STAT 415 — Final (Spring 2020)

Name: Solutions

Exam rules:

- You have 2hrs 45 minutes to to complete the exam, scan it and upload onto Gradebook.
- There are **5** Questions (The exam is worth 40pts in total).
- This exam is open book. You are free to use your class notes and HW solutions to solve the problems.

  Do not do a brain dump. Answers which are irrelevant will be penalized.

  Do not blindly copy answers from the HW solutions.
- State precisely in the derivations all the results that you use.
- Explain your arguments carefully, show all your work, and write out all your steps. Credit will be given for clarity. No credit will be given for a solution without working.
- If you are caught cheating or helping someone to cheat on this exam, you will both receive a grade of zero on the exam and you will be turned in.
- If a problem requires a numerical answer, you may express answer in terms of elementary functions or cdfs (e.g. $\Gamma(r+1)$ or $r!$).
- Presentation and scanning rules (20% will be knocked off if these rules are not adhered too).
  1. No light pencils. Black is easiest to read.
  2. Write big; scrawls are painful to read.
  3. Name at top of exam paper (this makes it easier to identify them).
  4. Name the submitted file LASTNAMEFIRSTNAME.pdf
  5. Submit only **one** pdf file.
  6. The solutions should be on separate pages.
  7. Check clarity of scan before submission.

(1) Suppose that $\{X_i\}_{i=1}^n$ are iid normal random variables with mean μ and variance σ^2. Let

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

To answer the below you can use that for normal random variables with mean μ and variance σ^2, then $\mathrm{E}[X] = \mu$, $\mathrm{E}[X^2] = \sigma^2 + \mu^2$, $\mathrm{E}[X^4] = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$.

(i) What is the distribution of $\bar{X}$?

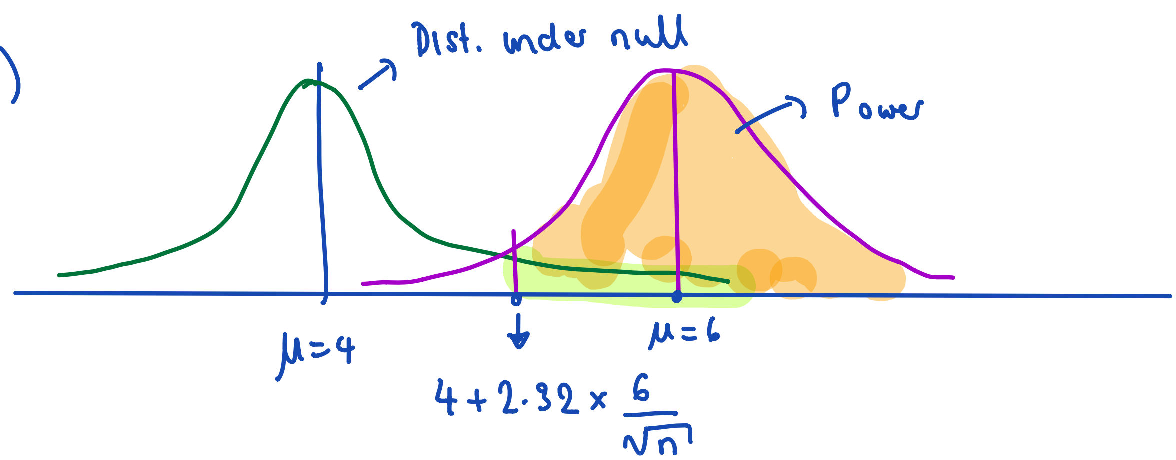
(ii) Suppose we test the hypothesis $H_0 : \mu = 4$ vs $H_A : \mu > 4$ using the classical z-test (with σ known). Obtain the power for the alternative $\mu = 6$ (as a function of σ and n).

(iii) Show that $\bar{X}^2$ is a biased estimator of μ^2 and obtain the bias.

(iv) Obtain the mean squared error $\mathrm{E}[(\bar{X}^2 - \mu^2)^2]$.
Hint: In the above calculation, you can use that $\bar{X}$ is normal.

