A test for stationarity for irregularly spaced spatial data - Supplementary material

Soutir Bandyopadhyay and Suhasini Subba Rao

November 19, 2015

1 Introduction

In the supplementary material we include material that is not included in the main body of the paper. In particular,

1. The proof of Theorem 2.1.
2. The proofs of the results in Section 3 (the stationary case).
3. The proofs of the results in Section 4 (the nonstationary case).
4. A test designed specifically for detecting changes in the spatial variance. This test is based on the test we proposed in the main body, however simulations suggest it has less power than the original test. However, we include it here, as we believe it may have some independent interest and is worth future investigation.
5. The diagnostic plots for the ozone example considered in Section 7.2.

2 Properties of the irregular sampled DFT

PROOF of Theorem 2.1(i) To reduce cumbersome notations we first give the proof for $d = 1$.

The proof of (ia) is clear since $\mathrm{E}[Z(\boldsymbol{s})] = \mu$ and $\mathrm{E}[e^{is\omega_k}] = 0$ when $k \neq 0$. To prove (ib) and (ic) we use the straightforward observation $\mathrm{cov}[J_n^{\mu}(\omega_{k_1}), J_n^{\mu}(\omega_{k_2})] = \mathrm{E}[J_n(\omega_{k_1})\overline{J_n(\omega_{k_2})}]$. To prove (ib) and (ic) we use that $Z_1Z_2 = (Z_1 - \mu)(Z_2 - \mu) + \mu(Z_1 - \mu) + \mu(Z_2 - \mu) + \mu^2$, which gives the decomposition

$$J_n(\omega_{k_1})\overline{J_n(\omega_{k_2})} = A_1 + A_2 + A_3 + A_4 + A_5, \tag{1}$$

where

$$
\begin{aligned}
A_1 &= \frac{\lambda}{n^2}\sum_{j_1,j_2=1}^{n}[Z(s_{j_1})-\mu][Z(s_{j_2})-\mu]e^{is_{j_1}\omega_{k_1}}e^{-is_{j_2}\omega_{k_2}},\\
A_2 &= \frac{\lambda\mu}{n^2}\sum_{j_1,j_2=1}^{n}[Z(s_{j_1})-\mu]e^{is_{j_1}\omega_{k_1}}e^{-is_{j_2}\omega_{k_2}}, \qquad A_3=\frac{\lambda\mu}{n^2}\sum_{j_1,j_2=1}^{n}[Z(s_{j_2})-\mu]e^{is_{j_1}\omega_{k_1}}e^{-is_{j_2}\omega_{k_2}}\\
A_4 &= \frac{\lambda\mu^2}{n^2}\sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}}^{n}e^{is_{j_1}\omega_{k_1}}e^{-is_{j_2}\omega_{k_2}} \quad \text{and} \quad A_5=\frac{\lambda\mu^2}{n^2}\sum_{j=1}^{n}e^{is_j(\omega_{k_1}-\omega_{k_2})}.
\end{aligned}
$$

We observe that $\mathrm{E}[A_2]=0$ and $\mathrm{E}[A_3]=0$. If $k_1=0$ and $k_2=0$, then $\mathrm{E}[A_4+A_5]=\lambda\mu^2$, if $k_1\neq k_2$, then $\mathrm{E}[A_4+A_5]=0$, finally if $k_1=k_2$ (but $k_1\neq 0$) then $\mathrm{E}[A_4]=0$ and $\mathrm{E}[A_5]=\lambda\mu^2/n$. Since $A_4+A_5=\mathrm{E}[J_n(\omega_{k_1})]\mathrm{E}[\overline{J_n(\omega_{k_2})}]$ using the above immediately proves (ib) and (ic).

To prove the second part of (i) for convenience we assume $\mu=0$ and make the decomposition

$$
\mathrm{cov}\left[J_n(\omega_{k_1}),J_n(\omega_{k_2})\right]=T_1+T_2,
$$

where

$$
\begin{aligned}
T_1 &= \frac{\lambda}{n}\mathrm{cov}[Z(s)\exp(is\omega_{k_1}),Z(s)\exp(is\omega_{k_2})]\\
T_2 &= \lambda c_2\mathrm{cov}[Z(s_1)\exp(is_1\omega_{k_1}),Z(s_2)\exp(is_2\omega_{k_2})]
\end{aligned}
$$

and $c_2=n(n-1)/2$. We first analyse T_1. We see by using conditional expectations and second order stationarity that

$$
T_1=\frac{\lambda c(0)}{n}\mathrm{E}\left[\exp(is(\omega_{k_1}-\omega_{k_2}))\right]=\frac{\lambda c(0)}{n}\frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2}e^{is(\omega_{k_1}-\omega_{k_2})}ds=\begin{cases}\frac{\lambda c(0)}{n} & k_1=k_2\\ 0 & k_1\neq k_2\end{cases} \tag{2}
$$

Next we analyse T_2. By using conditional expectations we have

$$
\mathrm{cov}[Z(s_1)\exp(is_1\omega_{k_1}),Z(s_2)\exp(is_2\omega_{k_2})]=\mathrm{E}[c(s_1-s_2)e^{i(s_1\omega_{k_1}-s_2\omega_{k_2})}].
$$

Writing the expectation as an integral, making a change of variables $t=s_1-s_2$ and decomposing the integral over t into three parts give us

$$\begin{aligned} T_2 &= \frac{1}{\lambda}\int_{[-\lambda/2,\lambda/2]^2} c(s_1-s_2)\exp(is_1\omega_{k_1}-is_2\omega_{k_2})ds_1ds_2 \\ &= \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2}\exp(is_2(\omega_{k_1}-\omega_{k_2}))\int_{-\lambda/2-s_2}^{\lambda/2-s_2} c(t)\exp(it\omega_{k_1})dtds_2 \\ &= T_{21}+T_{22}+T_{23} \end{aligned}$$

where

$$\begin{aligned} T_{21} &= \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2}\exp(is_2(\omega_{k_1}-\omega_{k_2}))\int_{-\lambda/2}^{\lambda/2} c(t)\exp(it\omega_{k_1})dtds_2 \\ T_{22} &= \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2}\exp(is_2(\omega_{k_1}-\omega_{k_2}))\int_{-\lambda/2-s_2}^{-\lambda/2} c(t)\exp(it\omega_{k_1})dtds_2 \\ T_{23} &= \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2}\exp(is_2(\omega_{k_1}-\omega_{k_2}))\int_{\lambda/2-s_2}^{\lambda/2} c(t)\exp(it\omega_{k_1})dtds_2. \end{aligned}$$

Applying the same argument used on T_1 to T_{21}, it is clear that

$$T_{21} = \begin{cases} 0 & k_1 \neq k_2 \\ \int_{-\lambda/2}^{\lambda/2} c(t)\exp(it\omega_{k_1})dt = f(\omega_k)+O(\frac{1}{\lambda}) & k_1=k_2=k \end{cases}. \quad (3)$$

T_{21} is the leading term, we now show that T_{22} and T_{23} are the approximation errors, where $T_{22} = O(\frac{1}{\lambda})$ and $T_{23} = O(\frac{1}{\lambda})$. Partitioning the outer integrals in T_{22} and T_{23} into positive and negative parts gives $T_{22} = T_{221}+T_{222}$ and $T_{23} = T_{231}+T_{232}$, where

$$\begin{aligned} T_{221} &= \frac{1}{\lambda}\int_{-\lambda/2}^{0}\exp(is_2(\omega_{k_1}-\omega_{k_2}))\int_{-\lambda/2-s_2}^{-\lambda/2} c(t)\exp(it\omega_{k_1})dtds_2 \\ T_{222} &= \frac{1}{\lambda}\int_{0}^{\lambda/2}\exp(is_2(\omega_{k_1}-\omega_{k_2}))\int_{-\lambda/2-s_2}^{-\lambda/2} c(t)\exp(it\omega_{k_1})dtds_2 \end{aligned}$$

and

$$\begin{aligned} T_{231} &= \frac{1}{\lambda}\int_{-\lambda/2}^{0}\exp(is_2(\omega_{k_1}-\omega_{k_2}))\int_{\lambda/2-s_2}^{\lambda/2} c(t)\exp(it\omega_{k_1})dtds_2 \\ T_{232} &= \frac{1}{\lambda}\int_{0}^{\lambda/2}\exp(is_2(\omega_{k_1}-\omega_{k_2}))\int_{\lambda/2-s_2}^{\lambda/2} c(t)\exp(it\omega_{k_1})dtds_2. \end{aligned}$$

For T_{221} we have

$$\begin{aligned} |T_{221}| &\leq \frac{1}{\lambda}\int_{-\lambda/2}^{0}\int_{-\lambda/2}^{-\lambda/2-s_2}|c(t)|dtds_2 = \frac{1}{\lambda}\int_{0}^{\lambda/2}\int_{\lambda/2-s_2}^{\lambda/2}|c(t)|dtds_2 = \frac{1}{\lambda}\underbrace{\int_{0}^{\lambda/2}|sc(s)|ds}_{\text{follows from integration by parts}} \\ &\leq \frac{1}{\lambda}\int_{0}^{\lambda/2}|s\beta_{2+\delta}(s)|ds = O(\lambda^{-1}). \end{aligned}$$

For T_{222} we have

$$|T_{222}| \leq \frac{1}{\lambda}\int_{0}^{\lambda/2}\int_{-\lambda/2}^{-\lambda/2-s_2}|\beta_{2+\delta}(t)|dtds_2 \leq \frac{1}{\lambda}\int_{0}^{\lambda/2}\int_{-\lambda/2}^{-\lambda/2-s_2}\frac{C}{t^{2+\delta}}dtds_2 = O(\lambda^{-1}).$$

Thus we have $T_{22} = O(\frac{1}{\lambda})$. Using identical methods we obtain $T_{23} = O(\frac{1}{\lambda})$. Altogether this gives Theorem 2.1(i) for the case $d = 1$.

We give a rough outline for the case $d = 2$, where we show

$$\begin{aligned} &\mathrm{cov}[J_n(\omega_{k_1},\omega_{k_2}), J_n(\omega_{k_1+r_1},\omega_{k_2+r_2})] \\ &= \begin{cases} O(\frac{1}{\lambda^2}) & r_1 \neq 0 \text{ and } r_2 \neq 0 \\ O(\frac{1}{\lambda}) & r_1 = 0 \text{ and } r_2 \neq 0 \text{ or } r_2 = 0 \text{ and } r_1 \neq 0. \\ f(\omega_{k_1},\omega_{k_2}) + O(\frac{1}{\lambda}+\frac{\lambda^2}{n}) & r_1 = 0 \text{ and } r_2 = 0. \end{cases} \end{aligned}$$

As in the one-dimensional case we make the expansion

$$\begin{aligned} &\mathrm{cov}[J_n(\omega_{k_1},\omega_{k_2}), J_n(\omega_{k_1+r_1},\omega_{k_2+r_2})] \\ &= \frac{\lambda^2 n(n-1)}{n^2}\mathrm{E}\left[c(u_1-u_2;v_1-v_2)\exp(i(u_1-u_2)\omega_{k_1} - iu_2\omega_{r_1} + i(v_1-v_2)\omega_{k_2} - iv_2\omega_{r_2})\right] \\ &\quad \underbrace{+\frac{\lambda^2}{n}\mathrm{E}\left[c(0,0)\exp(-iu_2\omega_{r_1} - iv_2\omega_{r_2})\right]}_{=\frac{\lambda^2 c(0,0)}{n}I(r_1=0)I(r_2=0)}. \end{aligned}$$

We first consider the case $r_1 = 0$ and $r_2 = 0$. By a change of variables $u = u_1 - u_2$ and $v = v_1 - v_2$ we have

$$\begin{aligned} &\mathrm{var}[J_n(\omega_{k_1},\omega_{k_2})] \\ &= \frac{n(n-1)}{n^2\lambda^2}\int_{-\lambda/2}^{\lambda/2}\int_{-\lambda/2-v_1}^{\lambda/2-v_1}e^{iv\omega_{k_2}}\int_{-\lambda/2}^{\lambda/2}\int_{-\lambda/2-u_1}^{\lambda/2-u_1}c(u;v)e^{iu\omega_{k_1}}dudu_1dv_1 + O(\frac{\lambda^2}{n}). \end{aligned}$$

Successively, changing the range of the integral $\int_{-\lambda/2-u_1}^{\lambda/2-u_1}$ to $\int_{-\lambda/2}^{\lambda/2}$ and $\int_{-\lambda/2-v_1}^{\lambda/2-v_1}$ to $\int_{-\lambda/2}^{\lambda/2}$ gives

$$\mathrm{var}[J_n(\omega_{k_1},\omega_{k_2})] = c_2 f^{(\lambda)}(\omega_{k_1},\omega_{k_2}) + R_1 + R_2 + O(\frac{\lambda^2}{n}),$$

where

$$\begin{aligned}
f^{(\lambda)}(\omega_{k_1},\omega_{k_2}) &= \frac{1}{\lambda^2}\int_{-\lambda/2}^{\lambda/2}\int_{-\lambda/2}^{\lambda/2}\int_{-\lambda/2}^{\lambda/2}\int_{-\lambda/2}^{\lambda/2} c(u,v)e^{iu\omega_{k_1}+iv\omega_{k_2}}dudv \\
R_1 &= \frac{c_2}{\lambda^2}\int_{-\lambda/2}^{\lambda/2}\int_{-\lambda/2-v_1}^{\lambda/2-v_2} e^{iv\omega_{k_2}}\int_{-\lambda/2}^{\lambda/2}\left(\int_{-\lambda/2-u_1}^{\lambda/2-u_1} - \int_{-\lambda/2}^{\lambda/2}\right)c(u,v)e^{iu\omega_{k_1}}dudu_1dvdv_1 \\
\text{and } R_2 &= \frac{c_2}{\lambda^2}\int_{-\lambda/2}^{\lambda/2}\int_{-\lambda/2}^{\lambda/2} e^{iu\omega_{k_1}}\int_{-\lambda/2}^{\lambda/2}\left(\int_{-\lambda/2-v_1}^{\lambda/2-v_1} - \int_{-\lambda/2}^{\lambda/2}\right)c(u,v)e^{iv\omega_{k_2}}dudu_1dvdv_1.
\end{aligned}$$

By using the equality

$$\int_{-\lambda/2-u_1}^{\lambda/2-u_1} c(u,v)e^{iu\omega_{k_1}}du - \int_{-\lambda/2}^{\lambda/2} c(u,v)e^{iu\omega_{k_1}}du = \left(\int_{-\lambda/2-u_1}^{-\lambda/2} + \int_{\lambda/2}^{\lambda/2-u_1}\right) c(u,v)e^{iu\omega_{k_1}}du \quad (4)$$

and identical methods to those in the case $d = 1$, it can be shown that $R_1 = O(\lambda^{-1})$. It is also straightforward to show that $f^{(\lambda)}(\omega_{k_1},\omega_{k_2}) = f(\omega_{k_1},\omega_{k_2}) + O(\frac{1}{\lambda})$. Altogether this gives $\mathrm{var}[J_n(\omega_{k_1},\omega_{k_2})] = f(\omega_{k_1},\omega_{k_2}) + O(\frac{1}{\lambda} + \frac{\lambda^2}{n})$.

