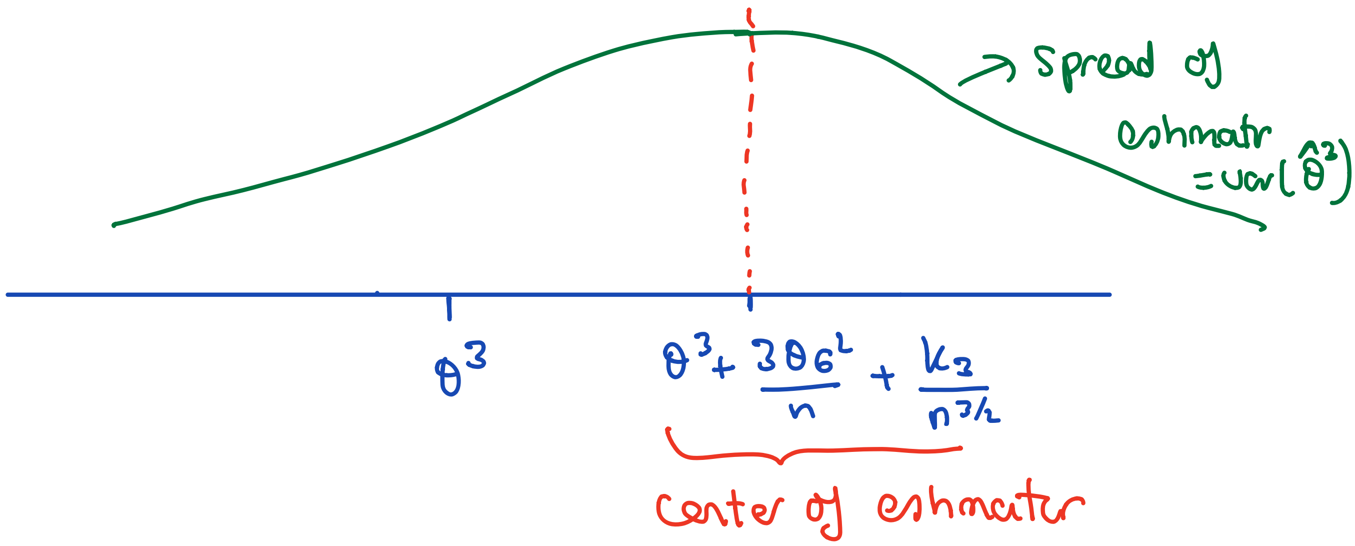
$$= E[\hat{\theta}_n^3] - 3\theta\left[\frac{\sigma^2}{n} + \theta^2\right] + 2\theta^3$$

$$= \underbrace{E[\hat{\theta}_n^3] - \frac{3\theta\sigma^2}{n} - \theta^3}_{\kappa_3/n^{3/2}}$$

$$\Rightarrow \quad E[\hat{\theta}_n^3] = \theta^3 + \underbrace{\frac{3\theta\sigma^2}{n} + \frac{\kappa_3}{n^{3/2}}}_{\text{Bias}}$$

$$\text{Bias} = E[\hat{\theta}_n^3] - \theta^3 = \frac{3\theta\sigma^2}{n} + \frac{K_3}{n^{3/2}}$$

(*) Remember Bias the centrality of the estimator




$$\lim_{n\to\infty}(\text{Bias}) = \lim_{n\to\infty}\left[\frac{3\theta\sigma^2}{n} + \frac{K_3}{n^{3/2}}\right] = 0$$

Thus $\hat{\theta}_n^3$ is an asymptotically unbiased estimator of $\theta^3$

(iii) Asymptotic distribution

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{D} N(0, \sigma^2)$$

$$g(\hat{\theta}) = \hat{\theta}^3 \qquad g'(\theta) = 3\theta^2 \qquad g'(\theta) \neq 0$$

$$\Rightarrow \sqrt{n}[\hat{\theta}^3 - \theta^3] \xrightarrow{D} N(0, [3\theta^2]^2\sigma^2)$$

$$= N\left(0,\ 9\theta^4 \sigma^2\right)$$

Lite version $\quad \hat{\theta}^3 \longrightarrow N\left(\theta^3,\ \dfrac{9\theta^4\sigma^2}{n}\right)$

4. Suppose that

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N\left( \begin{pmatrix} \mu \\ \mu \\ \mu \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix} \right)$$

Define the two random variables

$$\begin{aligned} Y_1 &= (X_2 - X_3) \\ Y_2 &= (-2X_1 + X_2 + X_3)^2 \end{aligned}$$

(i) Show that $Y_1$ and $Y_2$ are independent. [3]

(ii) Are $Y_1$ and $Y_2$ independent if $X_1, X_2, X_3$ are iid (non-normal) random variables with $\mathrm{E}[X_i] = 0$ and $\mathrm{var}[X_i] = \sigma^2$? Explain why. Marks will be given for the correct answer not yes or no. [2]

(i) Define the random variable

$Z = -2X_1 + X_2 + X_3$

Since $(X_1, X_2, X_3)$ are jointly normal, then $(Y_1, Z)$ are joint normal (linear combination of normal random variables are normal).

Aim Show $\mathrm{cov}[Y_1, Z] = 0 \Rightarrow Y_1$ and $Z$ are independent (since $Y_1$ and $Z$ are normal).

$\mathrm{cov}[Y_1, Z] = \mathrm{var}(X_2) - \mathrm{var}(X_3) = \sigma^2 - \sigma^2 = 0$

$\Rightarrow Y_1$ and $Z$ are independent (by normality)

$\Rightarrow Y_1$ and $Z^2$ are independent. Since

$Z^2 = Y_2$, this implies $Y_1$ and $Y_2$ are independent. Thus proving the claim in (i)

(ii) If $(X_1, X_2, X_2)$ are iid but not normal, then

$$\text{cov}[Y_1, Z] = 0$$

BUT $Y_1$ and $Z$ are not normal, thus this does not imply that $Y_1$ and $Z$ are independent. In general lack of correlation does not imply independence.

Think of it this way

$$Y_1 = (X_2 - X_3)$$

$$Z = -2X_1 + X_2 + X_3$$

Both $Y_1$ and $Z$ share the same variables, only in exceptional situations (such as normality) would $Y_1$ and $Z$ be independent.