(i) $\bar{X} \sim N(\mu, \sigma^2/n)$

(ii)



Reject null at 1% level if $\bar{x} > 4 + 2.32 \times \frac{\sigma}{\sqrt{n}}$

$$\text{Power} = P\left\{ \bar{X} \geq 4 + 2.32 \times \frac{\sigma}{\sqrt{n}} \,\middle|\, \mu = 6 \right\}$$

$$= P\left\{ \frac{\bar{X} - 6}{\sigma/\sqrt{n}} \geq \frac{4 + 2.32\,\sigma/\sqrt{n} - 6}{\sigma/\sqrt{n}} \,\middle|\, \mu = 6 \right\}$$

$$= P\left\{ Z \geq \sqrt{n}\,\frac{(4-6)}{\sigma} + 2.32 \right\}$$

Z ← standard normal

$$= 1 - \Phi\left(-\frac{2\sqrt{n}}{6} + 2.32\right)$$

(iii) $E[\bar{X}^2] = \text{var}[\bar{X}] + (E[\bar{X}])^2$

$$= \frac{6^2}{n} + \mu^2$$

$\Rightarrow$ Bias $= \frac{6^2}{n}$

(iv) Since $\bar{X}$ is normal we have that

$E[\bar{X}] = \mu$, $\text{var}[\bar{X}] = \frac{6^2}{n}$, $E[\bar{X}^4] = \mu^4 + 6\mu^2\frac{6^2}{n} + 3\left(\frac{6^2}{n}\right)^2$

$\Rightarrow \text{var}[\bar{X}^2] = E[\bar{X}^4] - (E[\bar{X}^2])^2$

$$= \mu^4 + 6\mu^2\frac{6^2}{n} + 3\left(\frac{6^2}{n}\right)^2$$

$$= \mu^4 + 6\mu^2\frac{6^2}{n} + 3\left(\frac{6^2}{n}\right)^2 - \left(\mu^2 + \frac{6^2}{n}\right)^2$$

$$= \cancel{\mu^4} + 6\mu^2\frac{6^2}{n} + 3\frac{6^4}{n^4} - \cancel{\mu^4} - 2\mu^2\frac{6^2}{n} + \frac{6^4}{n^2}$$

$$= 4\frac{\mu^2 6^2}{n} + 2\frac{6^4}{n^2}$$

Thus the mean squared error is

$$\operatorname{Var}[\bar{X}^2] + (\text{Bias})^2$$

$$= 4\mu^2 \frac{\sigma^2}{n} + \frac{2\sigma^4}{n^2} + \frac{\sigma^4}{n^2}$$

$$= 4\mu^2 \frac{\sigma^2}{n} + \frac{3\sigma^4}{n^2}$$

(2) Suppose that X_i is exponentially distributed with density $f(x;\theta) = \theta \exp(-\overset{\theta}{x})$ for $x \geq 0$. We define a new random variable

$$Y_i = \begin{cases} 0 & 0 \leq X_1 \leq 1 \\ 1 & X_i > 1. \end{cases}$$

- (i) Calculate $\mathrm{E}[Y_i]$.
- (ii) Obtain the MLE of θ based on $\{Y_i\}$. Call this estimator $\widehat{\theta}_Y$

  Hint: Be very careful with differentiation and to use the correct steps in the chain rule.
- (iii) Obtain the asymptotic distribution of $\widehat{\theta}_Y$.
- ~~(iv) Obtain a 95% confidence interval for θ.~~
- (v) Obtain the MLE of θ based on $\{X_i\}$. Call this estimator $\widehat{\theta}_X$
- (vi) Obtain the asymptotic distribution of $\widehat{\theta}_X$.
- ~~(vii) Compare the asymptotic variance of both estimators, which estimator has the smallest variance~~?

(2i) $$E[Y_i] = \int_1^\infty \theta e^{-\theta x}\, dx \quad \left[-e^{-\theta x}\right]_1^\infty = e^{-\theta}$$

(ii) $$Y_i = \begin{cases} 1 \\ 0 \end{cases} \qquad \begin{aligned} P(Y_i = 1) &= e^{-\theta} \\ P(Y_i = 0) &= 1 - e^{-\theta} \end{aligned}$$

