STAT 415 — Midterm 1 (Spring 2020)

Answer: Solutions should be written below each question.

Name: ______________________________

Exam rules:

- You have 75 minutes to complete the exam.
- There are **5** Questions.
- You will get a formula sheet.
- You are only allowed to use a pen and pencil.
- Explain your arguments carefully, show all your work, and write out all your steps. Credit will be given for clarity. No credit will be given for a solution without working.
- If you are caught cheating or helping someone to cheat on this exam, you will both receive a grade of zero on the exam and you will be turned in.
- If a problem requires a numerical answer, you may express answer in terms of elementary functions or cdfs (e.g. $\Gamma(r+1)$ or $r!$).

1. Suppose

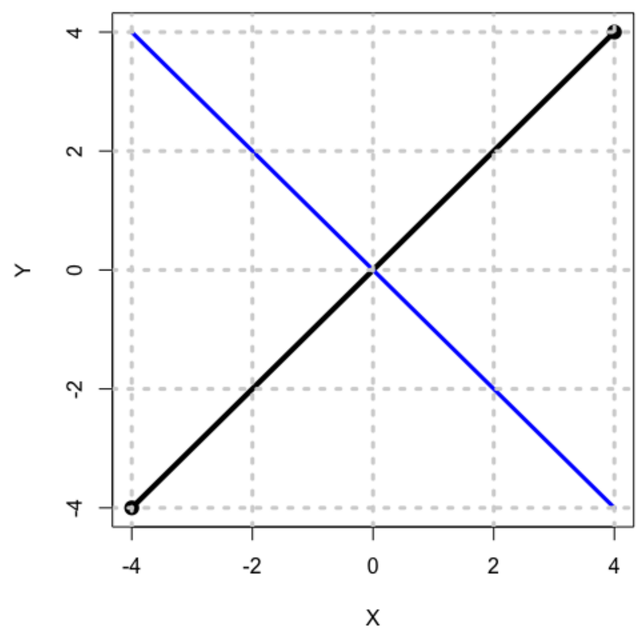
$$\underline{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \underline{e}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Define the bivariate random vector

$$\underline{X} = \begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{0.2} Z_1 \underline{e}_1 + \mathbf{2} Z_2 \underline{e}_2$$
$$= 0.2 \begin{pmatrix} Z_1 \\ Z_1 \end{pmatrix} + 2 \begin{pmatrix} Z_2 \\ -Z_2 \end{pmatrix}$$

where Z_1, Z_2 are iid normal random variables.

(i) Suppose that $\{\underline{X}\}_{i=1}^{n}$ are iid random variables where $\underline{X}_i \sim \underline{X}$. Draw on the plot below 20 or so typical realisations of $\{\underline{X}\}_{i=1}^{n}$ (plotting X against Y), enough to get the general behaviour of $\underline{X}$. [2]



(ii) Explain your answer. [2]

2. Suppose $\{X_i\}$ and U are independent random variables, where $\mathrm{E}[X_i] = 0$, $\mathrm{var}[X_i] = \sigma_X^2$, $\mathrm{E}[U] = \mu$ and $\mathrm{var}[U] = \sigma_U^2$. Let $Z_i = X_i + U$ and $S_n = \frac{1}{n}\sum_{i=1}^{n} Z_i$.

(i) Calculate $\mathrm{E}[S_n]$. [2]

(ii) Calculate $\mathrm{var}[S_n]$. [4]

(iii) Calculate the mean squared error $\mathrm{E}[S_n - \mu]^2$. What happens to the mean squared error as $n \to \infty$? [2]

(iv) Is S_n an asymptotically consistent estimator of μ, either in probability or mean squared error. Give a reason for your answer. [2]

(3) Suppose that $\{Z_i\}_{i=1}^{3}$ are iid standard normal random variables ($Z_i \sim N(0,1)$).

(i) Let $Y_1 = Z_1 + Z_2$, $Y_2 = Z_2$ and $Y_3 = Z_2 + Z_3$.

Calculate the variance matrix of $\underline{Y}' = (Y_1, Y_2, Y_3)$. Draw the associated covariance network. [6]

(ii) Let $X_1 = Z_1 + Z_2Z_3$, $X_2 = Z_2$ and $X_3 = Z_2 + Z_3$.

Calculate the variance matrix of $\underline{X}' = (X_1, X_2, X_3)$. Draw the associated covariance network. [5]

4. Suppose that X_1 and X_2 are iid normal random variables with mean μ and variance σ^2. Let $\bar{X} = 2^{-1}(X_1 + X_2)$ and $s_2^2 = (X_1 - \bar{X})^2 + (X_2 - \bar{X})^2$.

(i) Show that $\bar{X}$ and s_2^2 are independent random variables (credit will be given for a coherent argument, state precisely all results that you use). [6]

Hint: Use that $s_2^2 = (X_1 - X_2)^2/2$.

(ii) Define the T and Z-transforms: $T_2 = \frac{\sqrt{2}(\bar{X}-\mu)}{s_2}$ and $Z = \frac{\sqrt{2}(\bar{X}-\mu)}{\sigma}$. Calculate

$$P\left(T_2 > \frac{3}{2}Z\right).$$

Give your answer in terms of an appropriate χ^2 distribution. You may use that $s_2^2 \sim \sigma^2\chi_1^2$. [3]

5. Suppose that $\{X_i\}$ are iid random variables drawn from an exponential distribution with density $\lambda \exp(-\lambda x)$ where $x \geq 0$.

(i) Let $\bar{X}_n = n^{-1} \sum_{i=1}^{n} X_i$. State the limiting distribution of $\bar{X}_n$ (giving the correct mean and variance). [3]

(ii) Obtain the first moment of X_i and the method of moments estimator of λ (based on the first moment)? [3]