Using the same method we can show that $\mathrm{cov}\left[J_n(\omega_{k_1},\omega_{k_2}), J_n(\omega_{k_1+r_1},\omega_{k_2+r_2})\right] = O(\lambda^{-1})$, if either $r_1 = 0$ and $r_2 \neq 0$ or $r_1 \neq 0$ and $r_2 = 0$.

However in the case that $r_1 \neq 0$ and $r_2 \neq 0$ we can show that the correlation decays at a faster rate. Again by changing the limits of the integral we have

$$\mathrm{cov}\left[J_n(\omega_{k_1},\omega_{k_2}), J_n(\omega_{k_1+r_1},\omega_{k_2+r_2})\right] = R_1 + R_2 - R_3$$

where

$$\begin{aligned}
R_1 &= \frac{c_2}{\lambda^2}\int_{-\lambda/2}^{\lambda/2}\int_{-\lambda/2}^{\lambda/2} e^{-iu_1\omega_{r_1}+iv_1\omega_{r_2}}\int_{-\lambda/2}^{\lambda/2}\int_{-\lambda/2}^{\lambda/2} c(u,v)e^{iu\omega_{k_1}+iv\omega_{k_2}}dudvdu_1dv_1 \\
R_2 &= \frac{c_2}{\lambda^2}\int_{-\lambda/2}^{\lambda/2} e^{-iv_1\omega_{r_2}}\int_{-\lambda/2-v_1}^{\lambda/2-v_1} e^{iv\omega_{k_2}}\int_{-\lambda/2}^{\lambda/2} e^{-iu_1\omega_{r_1}}\left(\int_{-\lambda/2-u_1}^{\lambda/2-u_1} - \int_{-\lambda/2}^{\lambda/2}\right)c(u,v)e^{iu\omega_{k_1}}dudu_1dvdv_1 \\
R_3 &= \frac{c_2}{\lambda^2}\int_{-\lambda/2}^{\lambda/2} e^{-iu_1\omega_{r_1}}\int_{-\lambda/2}^{\lambda/2}\int_{-\lambda/2}^{\lambda/2} e^{iv_1\omega_{r_2}}\left(\int_{-\lambda/2-v_1}^{\lambda/2-v_1} - \int_{-\lambda/2}^{\lambda/2}\right)c(u,v)e^{iu\omega_{k_1}+iv\omega_{k_2}}dudu_1dvdv_1.
\end{aligned}$$

Since $r_1 \neq 0$ and $r_2 \neq 0$, it is clear that $R_1 = 0$. Furthermore, because $r_1 \neq 0$ we also have

$R_3 = 0$. Therefore the only non-zero term is R_2. By using (4) we decompose $R_2 = R_{21} + R_{22}$, where

$$
\begin{aligned}
R_{21} &= \frac{c_2}{\lambda^2}\int_{-\lambda/2}^{\lambda/2} e^{-iv_1\omega_{r_2}} \int_{-\lambda/2-v_1}^{\lambda/2-v_1} e^{iv\omega_{k_2}} \int_{-\lambda/2}^{\lambda/2} e^{-iu_1\omega_{r_1}} \int_{-\lambda/2-u_1}^{-\lambda/2} c(u,v)e^{iu\omega_{k_1}} du du_1 dv dv_1 \\
R_{22} &= \frac{c_2}{\lambda^2}\int_{-\lambda/2}^{\lambda/2} e^{-iv_1\omega_{r_2}} \int_{-\lambda/2-v_1}^{\lambda/2-v_1} e^{iv\omega_{k_2}} \int_{-\lambda/2}^{\lambda/2} e^{-iu_1\omega_{r_1}} \int_{\lambda/2}^{\lambda/2-u_1} c(u,v)e^{iu\omega_{k_1}} du du_1 dv dv_1.
\end{aligned}
$$

We now bound R_{21}. We exploit that $r_2 \neq 0$, by subtracting off the same term but with $\int_{-\lambda/2-v_1}^{\lambda/2-v_1}$ replacing $\int_{-\lambda/2}^{\lambda/2}$ (since this is zero) into R_{21}

$$
\begin{aligned}
R_{21} &= \frac{c_2}{\lambda^2}\int_{-\lambda/2}^{\lambda/2} e^{-iv_1\omega_{r_2}} \int_{-\lambda/2-v_1}^{\lambda/2-v_1} e^{iv\omega_{k_2}} \int_{-\lambda/2}^{\lambda/2} e^{-iu_1\omega_{r_1}} \int_{-\lambda/2-u_1}^{-\lambda/2} c(u,v)e^{iu\omega_{k_1}} du du_1 dv dv_1 \\
&\quad -\frac{c_2}{\lambda^2}\int_{-\lambda/2}^{\lambda/2} e^{-iv_1\omega_{r_2}} \int_{-\lambda/2}^{\lambda/2} e^{iv\omega_{k_2}} \int_{-\lambda/2}^{\lambda/2} e^{-iu_1\omega_{r_1}} \int_{-\lambda/2-u_1}^{-\lambda/2} c(u,v)e^{iu\omega_{k_1}} du du_1 dv dv_1 \\
&= \frac{c_2}{\lambda^2}\int_{-\lambda/2}^{\lambda/2} e^{-iu_1\omega_{r_1}} \int_{-\lambda/2-u_1}^{-\lambda/2} \int_{-\lambda/2}^{\lambda/2} e^{-iv_1\omega_{r_2}} \left(\int_{-\lambda/2-v_1}^{\lambda/2-v_1} - \int_{-\lambda/2}^{\lambda/2} \right) c(u,v)e^{iv\omega_{k_2}-iu\omega_{k_1}} du du_1 dv dv_1.
\end{aligned}
$$

By substituting (4) into the last integral above we have

$$
R_{21} = \frac{c_2}{\lambda^2}\int_{-\lambda/2}^{\lambda/2} e^{-iu_1\omega_{r_1}} \int_{-\lambda/2-u_1}^{-\lambda/2} \int_{-\lambda/2}^{\lambda/2} e^{-iv_1\omega_{r_2}} \left(\int_{-\lambda/2-v_1}^{-\lambda/2} + \int_{\lambda/2-v_1}^{\lambda/2} \right) c(u,v)e^{iv\omega_{k_2}-iu\omega_{k_1}} du du_1 dv dv_1. \tag{5}
$$

Using the same method we can show that

$$
R_{22} = \frac{c_2}{\lambda^2}\int_{-\lambda/2}^{\lambda/2} e^{-iu_1\omega_{r_1}} \int_{\lambda/2}^{\lambda/2-u_1} \int_{-\lambda/2}^{\lambda/2} e^{-iv_1\omega_{r_2}} \left(\int_{-\lambda/2-v_1}^{-\lambda/2} + \int_{\lambda/2-v_1}^{\lambda/2} \right) c(u,v)e^{iv\omega_{k_2}-iu\omega_{k_1}} du du_1 dv dv_1. \tag{6}
$$

Thus by using that $\mathrm{cov}\big[J_n(\omega_{k_1},\omega_{k_2}), J_n(\omega_{k_1+r_1},\omega_{k_2+r_2})\big] = R_{21} + R_{22}$, (5) and (6) we have

$$
\begin{aligned}
&\mathrm{cov}\big[J_n(\omega_{k_1},\omega_{k_2}), J_n(\omega_{k_1+r_1},\omega_{k_2+r_2})\big] \\
&= \frac{c_2}{\lambda^2}\int_{-\lambda/2}^{\lambda/2}\int_{-\lambda/2}^{\lambda/2} e^{-iu_1\omega_{r_1}} e^{-iv_1\omega_{k_2}} \left(\int_{-\lambda/2-u_1}^{-\lambda/2} + \int_{\lambda/2}^{\lambda/2-u_1} \right) \times \left(\int_{-\lambda/2-v_1}^{-\lambda/2} + \int_{\lambda/2}^{\lambda/2-v_1} \right) \\
&\quad \times c(u,v)e^{iu\omega_{k_1}+iv\omega_{k_2}} du dv du_1 dv_1.
\end{aligned}
$$

Thus we have

$$\begin{aligned}
& \left|\mathrm{cov}(J_n(\omega_{k_1},\omega_{k_2}),J_n(\omega_{k_1+r_1},\omega_{k_2+r_2})\right| \\
\leq\ & \frac{c_2}{\lambda^2}\int_{-\lambda/2}^{\lambda/2}\int_{-\lambda/2}^{\lambda/2}\left(\int_{-\lambda/2-u_1}^{-\lambda/2}+\int_{\lambda/2}^{\lambda/2-u_1}\right)\left(\int_{-\lambda/2-v_1}^{-\lambda/2}+\int_{\lambda/2}^{\lambda/2-v_1}\right)\beta_{2+\delta}(u)\beta_{2+\delta}(v)dudvdu_1dv_1 \\
\leq\ & \frac{4c_2}{\lambda^2}\int_{-\lambda/2}^{\lambda/2}\int_{-\lambda/2}^{\lambda/2}|uv|\beta_{2+\delta}(u)\beta_{2+\delta}(v)dudv = O(\lambda^{-2}).
\end{aligned}$$

Thus, we have shown the result holds for $d=2$. It is straightforward to generalize these arguments to $d>2$, which gives the desired result. $\square$

PROOF of Theorem 2.1(ii) We prove the result in the case $d=1$ (noting that the proof for $d>1$ is the same). Using that $\mathrm{cov}[Z(s_1),Z(s_2)|s_1,s_2]=\kappa_{s_1}(s_1-s_2)$ we have

$$\begin{aligned}
\lambda\mathrm{cov}(J_n(\omega_{k_1}),J_n(\omega_{k_2})) &= \lambda c_2\mathrm{E}\Big[\kappa_{s_1}(s_1-s_2)\exp(i(s_1-s_2)\omega_{k_1})+is_2(\omega_{k_1}-\omega_{k_2}))\Big] \\
&\quad +\frac{\lambda}{n}\mathrm{E}\left[\kappa_{s_1}(0)\exp(is_1(\omega_{k_1}-\omega_{k_2})\right].
\end{aligned}$$

By writing the expectation in terms of integral, changing the limits of the integral and using the same arguments as those used in the proof of Theorem 2.1(i) we have

$$\begin{aligned}
& \lambda\mathrm{cov}\left[J_n(\omega_{k_1}),J_n(\omega_{k_2})\right] = \\
=\ & \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2}e^{is\omega_{k_1-k_2}}\int_{-\lambda/2}^{\lambda/2}\kappa_s(t)e^{it\omega_{k_1}}dsdt+\frac{\lambda}{n}\left(\frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2}e^{is\omega_{k_1-k_2}}\kappa_s(0)ds\right)+O(\frac{1}{\lambda}) \\
=\ & \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2}e^{i2\pi(k_1-k_2)\frac{s}{\lambda}}f(\omega;s)ds+\frac{\lambda}{n}\left(\frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2}e^{is\omega_{k_1-k_2}}\kappa_s(0)ds\right)+O(\frac{1}{\lambda}),
\end{aligned}$$

which gives the desired result. $\square$

3 Proof of results in Section 3 (of main body)

PROOF of Theorem 3.1 The result follows immediately from Subba Rao [2015], Lemma 3.1(ii) for the case $\boldsymbol{r}\in\mathbb{Z}^d/\{\boldsymbol{0}\}$. $\square$

PROOF of Theorem 3.2 The proof follows immediately from Subba Rao [2015], Corollary 3.1 (see also Section 3.4 of the same paper). $\square$

PROOF of Theorem 3.3 Asymptotic normality of $\widehat{A}_\lambda(g;\boldsymbol{r})$ follows from Subba Rao [2015], Theorem 3.7, Subba Rao [2015]. $\square$