The log-likelihood is

$$\mathcal{L}(\theta) = \sum_{i=1}^n Y_i \log e^{-\theta} + \sum_{i=1}^n (1 - Y_i) \log(1 - e^{-\theta})$$

$$= -\theta \sum Y_i + \log(1 - e^{-\theta}) \sum_{i=1}^n (1 - Y_i)$$

$$= -\theta S_n + \left[\log(1 - e^{-\theta})\right](n - S_n)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -S_n + \frac{e^{-\theta}}{1-e^{-\theta}}(n-S_n) = 0$$

$$-(1-e^{-\theta})S_n + (n-S_n)e^{-\theta} = 0$$

$$-S_n + \cancel{e^{-\theta}S_n} + ne^{-\theta} - \cancel{S_n e^{-\theta}} = 0$$

$$\Rightarrow \quad e^{-\theta} = \frac{S_n}{n}$$

$$\Rightarrow \quad -\hat{\theta}_Y = \log S_n/n$$

$$= \log \bar{Y} \qquad E[\bar{Y}] = e^{-\theta}$$

$$\Rightarrow \quad \hat{\theta}_Y = \log \frac{1}{\bar{Y}}.$$

(iii) To obtain the asymptotic distribution use either Lemma 1.2 or Theorem 3.2

$$g(x) = -\log x$$

$$g'(x) = -\frac{1}{x} \qquad [g'(x)]^2 = \left(\frac{1}{x}\right)^2$$

(*) $E(\bar{Y}) = \mu = p = e^{-\theta}$

$$g(\mu) = -\log p = \theta \quad \text{and} \quad [g'(\mu)]^2 = \left(\frac{1}{p^2}\right) = e^{2\theta}$$

$$\Rightarrow \quad \bar{Y} \to N\left(p, \frac{p(1-p)}{n}\right)$$

$$= N\left(e^{\theta}, \frac{e^{\theta}(1-e^{\theta})}{n}\right)$$

$$\Rightarrow \quad \hat{\theta}_Y \to N\left(-\log p, \left(\frac{1}{p}\right)^2 \cdot \frac{p(1-p)}{n}\right)$$

$$= N\left(\theta, \frac{(1-p)}{pn}\right)$$

$$= N\left(\theta, \frac{(1-e^{-\theta})}{e^{-\theta} n}\right)$$

(iv) Now we return to $\{x_i\}$.

The likelihood is

$$\mathcal{L}(\theta) = \sum_{i=1}^{n} \log\theta - \theta \sum_{i=1}^{n} x_i$$

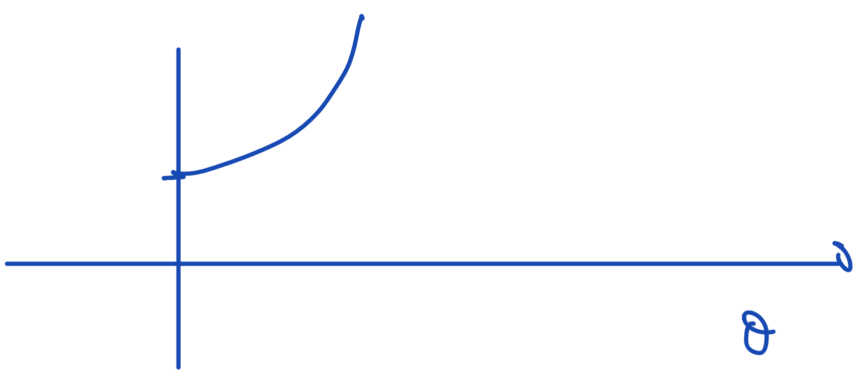
$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i \Rightarrow \hat{\theta}_X = \frac{1}{\bar{X}}$$

(iii) To obtain the distribution of $\hat{\theta}_X$ we obtain the Fisher information

$$\frac{\partial^2 \mathcal{L}}{\partial \theta^2} = -\frac{n}{\theta^2} \Rightarrow I(\theta) = \theta^2/n$$

$$\Rightarrow \hat{\theta}_X \to N\left(\theta, \frac{\theta^2}{n}\right)$$

$$\theta^2 < \frac{(1-e^{-\theta})}{e^{-\theta}} = \underbrace{e^{\theta} - 1}_{>0} \qquad \theta > 0$$



(3) Suppose that $\{X_i\}$ follows a double exponential distribution random variables with density

$$f(x;\theta) = \frac{1}{2}\theta \exp(-\theta|x|) \qquad x \in (-\infty, \infty),$$

where $\theta > 0$.

(i) Obtain the maximum likelihood estimator of θ.

(ii) Suppose we test the hypothesis $H_0 : \theta = \theta_0$ vs $H_A : \theta \neq \theta_0$. Construct the generalized log-likelihood ratio statistic $\log \Lambda(\underline{x})$.

(iii) Construct the asymptotic rejection region for $2 \log \Lambda(\underline{x})$ at the 5% level.

