STAT 415 — Midterm 1 (Spring 2021)

Name: ______________________________

Exam rules:

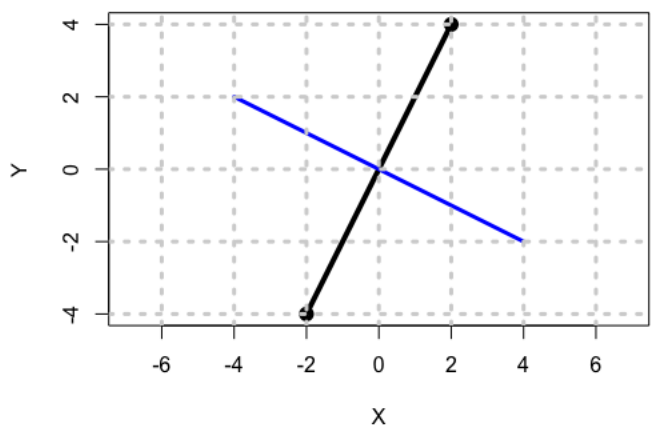
- You have 90 minutes to complete the exam.
- There are **4** Questions.
- You may use the provided formula sheet.
- You are only allowed to use a pen and pencil (or ipad for writing). You cannot use the internet to find anwers and you cannot confer with class mates.
- Explain your arguments carefully, show all your work, and write out all your steps. Credit will be given for clarity. No credit will be given for a solution without working.
- If you are caught cheating or helping someone to cheat on this exam, you will both receive a grade of zero on the exam and you will be turned in.

1. Define the bivariate random vector

$$\underline{U} = \begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{0.2} Z_1 \underline{e}_1 + \mathbf{2} Z_2 \underline{e}_2$$
$$= 0.2 \begin{pmatrix} Z_1/2 \\ Z_1 \end{pmatrix} + 2 \begin{pmatrix} Z_2 \\ -Z_2/2 \end{pmatrix}$$

where Z_1, Z_2 are iid normal standard random variables, $\underline{e}_1' = (1/2, 1)$ and $\underline{e}_2' = (1, -1/2)$.

(i) Suppose that $\{\underline{U}\}_{i=1}^n$ are iid random variables where $\underline{U}_i \sim \underline{X}$. Draw a plot (a plot as a guide is given) of 30 or so typical realisations of $\{\underline{U}\}_{i=1}^n$ (plotting X against Y); enough to get the general behaviour of $\underline{U}$. [2]



(ii) Derive the distribution of (X, Y). [2]

2. Suppose that $\{Z_i\}_{i=1}^3$ are iid random variables with mean 0 and variance 1. Let

$$Y_1 = Z_1 + Z_2, \quad Y_2 = Z_1 - Z_2, \quad Y_3 = Z_1 + Z_2 Z_3.$$

Obtain the covariance matrix of (Y_1, Y_2, Y_3, Y_4) and make a network plot based on its covariance structure. [3]

3. Suppose the estimator $\widehat{\theta}_n$ is an *unbiased* estimator of θ (i.e. $\mathrm{E}[\widehat{\theta}_n] = \theta$), $\mathrm{var}[\widehat{\theta}_n] = \mathrm{E}[(\widehat{\theta}_n - \theta)^2] = \sigma^2/n$ and $\mathrm{E}[(\widehat{\theta}_n - \theta)^3] = \kappa_3/n^{3/2}$.

   (i) Show that $\widehat{\theta}_n^3$ is a biased estimator of θ^3, and obtain the bias. [3]

   (ii) Shows that asymptotically $\widehat{\theta}_n^3$ is an unbiased estimator of θ^3. [2]

   (iii) Using that $\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow{\mathcal{D}} N(0, \sigma^2)$ (or $\widehat{\theta}_n \xrightarrow{\mathcal{D}} N(\theta, \sigma^2/n)$). Obtain the asymptotic distribution of $\widehat{\theta}^3$.
   (State the main condition under which it will hold). [3]

4. Suppose that

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu \\ \mu \\ \mu \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix}\right)$$

Define the two random variables

$$\begin{aligned} Y_1 &= (X_2 - X_3) \\ Y_2 &= (-2X_1 + X_2 + X_3)^2 \end{aligned}$$

(i) Show that Y_1 and Y_2 are independent. [3]

(ii) Are Y_1 and Y_2 independent if X_1, X_2, X_3 are iid (non-normal) random variables with $\mathrm{E}[X_i] = 0$ and $\mathrm{var}[X_i] = \sigma^2$? Explain why. Marks will be given for the correct answer not yes or no. [2]