PROOF of Lemma 3.1 Using the standard bias/variance decomposition we have

$$\mathrm{E}\left[\widehat{c}_\lambda(\mathcal{S}') - c_{\lambda,1}\right]^2 = \mathrm{var}\left[\widehat{c}_\lambda(\mathcal{S}')\right] + \left(\mathrm{E}[\widehat{c}_\lambda(\mathcal{S}')] - c_{\lambda,1}\right)^2 \quad (7)$$

To simplify notation, we let $A(\boldsymbol{r}) = \widehat{A}_\lambda(g;\boldsymbol{r})$ and write $\widehat{c}_\lambda(\mathcal{S}')$ as

$$\widehat{c}_\lambda(\mathcal{S}') = \frac{\lambda^d}{2|\mathcal{S}'|-1}\sum_{\boldsymbol{r}\in\mathcal{S}'}|A(\boldsymbol{r})|^2 - \frac{2|\mathcal{S}'|}{2|\mathcal{S}'|-1}\lambda^d\bar{A}^2.$$

Substituting this into (7) gives the bound

$$\begin{aligned}\mathrm{E}\left[\widehat{c}_\lambda(\mathcal{S}') - c_{\lambda,1}\right]^2 \quad \leq \quad & 2\mathrm{var}\left(\frac{\lambda^d}{2|\mathcal{S}'|-1}\sum_{\boldsymbol{r}\in\mathcal{S}'}|A(\boldsymbol{r})|^2\right) + 2c_{\mathcal{S}}^2\mathrm{var}\left(\lambda^d\bar{A}^2\right) \\ & +2\left[\mathrm{E}\left(\frac{\lambda^d}{2|\mathcal{S}'|-1}\sum_{\boldsymbol{r}\in\mathcal{S}'}|A(\boldsymbol{r})|^2\right) - c_{\lambda,1}\right]^2 + 2c_{\mathcal{S}'}^2\left[\lambda^d\mathrm{E}\left(\bar{A}^2\right)\right]^2, \quad (8)\end{aligned}$$

where $c_{\mathcal{S}'} = [1-(2|\mathcal{S}'|-1)^{-1}] \to 1$ as $|\mathcal{S}'| \to \infty$. We first show that $\lambda^d\mathrm{E}|\bar{A}|^2 \to 0$. Partitioning $A(\boldsymbol{r})$ into real and imaginary parts we observe that

$$\begin{aligned}\lambda^d\mathrm{E}(\bar{A}^2) &= \frac{\lambda^d}{(2|\mathcal{S}'|)^2}\sum_{\boldsymbol{r}_1,\boldsymbol{r}_2\in\mathcal{S}'}\left(\mathrm{E}[\Re A(\boldsymbol{r}_1)\Re A(\boldsymbol{r}_2)] + \mathrm{E}[\Im A(\boldsymbol{r}_1)\Im A(\boldsymbol{r}_2)] + 2\mathrm{E}[\Re A(\boldsymbol{r}_1)\Im A(\boldsymbol{r}_2)]\right) \\ &= I + II + III.\end{aligned}$$

We obtain a bound for I, noting that similar bounds can be obtained for II and III. Writing the expectation in terms of first and second order cumulants we have

$$I = \frac{\lambda^d}{(2|\mathcal{S}'|)^2}\sum_{\boldsymbol{r}_1,\boldsymbol{r}_2\in\mathcal{S}'}\left(\mathrm{cov}[\Re A(\boldsymbol{r}_1),\Re A(\boldsymbol{r}_2)] + \mathrm{E}[\Re A(\boldsymbol{r}_1)]\mathrm{E}[\Re A(\boldsymbol{r}_2)]\right) = I_1 + I_2.$$

To bound I_1 we use Theorem 3.2 to give

$$I_1 = O\left(\frac{1}{|\mathcal{S}|'} + \ell_{\lambda,a,n}\right).$$

The bounds for $\mathrm{E}[\Re A(\boldsymbol{r}_1)]\mathrm{E}[\Re A(\boldsymbol{r}_2)]$ (elements in I_2) depend on whether the vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ contain zero (noting that we have placed the constraint that at least half the elements in the vector $\boldsymbol{r}$ are non-zero). Therefore we partition $\mathcal{S}'$ into the set of vectors which contain zeros, $\mathcal{S}_0$ and its complement $\mathcal{S}_1$. Using this decomposition and Theorem 3.1 we have

$$I_2 = O\left(\frac{1}{\lambda^d}\frac{|\mathcal{S}_1|^2}{|\mathcal{S}'|^2} + \frac{|\mathcal{S}_0||\mathcal{S}_1|}{\lambda^{\lceil d/2\rceil}|\mathcal{S}'|^2} + \frac{|\mathcal{S}_0|^2}{|\mathcal{S}'|^2}\right).$$

Therefore $I = O(\frac{1}{|\mathcal{S}|'} + \ell_{\lambda,a,n} + \frac{1}{\lambda^d}\frac{|\mathcal{S}_1|^2}{|\mathcal{S}'|^2} + \frac{|\mathcal{S}_0||\mathcal{S}_1|}{\lambda^{\lceil d/2\rceil}|\mathcal{S}'|^2} + \frac{|\mathcal{S}_0|^2}{|\mathcal{S}'|^2})$. Similar bounds can be obtained for II and III. Altogether this gives

$$\lambda^d \mathrm{E}(\bar{A}^2) = O\left(\frac{1}{|\mathcal{S}|'} + \ell_{\lambda,a,n} + \frac{1}{\lambda^d}\frac{|\mathcal{S}_1|^2}{|\mathcal{S}'|^2} + \frac{|\mathcal{S}_0||\mathcal{S}_1|}{\lambda^{\lceil d/2\rceil}|\mathcal{S}'|^2} + \frac{|\mathcal{S}_0|^2}{|\mathcal{S}'|^2}\right). \tag{9}$$

By using a similar method and Theorem 3.2 we can show that

$$\left|\mathrm{E}\left(\frac{\lambda^d}{2|\mathcal{S}'|-1}\sum_{\boldsymbol{r}\in\mathcal{S}'}|A(\boldsymbol{r})|^2\right) - c_{\lambda,1}\right| = O\left(\left[\frac{1}{|\mathcal{S}|'}\sum_{\boldsymbol{r}\in\mathcal{S}'}\frac{\|\boldsymbol{r}\|_1}{\lambda}\right] + \frac{1}{\lambda^d} + \frac{|\mathcal{S}_0|}{|\mathcal{S}'|} + \ell_{\lambda,a,n}\right). \tag{10}$$

Now we bound the two variance terms in (8). Expanding $\mathrm{var}(\frac{\lambda^d}{2|\mathcal{S}'|-1}\sum_{\boldsymbol{r}\in\mathcal{S}'}|A(\boldsymbol{r})|^2)$ gives

$$\begin{aligned}
&\mathrm{var}\left(\frac{\lambda^d}{2|\mathcal{S}'|-1}\sum_{\boldsymbol{r}\in\mathcal{S}'}|A(\boldsymbol{r})|^2\right)\\
= \;& \frac{\lambda^{2d}}{(2|\mathcal{S}'|-1)^2}\sum_{\boldsymbol{r}_1,\boldsymbol{r}_2\in\mathcal{S}'} 2\left|\mathrm{cov}(A(\boldsymbol{r}_1),A(\boldsymbol{r}_2))\right|^2 + \frac{\lambda^{2d}}{(2|\mathcal{S}'|-1)^2}\sum_{\boldsymbol{r}_1,\boldsymbol{r}_2}\mathrm{cum}\left[A(\boldsymbol{r}_1),\overline{A(\boldsymbol{r}_1)},A(\boldsymbol{r}_2),\overline{A(\boldsymbol{r}_2)}\right]\\
&+\frac{\lambda^{2d}}{(2|\mathcal{S}'|-1)^2}\sum_{\boldsymbol{r}_1,\boldsymbol{r}_2\in\mathcal{S}'}\left(\mathrm{cum}[A(\boldsymbol{r}_{j_1}),\overline{A(\boldsymbol{r}_{j_1})},A(\boldsymbol{r}_{j_2})]\mathrm{E}[\overline{A(\boldsymbol{r}_{j_2})}] + \mathrm{cum}[A(\boldsymbol{r}_{j_1}),\overline{A(\boldsymbol{r}_{j_1})},\overline{A(\boldsymbol{r}_{j_2})}]\mathrm{E}[A(\boldsymbol{r}_{j_2})]\right).
\end{aligned}$$

By applying Theorem 3.2 and Subba Rao [2015], Theorem 3.4, to the terms above we have

$$\mathrm{var}\left(\frac{\lambda^d}{2|\mathcal{S}'|-1}\sum_{\boldsymbol{r}\in\mathcal{S}'}|A(\boldsymbol{r})|^2\right) = O\left(\ell_{\lambda,a,n} + \frac{1}{|\mathcal{S}'|} + \frac{\log^{4d}(a)}{\lambda^d} + \frac{|\mathcal{S}_0|}{|\mathcal{S}'|\lambda^{\lceil d/2\rceil}}\right). \tag{11}$$

We note that we obtain similar bounds as the above when analysing $\mathrm{var}\left(\lambda^d\bar{A}^2\right)$. Therefore, by using (9)-(11) we obtain the desired result. □

4 Proof of results in Section 4 (in the main body)

In the following proof we require the following result

$$\int|\mathrm{sinc}(u)\mathrm{sinc}(u+x)|du \leq \ell_1(x) \text{ and } \sum_m \ell_p(m)\ell_q(m+r) \leq \ell_{p+q+1}(r), \tag{12}$$

where for some C, and $p \geq 0$, $\ell_p(x) = \log^p |x|/|x|$ for $|x| \geq Ce$ and $\ell_p(x) = Ce$ for $|x| \leq e$ (a proof can be found in Subba Rao [2015], Lemma C.1). We will also make heavy use of the representations

$$\widetilde{\kappa}\left(\boldsymbol{s}_1 - \boldsymbol{s}_2; \frac{\boldsymbol{s}_2}{\lambda}\right) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widetilde{f}\left(\boldsymbol{\omega}; \frac{\boldsymbol{s}_2}{\lambda}\right) \exp[i\boldsymbol{\omega}'(\boldsymbol{s}_1 - \boldsymbol{s}_2)] d\boldsymbol{\omega} \tag{13}$$

and

$$\widetilde{f}\left(\boldsymbol{\omega}; \frac{\boldsymbol{s}}{\lambda}\right) = \sum_{\boldsymbol{j} \in \mathbb{Z}^d} \zeta_{\boldsymbol{j}}(\boldsymbol{\omega}) \exp\left(\frac{i2\pi \boldsymbol{s}'\boldsymbol{j}}{\lambda}\right) = \sum_{\boldsymbol{j} \in \mathbb{Z}^d} \zeta_{\boldsymbol{j}}(\boldsymbol{\omega}) \exp\left(i2\pi \boldsymbol{s}'\boldsymbol{\omega}_{\boldsymbol{j}}\right). \tag{14}$$

PROOF of Theorem 4.1, equation (8) Taking expectation gives

$$\mathrm{E}\left[\widehat{A}_\lambda(g; \boldsymbol{r})\right] = c_2 \sum_{k_1,\ldots,k_d=-a}^{a} g(\boldsymbol{\omega_k}) \mathrm{E}\left[\widetilde{\kappa}\left(\boldsymbol{s}_1 - \boldsymbol{s}_2; \frac{\boldsymbol{s}_2}{\lambda}\right) \exp(i\boldsymbol{\omega}_k'(\boldsymbol{s}_1 - \boldsymbol{s}_2) \exp(-i\boldsymbol{\omega}_{\boldsymbol{r}}'\boldsymbol{s}_2)\right], \tag{15}$$

where $c_2 = n(n-1)/2$. To simplify notation we prove the rest of the result in the case $d = 1$. We first replace $\widetilde{\kappa}(s_1 - s_2, s_2/\lambda)$ in (15) with its representation (13) to give

$$\begin{aligned}
&\mathrm{E}[\widehat{A}_\lambda(g; r)] \\
&= \frac{c_2}{2\pi} \sum_{k=-a}^{a} g(\omega_k) \int_{-\infty}^{\infty} \frac{1}{\lambda^2} \int_{[-\lambda/2,\lambda/2]^2} \widetilde{f}(x; s_2/\lambda) \exp(is_1(\omega_k - x)) \exp(-is_2(\omega_{k+r} + x)) dx ds_1 ds_2.
\end{aligned}$$

