STAT 415 — Final (Spring 2021)

Name: ____________________________

Exam rules:

- You have 2hrs 45 minutes to to complete the exam, scan it and upload onto Gradebook.
- There are **4** Questions (The exam is worth 40pts in total).
- This exam is open book. You are free to use your class notes and HW solutions to solve the problems.

  Do not do a brain dump. Answers which are irrelevant will be penalized.

  Do not blindly copy answers from the HW solutions.
- State precisely in the derivations all the results that you use.
- Explain your arguments carefully, show all your work, and write out all your steps. Credit will be given for clarity. No credit will be given for a solution without working.
- If you are caught cheating or helping someone to cheat on this exam, you will both receive a grade of zero on the exam and you will be turned in.
- If a problem requires a numerical answer, you may express answer in terms of elementary functions or cdfs (e.g. $\Gamma(r+1)$ or $r!$).
- Presentation and scanning rules (20% will be knocked off if these rules are not adhered too).
  1. No light pencils. Black is easiest to read.
  2. Write big; scrawls are painful to read.
  3. Submit only **one** pdf file.
  4. The solutions should be on separate pages. When uploading onto Gradescope tie each question to the number.
  5. Check clarity of scan before submission.

(1) Suppose that $\{X_i\}_{i=1}^n$ are iid normal random variables with mean μ and variance σ^2. Let

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

To answer the below you can use that for normal random variables with mean μ and variance σ^2, then $\mathrm{E}[X] = \mu$, $\mathrm{E}[X^2] = \sigma^2 + \mu^2$, $\mathrm{E}[X^4] = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$.

(i) What is the distribution of $\bar{X}$?

(ii) Suppose we test the hypothesis $H_0 : \mu = 4$ vs $H_A : \mu > 4$ using the classical z-test at the 1% level (with σ known). Obtain an expression for the power for the alternative $\mu = 6$ (as a function of σ and n).

(iii) Show that $\bar{X}^2$ is a biased estimator of μ^2 and obtain the bias.

(iv) Obtain the mean squared error $\mathrm{E}[(\bar{X}^2 - \mu^2)^2]$.

Hint: In the above calculation, you can use that $\bar{X}$ is normal.

[10]

(2) Suppose that X_i is exponentially distributed with density $f(x;\theta) = \theta \exp(-\theta x)$ for $x \geq 0$. We define a new random variable

$$Y_i = \begin{cases} 0 & 0 \leq X_1 \leq 1 \\ 1 & X_i > 1. \end{cases}$$

(i) Calculate $\mathrm{E}[Y_i]$.

(ii) Obtain the MLE of θ based on $\{Y_i\}$ (not $\{X_i\}$). Call this estimator $\widehat{\theta}_Y$

Hint: Either use Lemma 3.1, or be very careful with differentiation and use the chain rule correctly.

(iii) Obtain the asymptotic distribution of $\widehat{\theta}_Y$.

(vi) Obtain the MLE of θ based on $\{X_i\}$. Call this estimator $\widehat{\theta}_X$

(v) Obtain the asymptotic distribution of $\widehat{\theta}_X$.

[10]

(3) Suppose that $\{X_i\}$ follows a double exponential distribution random variables with density

$$f(x;\theta) = \frac{1}{2}\theta\exp(-\theta|x|) \qquad x \in (-\infty, \infty),$$

where $\theta > 0$.

(i) Obtain the maximum likelihood estimator of θ.

(ii) Suppose we test the hypothesis $H_0 : \theta = \theta_0$ vs $H_A : \theta \neq \theta_0$. Construct the generalized log-likelihood ratio statistic $\log\Lambda(\underline{x})$.

(iii) Construct the asymptotic rejection region for $2\log\Lambda(\underline{x})$ at the 5% level.

(iv) Suppose you test the hypothesis $H_0 : \theta = 1$ vs $H_A : \theta \neq 1$ at the 5% level using the generalized log-likelihood ratio test and the the data ($n = 15$)

$$-0.09166, 0.49225, 0.15384, -0.63384, 0.29354, 0.21230, -0.59322, -0.20680, 0.76531$$
$$0.22258, 0.08837, -0.98747, 0.06102, -0.38458, 0.75048.$$

with summary statistics $\sum_{i=1}^{15} x_i = 0.142$ and $\sum_{i=1}^{15} |x_i| = 5.9$. What are the conclusions of the test?

[10]

(4) Suppose that X is a continuous random variable defined on $[0, 1]$ with density f.

There are two potential candidates for the density of X, $f_0(x) = 1$ for $x \in [0, 1]$ and zero elsewhere or $f_1(x) = 2x$ for $x \in [0, 1]$ and zero elsewhere.

We test $H_0 : f(x) = f_0(x)$ versus $H_A : f(x) = f_1(x)$.

(i) Suppose we observe just one random variable (sample size is $n = 1$). Using the Likelihood Ratio test, obtain the rejection region for X at the α significance level.

(ii) Suppose that $\alpha = 0.1$. Do we reject the null for $x = 0.95$.

(iii) Obtain the power of the test at the α signficance level (the power will be a function of the α).

(iv) Show that for all $\alpha \in (0, 1)$ the power will be greater than α.

[10]