(iv) Suppose you test the hypothesis $H_0 : \theta = 1$ vs $H_A : \theta \neq 1$ at the 5% level using the generalized log-likelihood ratio test using the data ($n = 15$)

$$-0.09166, 0.49225, 0.15384, -0.63384, 0.29354, 0.21230, -0.59322, -0.20680, 0.76531$$
$$0.22258, 0.08837, -0.98747, 0.06102, -0.38458, 0.75048.$$

with summary statistics $\sum_{i=1}^{15} x_i = 0.142$ and $\sum_{i=1}^{15} |x_i| = 5.9$. What are the conclusions of the test?

$$\mathcal{L}(\theta) = -n \log 2 + n \log \theta - \theta \sum_{i=1}^{n} |x_i|$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} |x_i| = 0$$

$$\hat{\theta} = \frac{n}{\sum |x_i|}$$

(ii) $H_0 : \theta = \theta_0 \qquad H_A : \theta \neq \theta_0$

$$\log \Lambda(\underline{x}) = \cancel{-n \log 2} + n \log \hat{\theta} - \hat{\theta} \sum_{i=1}^{n} |x_i|$$

$$+ \cancel{n\log 2} - n\log\theta_0 + \theta_0 \sum_{i=1}^{n} |x_i|$$

$$= n\log\frac{\hat{\theta}}{\theta_0} + (\theta_0 - \hat{\theta})\sum |x_i|$$

$$2\log\Lambda(\underline{x}) = 2\left\{ n\log\frac{\hat{\theta}}{\theta_0} + (\theta_0 - \hat{\theta})\sum_{i=1}^{n} |x_i| \right\}$$

$$\to \chi^2_1 \quad \text{(under the null hypothesis)}$$

Reject null iff

$$2\log\Lambda(\underline{x}) \geq 3.82.$$

(iv) Data analysis $\sum_{i=1}^{15} |x_i| = 5.9$

$$\hat{\theta} = \frac{15}{5.9} = 2.54$$

$$2\log\Lambda(\underline{x}) = 2\left\{ 15 \times \log 2.54 + (1-2.54) \times 5.9 \right\}$$

$$= 9.79$$

Since $9.79 \geq 3.82$ we reject the null at the 5% level.

(4) Suppose that X is a continuous random variable defined on $[0,1]$ with density f.

There are two potential candidates for the density of X, $f_0(x) = 1$ for $x \in [0,1]$ and zero elsewhere or $f_1(x) = 2x$ for $x \in [0,1]$ and zero elsewhere.

We test $H_0 : f(x) = f_0(x)$ versus $H_A : f(x) = f_1(x)$.

(i) Suppose we observe just one random variable (sample size is $n = 1$). Using the Likelihood Ratio test, obtain the rejection region for x at the α significance level.

(ii) Obtain the power of the test at the α signficance level (the power will be a function of the α).

(iii) Suppose that $\alpha = 0.1$. Do we reject the null for $x = 0.9$.

(iv) Show that for all α, the power will be greater than α.

(i) $LR(X) = \dfrac{2X}{1}$

$$P\left\{ LR(X) \geq K_\alpha \mid H_0 \right\} = \alpha$$

$$= P\left\{ 2X \geq k_\alpha \mid H_0 \right\} = \alpha$$

$$= P\left\{ X \geq K_\alpha/2 \mid H_0 \right\} = \alpha$$

$$= \left[1 - \frac{K_\alpha}{2} \right] = \alpha$$

$$\Rightarrow K_\alpha = 2(1-\alpha)$$

Thus reject the null at the $\alpha\%$ level iff $2x \geq 2(1-\alpha)$.

(ii) Suppos $\alpha = 0.1$ and $x = 0.95$

Since $0.95 \geq (1-0.1) = 0.9$

we reject the null at 10% level.

(iii) Power calculation;

$$P\{2X \geq 2(1-\alpha) \mid H_A\}$$

$$= P\{X \geq (1-\alpha) \mid H_A\} = \int_{(1-\alpha)}^{1} (2x)\,dx$$

$$= \left[\frac{\cancel{2}x^2}{\cancel{2}}\right]_{(1-\alpha)}^{1} = 1-(1-\alpha)^2 = 1-(1-2\alpha+\alpha^2)$$

$$= \alpha(2-\alpha)$$

(iv) Because $\alpha \in (0,1)$ $2-\alpha \geq 1$

$\Rightarrow$ $\alpha(2-\alpha) \geq \alpha$. Thus power is greater the type I errors.