To show $\mathrm{E}[\widehat{A}_\lambda(g; r)] \to A(g; \boldsymbol{r})$ we replace $\widetilde{f}(x; s_2/\lambda)$ with its Fourier transform to give

$$\begin{aligned}
\mathrm{E}[\widehat{A}_\lambda(g; r)] = &= \frac{c_2}{2\pi} \sum_{k=-a}^{a} \sum_{j=-\infty}^{\infty} g(\omega_k) \int_{-\infty}^{\infty} \frac{1}{\lambda^2} \int_{[-\lambda/2,\lambda/2]^2} \zeta_j(x) e^{-ixs_2\omega_j} e^{is_1(\omega_k - x)} e^{-is_2(\omega_{k+r}+x)} dx ds_1 ds_2 \\
&= \frac{c_2}{2\pi} \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} \zeta_j(x) \sum_{k=-a}^{a} g(\omega_k) \mathrm{sinc}\left(\frac{\lambda x}{2} - k\pi\right) \mathrm{sinc}\left(\frac{\lambda x}{2} + (k + r - j)\pi\right) dx,
\end{aligned}$$

where $\mathrm{sinc}(x) = \sin(x)/x$. By using a change of variable, $u = \frac{\lambda x}{2} - k\pi$, we have

$$\begin{aligned}
\mathrm{E}[\widehat{A}_\lambda(g; r)] &= \frac{c_2}{\pi} \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\lambda} \sum_{k=-a}^{a} \zeta_j\left(\frac{2u}{\lambda} + \omega_k\right) g(\omega_k) \mathrm{sinc}(u) \mathrm{sinc}\big(u + (r - j)\pi\big) du \qquad (16) \\
&= \frac{c_2}{(2\pi)\pi} \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\int_{-2\pi a/\lambda}^{2\pi a/\lambda} \zeta_j\left(\frac{2u}{\lambda} + \omega\right) g(\omega) d\omega\right] \mathrm{sinc}(u) \mathrm{sinc}\big(u + (r - j)\pi\big) du + O(\frac{1}{\lambda}),
\end{aligned}$$

where the $O(\lambda^{-1})$ arises because we have replaced a sum with an integral. We decompose the above sum as $\mathrm{E}[\widehat{A}_\lambda(g;r)] = I_M + I_R + O(\lambda^{-1})$, where

$$
\begin{aligned}
I_M &= \frac{c_2}{(2\pi)\pi}\sum_{|j|\le c}\int_{-\infty}^{\infty}\int_{-2\pi a/\lambda}^{2\pi a/\lambda}\zeta_j\left(\frac{2u}{\lambda}+\omega\right)g(\omega)\mathrm{sinc}\big(u\big)\mathrm{sinc}\big(u+(r-j)\pi\big)dud\omega \\
I_R &= \frac{c_2}{(2\pi)\pi}\sum_{|j|>c}\int_{-\infty}^{\infty}\int_{-2\pi a/\lambda}^{2\pi a/\lambda}\zeta_j\left(\frac{2u}{\lambda}+\omega\right)g(\omega)\mathrm{sinc}\big(u\big)\mathrm{sinc}\big(u+(r-j)\pi\big)dud\omega
\end{aligned}
$$

and c is some integer value greater than $|r|$ (at the end of the proof we specify exactly what c should be). First we show that I_R is asymptotically negligible. Under the stated assumptions we have, $\big|\int_{-2\pi a/\lambda}^{2\pi a/\lambda}\zeta_j(\frac{2u}{\lambda}+\omega)g(\omega)d\omega\big| \le \sup_\omega|g(\omega)|\ell(j)$ (noting if $j\neq 0$, then $\ell(j)=C/j$). Substituting this into I_R and using (12) we have

$$
\begin{aligned}
|I_R| &\le C\frac{\sup_\omega|g(\omega)|}{(2\pi)\pi}\sum_{|j|>c}\frac{1}{|j|}\int_{-\infty}^{\infty}|\mathrm{sinc}\big(u\big)\mathrm{sinc}\big(u+(r-j)\pi\big)|du \\
&\le C\frac{\sup_\omega|g(\omega)|}{(2\pi)\pi}\sum_{|j|>c}\frac{\log(j-r)}{j(j-r)} = \sum_{|j|>1}\frac{\log(c+j-r)}{(c+j)(c+j-r)} \le C\frac{\log(c-r)}{(c-r)},
\end{aligned}
$$

for some finite constant C. Next we consider I_M. By adding and subtracting $\zeta_j(\omega)$ we have $I_M = I_{M1}+I_{M2}$, where

$$
\begin{aligned}
I_{M1} &= \frac{c_2}{(2\pi)\pi}\sum_{|j|\le c}\int_{-2\pi a/\lambda}^{2\pi a/\lambda}\zeta_j(\omega)g(\omega)d\omega\int_{-\infty}^{\infty}\mathrm{sinc}\big(u\big)\mathrm{sinc}\big(u+(r-j)\pi\big)du \\
I_{M2} &= \frac{c_2}{(2\pi)\pi}\sum_{|j|\le c}\int_{-\infty}^{\infty}\mathrm{sinc}\big(u\big)\mathrm{sinc}\big(u+(r-j)\pi\big)\int_{-2\pi a/\lambda}^{2\pi a/\lambda}\Big(\zeta_j(\frac{2u}{\lambda}+\omega)g(\omega)-\zeta_j(\omega)g(\omega)\Big)d\omega du.
\end{aligned}
$$

We will show that $I_{M2}\to 0$. To do this we start by bounding the integral inside the sum. By using Subba Rao [2015], Lemma C.2 we can show that

$$
\begin{aligned}
&\Big|\int_{-\infty}^{\infty}\mathrm{sinc}\big(u\big)\mathrm{sinc}\big(u+(r-j)\pi\big)\int_{-2\pi a/\lambda}^{2\pi a/\lambda}\Big(\zeta_j(\frac{2u}{\lambda}+\omega)g(\omega)-\zeta_j(\omega)g(\omega)\Big)d\omega du\Big| \\
&\le C\ell(j)\sup_\omega|g(\omega)|\left(\frac{\log\lambda+\log|r-j|}{\lambda}\right).
\end{aligned}
$$

Substituting the above into I_{M2} gives

$$\begin{aligned}|I_{M2}| &\leq \frac{c_2}{(2\pi)\pi}\sum_{|j|\leq c} C\ell(j)\sup_{\omega}|g(\omega)|\left(\frac{\log\lambda+\log|r-j|+\log(\lambda+|r-j|)}{\lambda}\right)\\ &= O\left(\log c\left[\frac{\log\lambda+\log(|r|+|c|)}{\lambda}\right]\right).\end{aligned}$$

Altogether the bounds on I_{M2} and I_R give

$$\begin{aligned}&\mathrm{E}\left[\widehat{A}_\lambda(g;r)\right]\\ &= \frac{c_2}{(2\pi)\pi}\sum_{|j|\leq c}\int_{-2\pi a/\lambda}^{2\pi a/\lambda}\zeta_j(\omega)g(\omega)d\omega\int_{-\infty}^{\infty}\mathrm{sinc}\big(u\big)\mathrm{sinc}\big(u+(r-j)\pi\big)du\\ &\quad +O\bigg(\log c\left[\frac{\log\lambda+\log(|r|+|c|)}{\lambda}\right]+\frac{\log|c-r|}{|c-r|}\bigg)\\ &= \frac{1}{2\pi}\int_{-2\pi a/\lambda}^{2\pi a/\lambda}\zeta_r(\omega)g(\omega)d\omega+O\bigg(\log c\left[\frac{\log\lambda+\log(|r|+|c|)}{\lambda}\right]+\frac{\log|c-r|}{|c-r|}\bigg),\end{aligned}$$

where the last line of the above follows from the identity $\int_{-\infty}^{\infty}\mathrm{sinc}^2(u)du=\pi$ and if $r\neq 0$ then $\int_{-\infty}^{\infty}\mathrm{sinc}(u)\mathrm{sinc}(u+r\pi)du=0$. Finally, by choosing $c=(\lambda+r)$ we obtain

$$\mathrm{E}\left[\widehat{A}_\lambda(g;r)\right]=\frac{1}{2\pi}\int_{-2\pi a/\lambda}^{2\pi a/\lambda}\zeta_r(\omega)g(\omega)d\omega+O\left(\frac{\log^2(\lambda+r)}{\lambda}\right),$$

which gives the desired result. □

PROOF of Theorem 4.1, equation (9) To simplify the notation we prove the result for $d=1$. Expanding $\mathrm{cov}[\widehat{A}_\lambda(g;r_1),\widehat{A}_\lambda(g;r_2)]$ gives

$$\lambda\mathrm{cov}[\widehat{A}_\lambda(g;r_1),\widehat{A}_\lambda(g;r_2)] = A_{11}+A_{12}+A_2,$$

where by conditioning on the locations we have

$$\begin{aligned}A_{11}=c_4\lambda\sum_{k_1,k_2=-a}^{a}&g(\omega_{k_1})\overline{g(\omega_{k_2})}\mathrm{E}\left[e^{i\omega_{k_1}(s_1-s_2)}e^{-i\omega_{k_2}(s_3-s_4)}e^{-is_2\omega_{r_1}+is_4\omega_{r_2}}\right.\\ &\left.\times\widetilde{\kappa}(s_1-s_3;\frac{s_3}{\lambda})\widetilde{\kappa}(s_2-s_4;\frac{s_3}{\lambda})\right]\end{aligned}$$

$$A_{12} = c_4\lambda \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})}\mathrm{E}\left[e^{i\omega_{k_1}(s_1-s_2)}e^{-i\omega_{k_2}(s_3-s_4)}e^{-is_2\omega_{r_1}+is_4\omega_{r_2}}\right.$$
$$\left. \times\widetilde{\kappa}(s_1-s_4;\frac{s_4}{\lambda})\widetilde{\kappa}(s_2-s_3;\frac{s_4}{\lambda})\right],$$

and

$$A_2 = \frac{\lambda}{n^4}\sum_{j_1,j_2,j_3,j_4\in\mathcal{S}_3}\sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})}\mathrm{cov}\left[Z(s_{j_1})Z(s_{j_2})e^{is_{j_1}\omega_{k_1}}e^{-is_{j_2}\omega_{k_1+r_1}},\right.$$
$$\left. Z(s_{j_1})Z(s_{j_2})e^{is_{j_1}\omega_{k_1}}e^{-is_{j_2}\omega_{k_1+r_1}}\right]$$

where $c_4 = n(n-1)(n-2)(n-3)/n^4$ and $\mathcal{S}_3 = \{(j_1,j_2,j_3,j_4); 1 \leq j_1,\ldots,j_4 \leq n, j_1 \neq j_2$ and $j_3 \neq j_4$ but at least two j's are same$\}$. By using similar arguments to those given in Subba Rao [2015], Lemma D.1, it can be shown that $A_2 = O(\lambda/n)$. Below we will show that A_{11} and A_{12} are the leading term.

We will obtain a bound for A_{11}. Replacing $\widetilde{\kappa}(s_1-s_3;\frac{s_3}{\lambda})$ and $\widetilde{\kappa}(s_2-s_4;\frac{s_4}{\lambda})$ in A_{11} with the representations given in (13) we have

$$\begin{aligned} A_{11} &= \frac{c_4\lambda}{\lambda^4(2\pi)^2}\int_{[-\lambda/2,\lambda/2]^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widetilde{f}(x;\frac{s_3}{\lambda})\widetilde{f}(y;\frac{s_4}{\lambda})\sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})} \\ &\quad \times e^{i\omega_{k_1}(s_1-s_2)}e^{is_1x}e^{is_2(y-\omega_{r_1})}e^{-is_3(x+\omega_{k_2})}e^{is_4(-y+\omega_{k_2}+\omega_{r_2})}ds_1ds_2ds_3ds_4dxdy. \end{aligned}$$

Next we use Assumption 4.1 to replace $\widetilde{f}(x;\frac{s_3}{\lambda})$ and $\widetilde{f}(y;\frac{s_4}{\lambda})$ in the above by their Fourier representations given in (14)

$$\begin{aligned} A_{11} &= \frac{c_4}{\lambda^3(2\pi)^2}\int_{[-\lambda/2,\lambda/2]^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{j_1,j_2=-\infty}^{\infty}\zeta_{j_1}(x)e^{is_3\omega_{j_1}}\zeta_{j_2}(y)e^{is_4\omega_{j_2}} \\ &\quad \times\sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})}e^{i\omega_{k_1}(s_1-s_2)}e^{is_1x}e^{is_2(y-\omega_{r_1})}e^{-is_3(x+\omega_{k_2})} \\ &\quad \times e^{is_4(-y+\omega_{k_2}+\omega_{r_2})}ds_1ds_2ds_3ds_4dxdy \\ &= \frac{c_4\lambda}{(2\pi)^2}\sum_{j_1,j_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{k_1,k_2=-a}^{a}\zeta_{j_1}(x)\zeta_{j_2}(y)g(\omega_{k_1})\overline{g(\omega_{k_2})}\mathrm{sinc}\left(\frac{\lambda x}{2}+k_1\pi\right)\times \\ &\quad \mathrm{sinc}\left(\frac{\lambda y}{2}-(k_1+r_1)\pi\right)\mathrm{sinc}\left(\frac{\lambda x}{2}+(k_2-j_1)\pi\right)\mathrm{sinc}\left(\frac{\lambda y}{2}-(k_2+r_2+j_2)\right)dxdy. \end{aligned}$$

Changing variables with $u = \frac{\lambda x}{2} + k_1\pi$, $v = \frac{\lambda y}{2} - (k_1 + r_1)\pi$ and $m = k_2 - k_1$ gives

$$
\begin{aligned}
A_{11} = {} & \frac{c_4}{\lambda\pi^2}\sum_{j_1,j_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{k_1,k_2=-a}^{a}\zeta_{j_1}\left(\frac{\lambda u}{2} - \omega_{k_1}\right)\zeta_{j_2}\left(\frac{\lambda v}{2} - \omega_{k_2+r_1}\right)g(\omega_{k_1})\overline{g(\omega_{k_2})} \\
& \quad \mathrm{sinc}(u)\mathrm{sinc}(v)\mathrm{sinc}(u + (k_2 - k_1 - j_1)\pi\big)\mathrm{sinc}(v - (k_2 - k_1 + r_2 - r_1 - j_2))dudv \\
= {} & \frac{c_4}{\pi^2}\sum_{m=-2a}^{2a}\sum_{j_1,j_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{sinc}(u)\mathrm{sinc}(v)\mathrm{sinc}(u + (m - j_1)\pi\big)\mathrm{sinc}(v - (m + r_2 - r_1 + j_2)) \\
& \quad \times H^{(m)}_{j_1,j_2}(u,v)dudv,
\end{aligned}
$$

where

$$
H^{(m)}_{j_1,j_2}(u,v) = \frac{1}{\lambda}\sum_{k_1=\max(-a,-a+m)}^{\min(a,a+m)}\zeta_{j_1}\left(\frac{\lambda u}{2} - \omega_{k_1}\right)\zeta_{j_2}\left(\frac{\lambda v}{2} - \omega_{k_1+r_1}\right)g(\omega_{k_1})\overline{g(\omega_{k_1+m})}.
$$

We start by bounding $H^{(m)}_{j_1,j_2}(u,v)$. Using that $\sup_\omega|\frac{\partial\zeta_j(\omega)}{\partial\omega}| \leq \ell(j)$, we replace the summand in $H^{(m)}_{j_1,j_2}(u,v)$ with its integral to give

$$
H^{(m)}_{j_1,j_2}(u,v) = \int_{-2\pi a/\lambda}^{2\pi a/\lambda}\zeta_{j_1}\left(\frac{\lambda u}{2} - \omega\right)\zeta_{j_2}\left(\frac{\lambda v}{2} - \omega - \omega_{r_1}\right)g(\omega)\overline{g(\omega + \omega_m)}d\omega + O\left(\frac{1}{\lambda\ell(j_1)\ell(j_2)}\right).
$$

Therefore, under Assumption 4.1(i) we can show that

$$
\begin{aligned}
\sup_{u,v}\left|H^{(m)}_{j_1,j_2}(u,v)\right| & \leq C\left|\int\zeta_{j_1}\left(\frac{\lambda u}{2} - \omega\right)\zeta_{j_2}\left(\frac{\lambda v}{2} - \omega - \omega_{r_1}\right)g(\omega)\overline{g(\omega + \omega_m)}d\omega\right| \\
& \leq C\sup_\omega|g(\omega)|^2\ell(j_1)\ell(j_2).
\end{aligned}
$$

Substituting the above into A_{11} gives the bound

$$
\begin{aligned}
|A_{11}| \leq {} & \frac{C}{\pi^2}\sum_{m=-2a}^{2a}\sum_{j_1,j_2}\ell(j_1)\ell(j_2) \\
& \times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\mathrm{sinc}(u)\mathrm{sinc}(v)\mathrm{sinc}(u + (m - j_1)\pi\big)\mathrm{sinc}(v - (m + r_2 - r_1 + j_2))|dudv.
\end{aligned}
$$

Finally, by using (12) we have

$$
\begin{aligned}
|A_{11}| &\leq C \sum_{j_1,j_2=-\infty}^{\infty} \sum_{m=-a}^{a} \ell(j_1)\ell(j_2)\ell_1(m-j_1)\ell_1(m+r_2-r_1+j_2) \\
&\leq C \sum_{m=-a}^{a} \left(\sum_{j_1=-\infty}^{\infty} \ell(j_1)\ell_1(m-j_1) \right) \left(\sum_{j_2=-\infty}^{\infty} \ell(j_2)\ell_1(m+r_2-r_1+j_2) \right) \\
&\leq \sum_{m=-a}^{a} \ell_2(m)\ell_2(m+r_2-r_1) \leq \ell_5(r_2-r_1).
\end{aligned}
$$

A similar bound can be obtained for A_{12}. Altogether this proves that in the nonstationary case (using rescaled asymptotics) we have $\lambda|\mathrm{cov}[\widehat{A}_\lambda(g;r), \widehat{A}_\lambda(g;r)]| \leq \ell_5(r_1-r_2) + O(\lambda/n)$. It is straightforward to extend the above argument to $d > 1$. Altogether this proves (ii). $\square$

We require the following corollary, which gives bounds on $\mathrm{E}|\widehat{A}_\lambda(g;\boldsymbol{r})|$, to prove Lemma 4.1. We note that we require this result because when $d > 1$, the expression for $\mathrm{E}|\widehat{A}_\lambda(g;\boldsymbol{r})|$ in Theorem 4.1, is not sufficient for proving Lemma 4.1.

Corollary 4.1 *Suppose Assumption 4.1 holds. Then we have*

(i) $\mathrm{E}|\widehat{A}_\lambda(g;\boldsymbol{r})| \leq C\ell_1(\boldsymbol{r})$

(ii) $\sum_{\boldsymbol{r}\in\mathcal{S}'} \left|\mathrm{E}\left(\widehat{A}_\lambda(g;\boldsymbol{r})\right)\right| \leq \sum_{\boldsymbol{r}} \prod_{i=1}^{d} \ell_1(j_i) \leq C\left[\log|\mathcal{S}'|_{\max}\left(\log|\mathcal{S}'|_{\max} - \log|\mathcal{S}'|_{\min}\right)\right]^d.$

PROOF. To prove (i) we use (16) to give

$$
\left|\mathrm{E}[\widehat{A}_\lambda(g;\boldsymbol{r})]\right| \leq \frac{C}{\pi^d} \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \frac{1}{\lambda^d} \sum_{k_1,\ldots,k_d=-a}^{a} \left|\zeta_{\boldsymbol{j}}(\frac{2\boldsymbol{u}}{\lambda} + \boldsymbol{\omega_k})g(\boldsymbol{\omega_k})\mathrm{Sinc}\big(\boldsymbol{u}\big)\mathrm{Sinc}\big(\boldsymbol{u}+(\boldsymbol{r}-\boldsymbol{j})\pi\big)\right| d\boldsymbol{u},
$$

where $\mathrm{Sinc}(\boldsymbol{u}) = \prod_{i=1}^{d}\mathrm{sinc}(u_i)$. By using Assumption 4.1 we have $|\frac{1}{\lambda^d}\sum_{\boldsymbol{k}=-a}^{a}|\zeta_{\boldsymbol{j}}(\frac{2\boldsymbol{s}}{\lambda} + \boldsymbol{\omega_k})g(\boldsymbol{\omega_k})| \leq \ell(\boldsymbol{j})$. Substituting this into the above gives

$$
\left|\mathrm{E}[\widehat{A}_\lambda(g;\boldsymbol{r})]\right| \leq C \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \ell(\boldsymbol{j}) \int_{\mathbb{R}^d} \left|\mathrm{Sinc}\big(\boldsymbol{u}\big)\mathrm{Sinc}\big(\boldsymbol{u}+(\boldsymbol{r}-\boldsymbol{j})\pi\big)\right| d\boldsymbol{u} \leq C\ell_1(\boldsymbol{r})
$$

where C is a finite constant and the last line follows from (12) (which holds for both integrals and summands). This proves (i).

To prove (ii) we use (i) to obtain

$$
\sum_{\boldsymbol{r}\in\mathcal{S}'} \left|\mathrm{E}[\widehat{A}_\lambda(g;\boldsymbol{r})]\right| \leq C \sum_{\boldsymbol{r}\in\mathcal{S}'} \ell_1(\boldsymbol{r}) \leq C \left(\sum_{r=|\mathcal{S}'|_{\min}}^{|\mathcal{S}'|_{\max}} \ell_1(r) \right)^d \leq C\left[\log|\mathcal{S}'|_{\max}\left(\log|\mathcal{S}'|_{\max} - \log|\mathcal{S}'|_{\min}\right)\right]^d,
$$

which proves (ii). $\square$

PROOF of Lemma 4.1 To prove (i) we use the notation $A(\boldsymbol{r}) = \widehat{A}_{\lambda}(g;\boldsymbol{r})$ and expand $\widehat{c}_{\lambda}(\mathcal{S}')$ this gives

$$\begin{aligned}
\mathrm{E}[\widehat{c}_{\lambda}(\mathcal{S}')] &= \mathrm{E}\left[\frac{\lambda^d}{(2|\mathcal{S}'|-1)}\sum_{\boldsymbol{r}\in\mathcal{S}'}\left([\Re\widehat{A}_{\lambda}(g;\boldsymbol{r})-\bar{A}]^2+[\Im\widehat{A}_{\lambda}(g;\boldsymbol{r})-\bar{A}]^2\right)\right] \\
&= \mathrm{E}\left(\frac{\lambda^d}{2|\mathcal{S}'|-1}\sum_{\boldsymbol{r}\in\mathcal{S}'}|A(\boldsymbol{r})|^2-\frac{2|\mathcal{S}'|}{2|\mathcal{S}'|-1}\lambda^d\bar{A}^2\right)\leq I+II,
\end{aligned}$$

where I and II are decompositions into expectations into first and second order cumulants

$$\begin{aligned}
I &= \frac{\lambda^d}{2|\mathcal{S}'|-1}\sum_{\boldsymbol{r}\in\mathcal{S}'}\mathrm{var}[A(\boldsymbol{r})] \\
&\quad+\frac{n_{\mathcal{S}}\lambda^d}{|\mathcal{S}'|^2}\sum_{\boldsymbol{r}_1,\boldsymbol{r}_2\in\mathcal{S}'}|\mathrm{cov}[\Re A(\boldsymbol{r}_1),\Re A(\boldsymbol{r}_2)]+\mathrm{cov}[\Im A(\boldsymbol{r}_1),\Im A(\boldsymbol{r}_2)]+2\mathrm{cov}[\Re A(\boldsymbol{r}_1),\Im A(\boldsymbol{r}_2)]| \\
II &= \frac{\lambda^d}{2|\mathcal{S}'|-1}\sum_{\boldsymbol{r}\in\mathcal{S}'}|\mathrm{E}[A(\boldsymbol{r})]|^2 \\
&\quad+\frac{c_{\mathcal{S}'}\lambda^d}{|\mathcal{S}'|^2}\sum_{\boldsymbol{r}_1,\boldsymbol{r}_2\in\mathcal{S}'}\left|\mathrm{E}[\Re A(\boldsymbol{r}_1)]\overline{\mathrm{E}[\Re A(\boldsymbol{r}_2)]}+\mathrm{E}[\Im A(\boldsymbol{r}_1)]\overline{\mathrm{E}[\Im A(\boldsymbol{r}_2)]}+2\mathrm{E}[\Re A(\boldsymbol{r}_1)]\overline{\mathrm{E}[\Im A(\boldsymbol{r}_2)]}\right|
\end{aligned}$$

and $c_{\mathcal{S}}' = 1-(2|\mathcal{S}'|-1)^{-1}$. By using Theorem 4.1(ii) it is straightforward to show that $I = O(1)$. To bound II we use Corollary 4.1. Corollary 4.1(ii) gives a bound for the second summand in II and Corollary 4.1(i) gives a bound for the first summand. Using these bounds we have

$$\begin{aligned}
II &\leq \frac{\lambda^d}{2|\mathcal{S}'|}\sum_{\boldsymbol{r}\in\mathcal{S}'}\ell_1(\boldsymbol{r})^2+\frac{\lambda^d}{|\mathcal{S}'|^2}\left[\log|\mathcal{S}'|_{\max}\left(\log|\mathcal{S}'|_{\max}-\log|\mathcal{S}'|_{\min}\right)\right]^{2d} \\
&\leq C\left[\frac{\lambda^d}{|\mathcal{S}'|}\times\frac{\log^2|\mathcal{S}'|_{\max}}{|\mathcal{S}'|_{\min}}+\frac{\lambda^d}{|\mathcal{S}'|^2}\left(\log|\mathcal{S}'|_{\max}\log\frac{|\mathcal{S}'|_{\max}}{|\mathcal{S}'|_{\min}}\right)^{2d}\right].
\end{aligned}$$

Thus the bounds for I and II altogether give

$$\mathrm{E}[\widehat{c}_{\lambda}(\mathcal{S}')] = O\left(1+\left[\frac{\lambda^d\log^2|\mathcal{S}'|_{\max}}{|\mathcal{S}'||\mathcal{S}'|_{\min}}+\frac{\lambda^d}{|\mathcal{S}'|^2}\left(\log|\mathcal{S}'|_{\max}\log\frac{|\mathcal{S}'|_{\max}}{|\mathcal{S}'|_{\min}}\right)^{2d}\right]\right),$$

this proves (i).

To prove (ii), we use similar expansions to those above to give

$$\mathrm{var}[\widehat{c}_{\lambda}(\mathcal{S}')] \leq 2\mathrm{var}\left(\frac{\lambda^d}{2|\mathcal{S}'|-1}\sum_{\boldsymbol{r}\in\mathcal{S}'}|A(\boldsymbol{r})|^2\right) + 2\mathrm{var}\left(\frac{2|\mathcal{S}'|}{2|\mathcal{S}'|-1}\lambda^d\bar{A}^2\right) = III + IV.$$

We first bound III, expanding the variance we have

$$\begin{aligned}
& \mathrm{var}\left(\frac{\lambda^d}{2|\mathcal{S}'|-1}\sum_{\boldsymbol{r}\in\mathcal{S}'}|A(\boldsymbol{r})|^2\right) \\
= & \frac{\lambda^{2d}}{(2|\mathcal{S}'|-1)^2}\sum_{\boldsymbol{r}_1,\boldsymbol{r}_2\in\mathcal{S}'}2\left|\mathrm{cov}(A(\boldsymbol{r}_1),A(\boldsymbol{r}_2))\right|^2 + \frac{\lambda^{2d}}{(2|\mathcal{S}'|-1)^2}\sum_{\boldsymbol{r}_1,\boldsymbol{r}_2}\mathrm{cum}\left[A(\boldsymbol{r}_1),\overline{A(\boldsymbol{r}_1)},A(\boldsymbol{r}_2),\overline{A(\boldsymbol{r}_2)}\right] \\
& + \frac{\lambda^{2d}}{(2|\mathcal{S}'|-1)^2}\sum_{\boldsymbol{r}_1,\boldsymbol{r}_2\in\mathcal{S}'}\left(\mathrm{cum}[A(\boldsymbol{r}_{j_1}),\overline{A(\boldsymbol{r}_{j_1})},A(\boldsymbol{r}_{j_2})]\mathrm{E}[\overline{A(\boldsymbol{r}_{j_2})}] + \mathrm{cum}[A(\boldsymbol{r}_{j_1}),\overline{A(\boldsymbol{r}_{j_1})},\overline{A(\boldsymbol{r}_{j_2})}]\mathrm{E}[A(\boldsymbol{r}_{j_2})]\right) \\
= & III_1 + III_2 + III_3.
\end{aligned}$$

To bound III_1 we use Theorem 4.1 to obtain the bound

$$|III_1| \leq \frac{2}{(2|\mathcal{S}'|-1)^2}\sum_{\boldsymbol{r}_1,\boldsymbol{r}_2\in\mathcal{S}'}\left(\ell_5(\boldsymbol{r}_1-\boldsymbol{r}_2)+\frac{\lambda^d}{n}\right)^2 = O\left(\frac{1}{|\mathcal{S}'|}+\frac{\lambda^d}{n}\right).$$

To bound III_2 and III_3 we require bounds on the third and fourth order cumulants. We use the same method used Subba Rao [2015], Lemma D.4 (section of higher order cumulants) and the techniques used to prove (9) to give

$$\mathrm{cum}_3\left[A(\boldsymbol{r}_1),A(\boldsymbol{r}_2),A(\boldsymbol{r}_3)\right] = O\left(\frac{\log^{4d}a}{\lambda^{2d}}\right) \quad \text{and} \quad \mathrm{cum}_4\left[A(\boldsymbol{r}_1),A(\boldsymbol{r}_2),A(\boldsymbol{r}_3),A(\boldsymbol{r}_4)\right] = O\left(\frac{\log^{6d}a}{\lambda^{3d}}\right). \quad (17)$$

Using (17) it is clear that $III_2 = O(\log^{6d}a\lambda^{-d})$. To bound III_3 we use (17) and Lemma 4.1(i) to give

$$III_3 \leq \frac{C\log^{4d}a}{|\mathcal{S}'|}\sum_{\boldsymbol{r}\in\mathcal{S}'}|\mathrm{E}[A(\boldsymbol{r})]| \leq \frac{C\log^{4d}(a)}{|\mathcal{S}'|}\left[\log|\mathcal{S}'|_{\max}\left(\log|\mathcal{S}'|_{\max}-\log|\mathcal{S}'|_{\min}\right)\right]^d.$$

Altogether the bounds give

$$III \leq C\left(\frac{1}{|\mathcal{S}'|}+\frac{\lambda^d}{n}+\frac{\log^{6d}(a)}{\lambda}+\frac{C\log^{4d}(a)}{|\mathcal{S}'|}\left[\log|\mathcal{S}'|_{\max}\left(\log\frac{|\mathcal{S}'|_{\max}}{|\mathcal{S}'|_{\min}}\right)\right]^d.\right)$$

Finally, we note that a similar bound can be obtained for IV. The bounds for III and IV give (ii). □

5 Detecting changes in the spatial variance

In the case that the locations $\{\boldsymbol{s}_j; j = 1, \ldots, n\}$ are not that dense on $[-\lambda/2, \lambda/2]^d$, $\mathcal{T}_{\mathcal{S},\mathcal{S}'}$ is not very sensitive to changes in the variance. To address this issue in this section we consider a method for simply detecting changes in the spatial variance. We do this by using Theorem 2.1 in the main body. We adapt the DFT $J_n(\boldsymbol{\omega}_k)$ to detect changes in the spatial variation and define

$$v_\lambda(\boldsymbol{r}) = \frac{1}{n}\sum_{j=1}^{n}|Z(\boldsymbol{s}_j)|\exp(-i\boldsymbol{s}_j'\boldsymbol{\omega}_{\boldsymbol{r}}),$$

where $\boldsymbol{\omega}_{\boldsymbol{r}} = 2\pi(\frac{r_1}{\lambda}, \ldots, \frac{r_d}{\lambda})'$. Under the assumption that the mean of $\{|Z(\boldsymbol{s})|; \boldsymbol{s} \in \mathbb{R}^2\}$ is constant over space, then by using Theorem 2.1(ia), for $\boldsymbol{r} \neq 0$ we have $\mathrm{E}[v_\lambda(\boldsymbol{r})] = 0$. Furthermore, under stationarity and sufficient mixing conditions, by using Bandyopadhyay and Lahiri [2009], Theorem 4.3, it can be shown if $1 \leq i < j \leq m$ with $\boldsymbol{r}_i \neq 0$, $\boldsymbol{r}_i \neq \boldsymbol{r}_j$ and $\boldsymbol{r}_i \neq -\boldsymbol{r}_j$ then

$$\frac{\lambda^{d/2}}{\sqrt{f_2}}\Big(\ \Re v_\lambda(\boldsymbol{r}_1), \Im v_\lambda(\boldsymbol{r}_1), \ldots, \Re v_\lambda(\boldsymbol{r}_m), \Im v_\lambda(\boldsymbol{r}_m)\ \Big)' \stackrel{\mathcal{D}}{\to} \mathcal{N}(\boldsymbol{0}, I_{2m})$$

as $\lambda^d/n \to 0$ when $\lambda \to \infty$ and $n \to \infty$, where $f_2 = \int_{\mathbb{R}^d} c_2(\boldsymbol{s})d\boldsymbol{s}$ and $\mathrm{cov}[|Z(\boldsymbol{s}_1)|, |Z(\boldsymbol{s}_2)|\big|\boldsymbol{s}_1, \boldsymbol{s}_2] = c_2(\boldsymbol{s}_1 - \boldsymbol{s}_2)$.

Based on this result, we estimate the variance f_2 using a similar variance estimation method proposed in Sections 3 and 4 of the main paper and define the maximal statistic as

$$\mathcal{V}_{\mathcal{S},\mathcal{S}'} = \frac{1}{\widehat{f}_2(\mathcal{S}')}\max_{\boldsymbol{r}\in\mathcal{S}}\Big(|\Re v_\lambda(\boldsymbol{r})|^2 + |\Im v_\lambda(\boldsymbol{r})|^2\Big),$$

where

$$\widehat{f}_2(\mathcal{S}') = \frac{1}{(2|\mathcal{S}'|-1)}\sum_{\boldsymbol{r}\in\mathcal{S}'}\left([\Re v_\lambda(\boldsymbol{r}) - \bar{J}]^2 + [\Im v_\lambda(\boldsymbol{r}) - \bar{J}]^2\right),$$

$\mathcal{S}'$ is defined in Section 3 of the main paper and $\bar{J} = \frac{1}{2|\mathcal{S}'|}\sum_{\boldsymbol{r}\in\mathcal{S}'}[\Re v_\lambda(\boldsymbol{r}) + \Im v_\lambda(\boldsymbol{r})]$. Using the same arguments as those given in Theorem 3.3 of the main paper we can show that

$$\mathcal{V}_{\mathcal{S},\mathcal{S}'} \stackrel{\mathcal{D}}{\to} \frac{\max_{1\leq i\leq|\mathcal{S}|}(|Z_{2i-1}^2 + Z_{2i}^2|)}{\frac{1}{2|\mathcal{S}'|-1}\sum_{j=2|\mathcal{S}|+1}^{2(|\mathcal{S}|+|\mathcal{S}'|)}(Z_j - \bar{Z})^2},$$

where $\{Z_j; 1 \leq j \leq 2(|\mathcal{S}| + |\mathcal{S}'|)\}$ are iid Gaussian random variables.

We now consider the power of $\mathcal{V}_{\mathcal{S},\mathcal{S}'}$ under the alternative of nonstationarity. We derive a result analogous to Theorem 4.1 in the main body.

Lemma 5.1 *Suppose* $\{Z(\boldsymbol{s}); \boldsymbol{s} \in \mathbb{R}^d\}$ *is a zero mean nonstationary spatial random process where the covariance of the absolute process satisfies the assumptions in Theorem 2.1(ii). Let* $\sigma(\boldsymbol{s}) = \mathrm{E}\big(|Z(\boldsymbol{s})|\big|\boldsymbol{s}\big)$*, then we have*

$$\mathrm{E}[v_\lambda(\boldsymbol{r})] = \frac{1}{\lambda^d}\int_{[-\lambda/2,\lambda/2]^d} \sigma(\boldsymbol{s}) \exp\left(i2\pi \boldsymbol{s}'\frac{\boldsymbol{r}}{\lambda}\right) d\boldsymbol{s},$$

and

$$\lambda^d \mathrm{cov}\left[v_\lambda(\boldsymbol{r}_1), v_\lambda(\boldsymbol{r}_2)\right] = O\left(\ell(\boldsymbol{r}_1 - \boldsymbol{r}_2)\left[1 + \frac{\lambda^d}{n}\right]\right) \text{ for all } \boldsymbol{r}_1, \boldsymbol{r}_2 \in \mathbb{Z}^d.$$

PROOF Identical to the proof of Theorem 2.1(ii) in the mainbody. □

Using the above lemma and under sufficient conditions on the rate of decay of the spatial cumulants, $\mathcal{V}_{\mathcal{S},\mathcal{S}'}$ gives power results similar to $\mathcal{T}_{\mathcal{S},\mathcal{S}'}$.

5.1 Simulations

We now consider the test $\mathcal{V}_{\mathcal{S},\mathcal{S}'}$ (to detect changes in the variance). The results are given in Table 1. Unlike $\mathcal{T}_{\mathcal{S},\mathcal{S}'}$, we observe that when n is relatively large ($n = 500, 1000$ and 2000) there is an inflation of the type I error for larger (or even moderate) values of ρ. However, the type I errors come close to the nominal for smaller values of ρ relative to λ.

For empirical power calculations, we recall that the models NS1 and NS2 in the main paper are such that the variance is constant over the spatial random field. Therefore the variance test $\mathcal{V}_{\mathcal{S},\mathcal{S}'}$ would not have any power for these models. Therefore, to assess the power of the test based on $\mathcal{V}_{\mathcal{S},\mathcal{S}'}$ we define a nonstationary spatial model where the stationary covariance is modulated by a spatially dependent variance, i.e., $Z(\boldsymbol{s}) = \sigma(\boldsymbol{u})X(\boldsymbol{s})$, where $X(\boldsymbol{s})$ is a spatially stationary process.

(NS3) We consider a piecewise stationary model where the variance structure changes abruptly. Suppose, $[-\lambda/2, \lambda/2]^2$ is partitioned into 2 rectangles each with their own stationary spatial process generated from the exponential covariance function, defined in Section 7.1.1 of the main paper, with $\rho = 1$, and variances 1 and 5, respectively.

ρ	Gaussian					log|Gaussian|					Non-uniform				
	n					n					n				
	50	100	500	1000	2000	50	100	500	1000	2000	50	100	500	1000	2000
2	0.04	0.06	0.09	0.11	0.16	0.05	0.06	0.10	0.15	0.20	0.03	0.04	0.04	0.05	0.04
	0.04	0.06	0.10	0.11	0.16	0.05	0.05	0.09	0.15	0.20	0.03	0.04	0.04	0.06	0.04
4/3	0.04	0.06	0.09	0.11	0.21	0.02	0.05	0.09	0.10	0.21	0.03	0.05	0.05	0.05	0.04
	0.03	0.06	0.09	0.11	0.19	0.02	0.05	0.08	0.10	0.21	0.03	0.04	0.05	0.05	0.04
1	0.04	0.07	0.10	0.15	0.21	0.04	0.04	0.08	0.09	0.15	0.04	0.04	0.05	0.05	0.05
	0.05	0.07	0.09	0.15	0.22	0.04	0.04	0.08	0.09	0.15	0.04	0.04	0.04	0.04	0.06
2/3	0.05	0.06	0.10	0.17	0.24	0.03	0.05	0.06	0.09	0.13	0.06	0.06	0.05	0.05	0.04
	0.05	0.05	0.10	0.16	0.23	0.03	0.05	0.06	0.09	0.13	0.06	0.06	0.05	0.05	0.04
1/3	0.05	0.05	0.09	0.12	0.17	0.04	0.06	0.07	0.07	0.09	0.04	0.05	0.06	0.05	0.05
	0.05	0.05	0.07	0.13	0.17	0.04	0.06	0.07	0.06	0.09	0.04	0.04	0.06	0.06	0.05
1/10	0.04	0.04	0.06	0.07	0.05	0.04	0.05	0.06	0.05	0.06	0.07	0.05	0.04	0.05	0.05
	0.04	0.04	0.06	0.07	0.06	0.04	0.04	0.06	0.07	0.06	0.05	0.05	0.04	0.05	0.04
1/20	0.04	0.04	0.06	0.06	0.04	0.04	0.06	0.06	0.06	0.04	0.03	0.04	0.06	0.05	0.06
	0.04	0.04	0.05	0.06	0.05	0.04	0.06	0.06	0.06	0.04	0.03	0.04	0.05	0.05	0.06

Table 1: Empirical type I errors (at 5% level) based on $\mathcal{V}_{\mathcal{S},\mathcal{S}'}$ with $\lambda = 5$ for Gaussian, $\log|\text{Gaussian}|$, and data with non-uniform locations based on stationary correlation functions. For each ρ the first row corresponds to the original data and the second row corresponds to the data with measurement error.

Model	λ	Gaussian					log|Gaussian|				
		n					n				
		50	100	500	1000	2000	50	100	500	1000	2000
NS3	5	0.05	0.07	0.17	0.36	0.57	0.04	0.07	0.08	0.12	0.14
		0.05	0.07	0.17	0.35	0.57	0.04	0.07	0.08	0.12	0.14

Table 2: Empirical powers based on $\mathcal{V}_{\mathcal{S},\mathcal{S}'}$ for the non-stationary model NS3 with uniform locations. The first row corresponds to the original data and the second row corresponds to the data with measurement error.

6 Ozone diagnostic plots

Below we give the diagnostic plots for the maximum daily ozone. We calculate $\widehat{c}_\lambda(\mathcal{S}')$ using $\widehat{A}_\lambda(g;\boldsymbol{r})$, where $\mathcal{S}' = \{\{-4,-3,-2,-1,0,1,2,3,4\} \times \{0,1,2,3,4\}\}/\{\{-4,-3,-2,-1,0\} \times \{0\}\}$, i.e., $\mathcal{S}'$ contains 4th order nearest points to the origin.

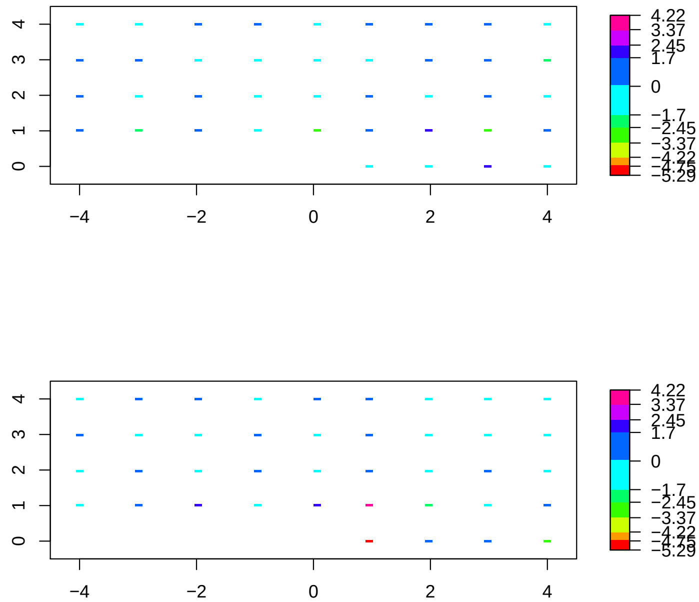


Figure 1: 6th April $\mathcal{T}_{\mathcal{S},\mathcal{S}'} = 18.79$

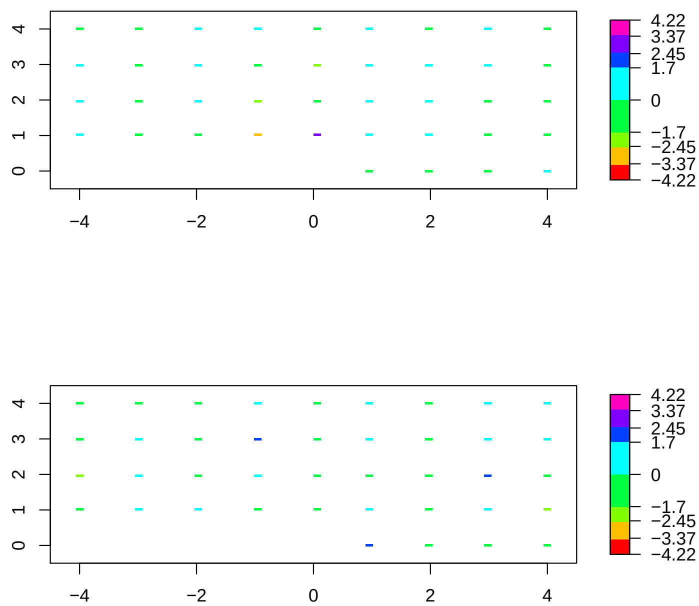


Figure 2: 15th April $\mathcal{T}_{\mathcal{S},\mathcal{S}'} = 21.9$

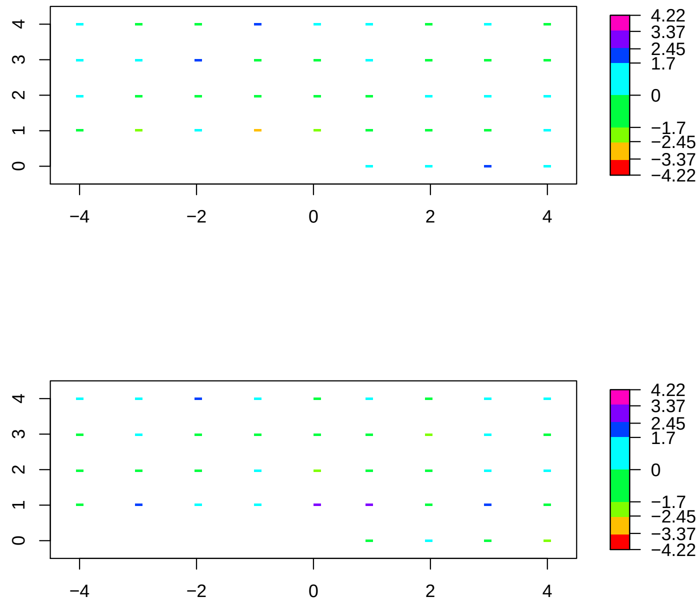


Figure 3: 20th April $\mathcal{T}_{\mathcal{S},\mathcal{S}'} = 12.11$

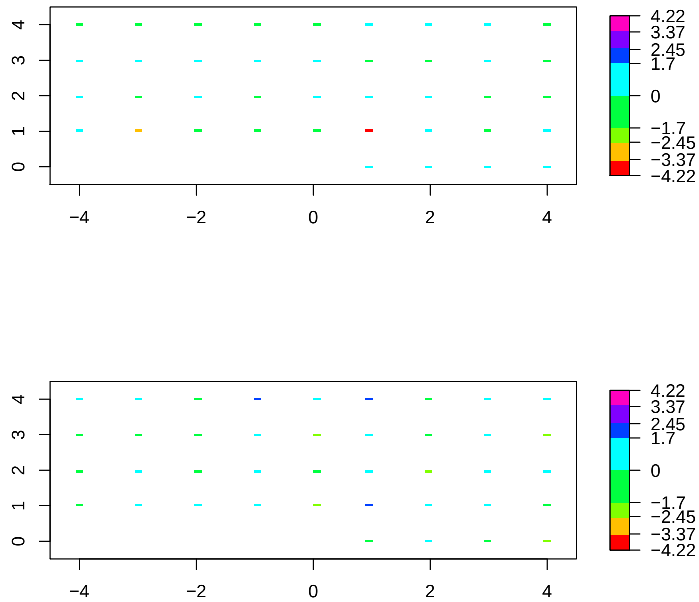


Figure 4: 24th May $\mathcal{T}_{\mathcal{S},\mathcal{S}'} = 13.1$

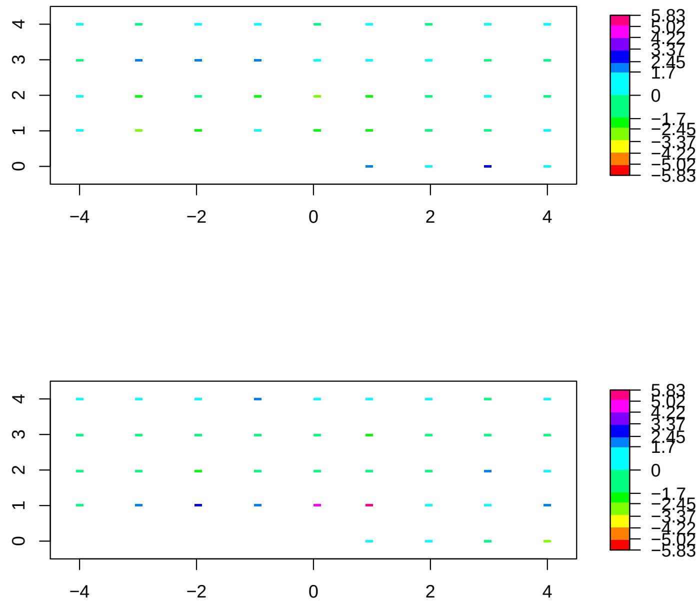


Figure 5: 26th June $\mathcal{T}_{\mathcal{S},\mathcal{S}'} = 11.39$

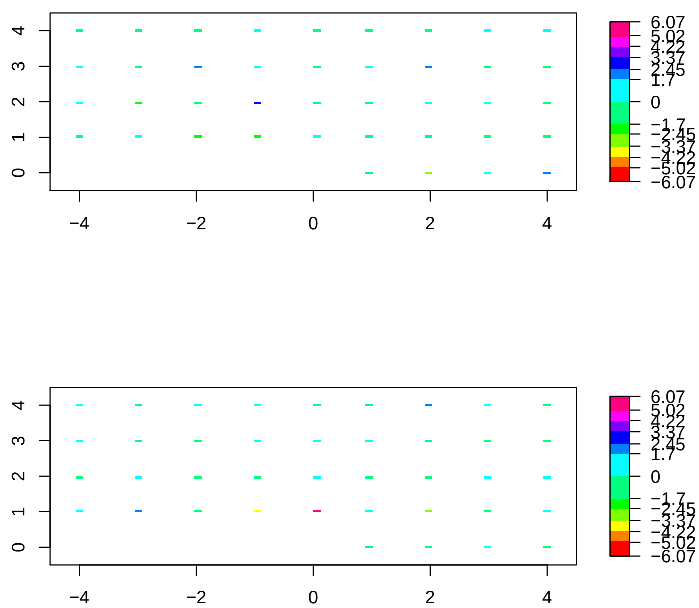


Figure 6: 21st July $\mathcal{T}_{\mathcal{S},\mathcal{S}'} = 14.7$

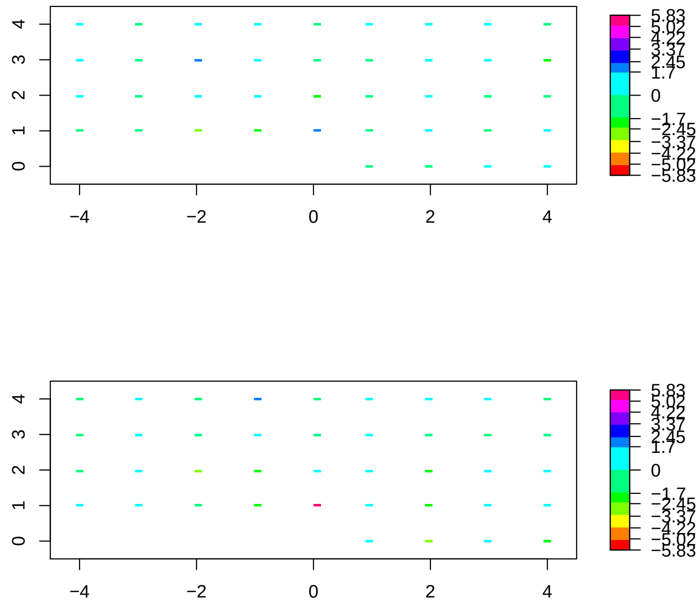


Figure 7: 22nd July $\mathcal{T}_{\mathcal{S},\mathcal{S}'} = 15.09$

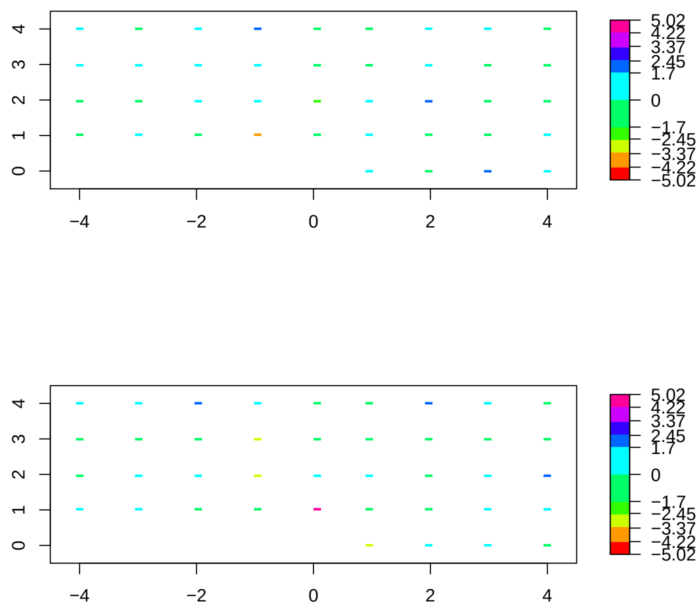


Figure 8: 3rd August $\mathcal{T}_{\mathcal{S},\mathcal{S}'} = 14.35$

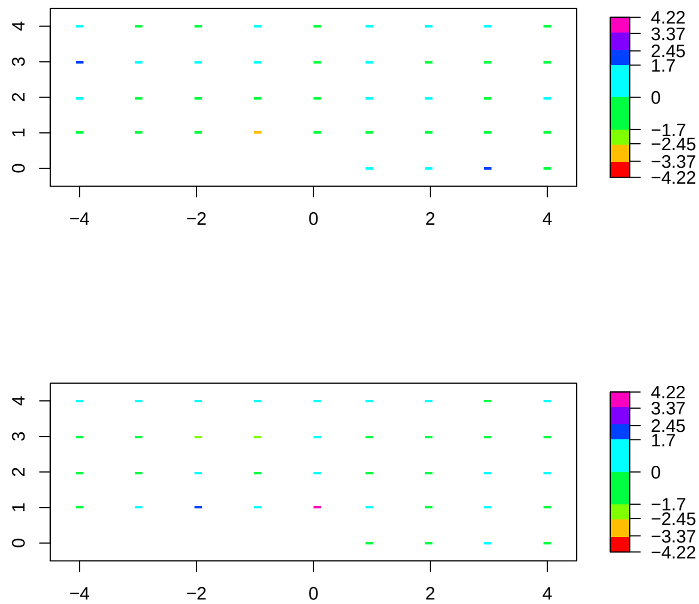


Figure 9: 4th August $\mathcal{T}_{\mathcal{S},\mathcal{S}'} = 15.21$

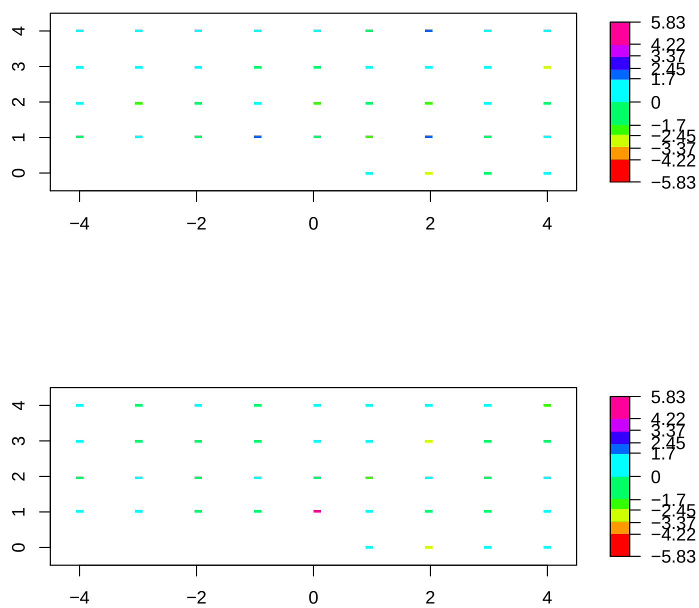


Figure 10: 6th August $\mathcal{T}_{\mathcal{S},\mathcal{S}'} = 15.71$

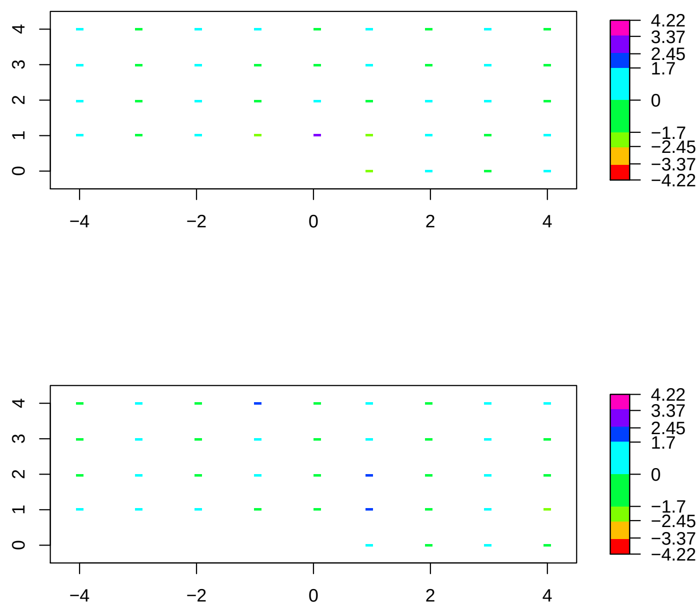


Figure 11: 24th September $\mathcal{T}_{\mathcal{S},\mathcal{S}'} = 11.57$

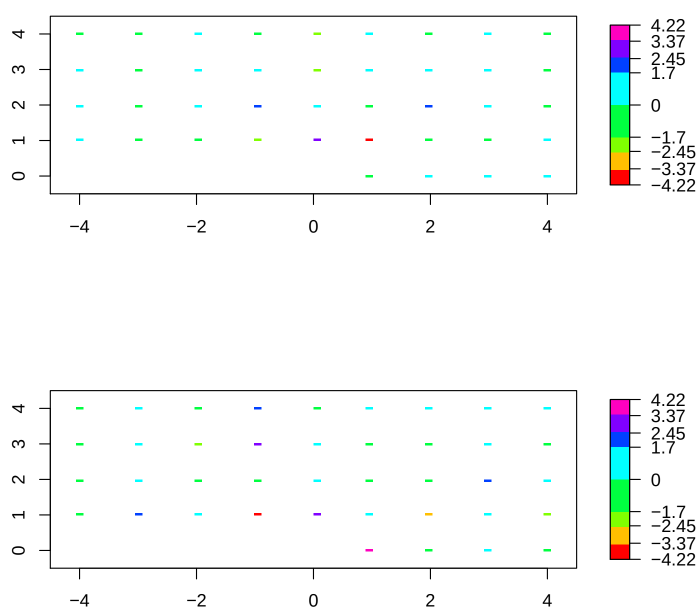


Figure 12: 25th September $\mathcal{T}_{\mathcal{S},\mathcal{S}'} = 12.17$

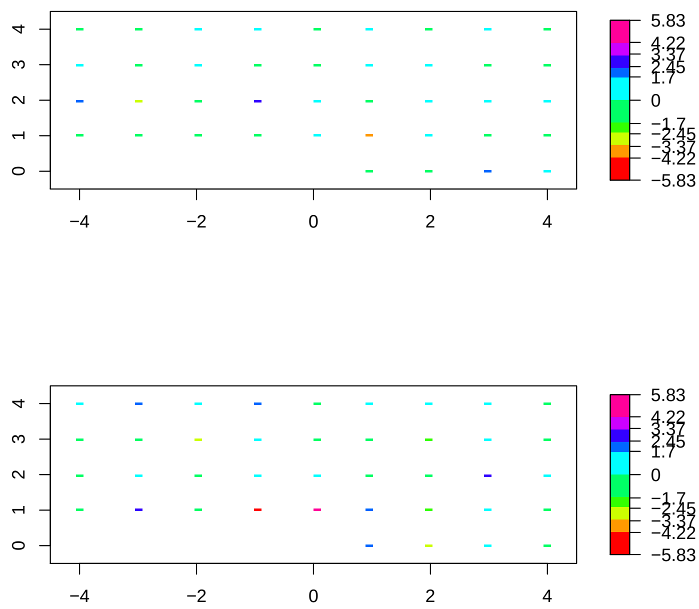


Figure 13: 26th September$\mathcal{T}_{\mathcal{S},\mathcal{S}'} = 14.28$

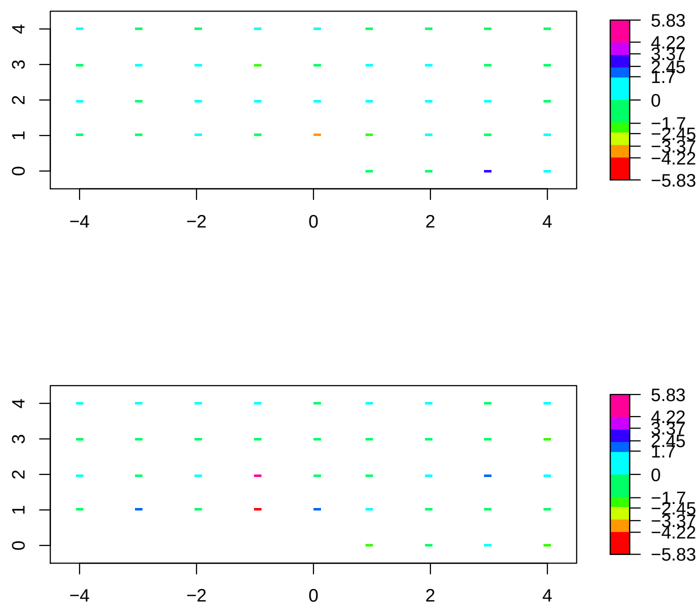


Figure 14: 27th September $\mathcal{T}_{\mathcal{S},\mathcal{S}'} = 19.76$

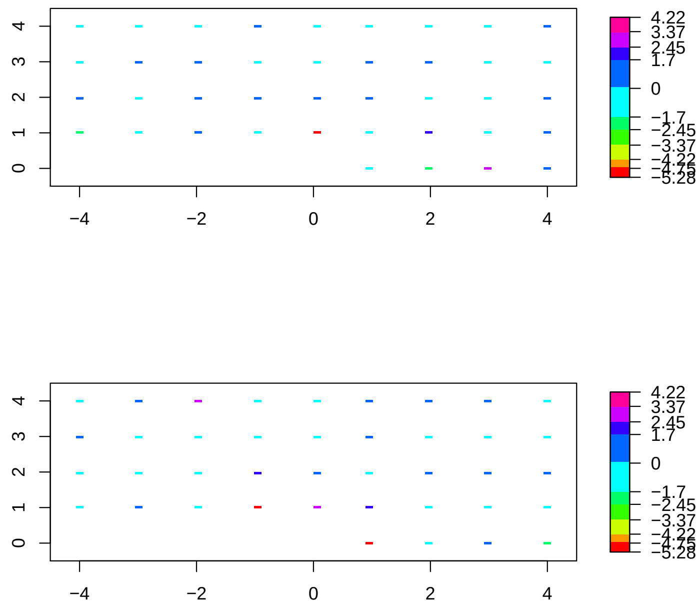


Figure 15: 28th September $\mathcal{T}_{\mathcal{S},\mathcal{S}'} = 33.43$

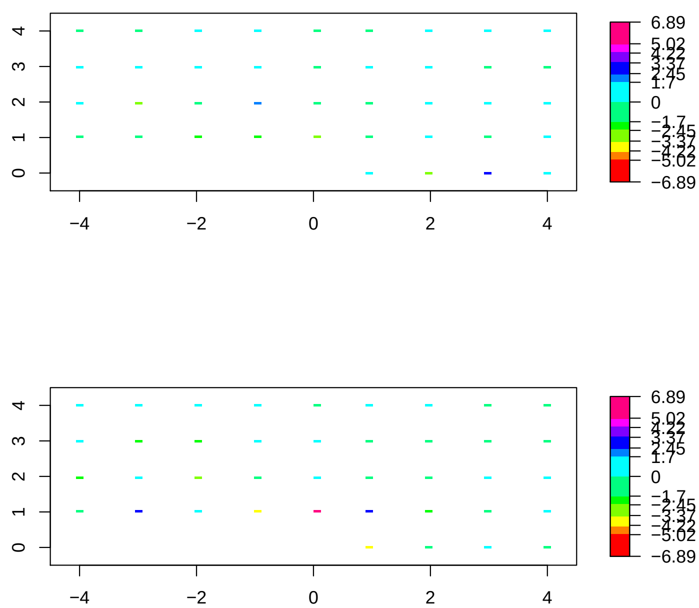
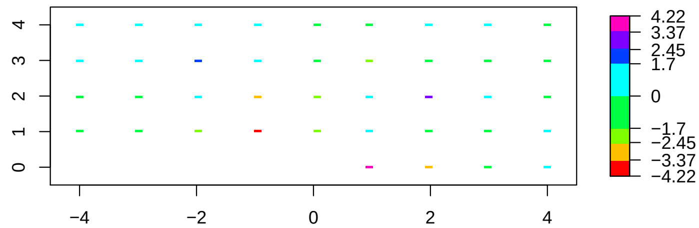


Figure 16: 29th September $\mathcal{T}_{\mathcal{S},\mathcal{S}'} = 24.82$



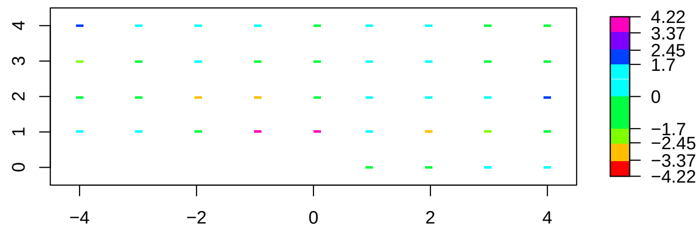


Figure 17: 30th September $\mathcal{T}_{\mathcal{S},\mathcal{S}'} = 7.89$

References

S. Bandyopadhyay and S. N. Lahiri. Asymptotic properties of discrete fourier transforms for spatial data. *Sankhya, Series A*, 71:221–259, 2009.

S. Subba Rao. Fourier based statistics for irregular spaced spatial data. *In revision*, 2015.