Statistical inference for spatial statistics defined in the Fourier domain

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Abstract

A class of Fourier based statistics for irregular spaced spatial data is introduced, examples include, the Whittle likelihood, a parametric estimator of the covariance function based on the L_2-contrast function and a simple nonparametric estimator of the spatial autocovariance which is a non-negative function. The Fourier based statistic is a quadratic form of a discrete Fourier-type transform of the spatial data. Evaluation of the statistic is computationally tractable, requiring $O(nb)$ operations, where b are the number of Fourier frequencies used in the definition of the statistic and n is the sample size. The asymptotic sampling properties of the statistic are derived using both increasing domain and fixed domain spatial asymptotics. These results are used to construct a statistic which is asymptotically pivotal.

Keywords and phrases: Fixed and increasing domain asymptotics, irregular spaced locations, quadratic forms, spectral density function, stationary spatial random fields.

1 Introduction

In recent years irregular spaced spatial data has become ubiquitous in several disciplines as varied as the geosciences to econometrics. The analysis of such data poses several challenges which do not arise in data which is sampled on a regular lattice. A major obstacle is the computational costs when dealing with large irregular sampled data sets. If spatial data are sampled on a regular lattice then algorithms such as the Fast Fourier transform can be employed to reduce the computational burden (see, for example, (Chen, Hurvich, & Lu, 2006)).

Unfortunately, such algorithms have little benefit if the spatial data are irregularly sampled. To address this issue, within the spatial domain, several authors, including, (Vecchia, 1988), (Cressie & Huang, 1999), (Stein, Chi, & Welty, 2004), have proposed estimation methods which are designed to reduce the computational burden.

In contrast to the above references, (Fuentes, 2007) and (Matsuda & Yajima, 2009) argue that working within the frequency domain often simplifies the computational burden. Both authors focus on parametric estimation using a Whittle-type likelihood. (Fuentes, 2007) assumes that the irregular spaced data can be embedded on a grid and the missing mechanism is deterministic and "locally smooth". A possible drawback of this construction, is that the local smooth assumption will not hold if the locations are extremely irregular. Therefore (Matsuda & Yajima, 2009) propose a Whittle likelihood approach to parameter estimation which takes into account the irregular nature of the locations. The focus of most Fourier domain estimators have been on the Whittle likelihood (the exception being the recent paper by (Bandyopadhyay, Lahiri, & Norman, 2015), which we discuss later). In this paper we argue that several estimators, both parametric and nonparametric can be defined within the Fourier domain. For example, within the Fourier domain, we propose a nonparametric, non-negative definite estimator of the spatial covariance. Nonparametric estimators of the spatial autocovariance are often defined using kernel smoothing methods (see (Hall, Fisher, & Hoffman, 1994)) or the empirical variogram (see (Cressie, 1993)). However, these "raw" covariance estimators may not be non-negative functions, and a second step is required, which involves taking the Fourier transform of a finite discretisation of the sample autocovariance, setting negative values to zero and inverting back, to ensure that the resulting estimator is a non-negative function. In contrast, by defining the covariance estimator within the Fourier domain the estimator is guaranteed to be a non-negative definite function. The purpose of this paper is two fold. The first is to demonstrate that several parameters can be estimated within the Fourier domain. The second is to obtain a comprehensive understanding of quadratic forms of irregular sampled spatial processes.

In order to define estimators within the Fourier domain we adopt the approach pioneered by (Matsuda & Yajima, 2009) and (Bandyopadhyay & Lahiri, 2009) who assume that the irregular locations are independent, identically distributed random variables (thus allowing the data to be extremely irregular) and define the irregular sampled discrete Fourier transform (DFT) as

$$J_n(\boldsymbol{\omega}) = \frac{\lambda^{d/2}}{n} \sum_{j=1}^{n} Z(\boldsymbol{s}_j) \exp(i \boldsymbol{s}_j' \boldsymbol{\omega}), \tag{1.1}$$

where $\boldsymbol{s}_j \in [-\lambda/2, \lambda/2]^d$ denotes the spatial locations observed in the space $[-\lambda/2, \lambda/2]^d$ and $\{Z(\boldsymbol{s}_j)\}$ denotes the spatial random field at these locations. It's worth mentioning a similar transformation on irregular sampled data goes back to (Masry, 1978), who defines the discrete Fourier transform of Poisson sampled continuous time series. Using this definition, (Matsuda & Yajima, 2009) define the Whittle likelihood by taking the weighted integral of the periodogram, $|J_n(\boldsymbol{\omega})|^2$. Of course, in practice the weighted integral needs to be approximated by a Riemann sum. Indeed in Remark 2, (Matsuda & Yajima, 2009), suggest using the frequency grid $\{\boldsymbol{\omega}_{\boldsymbol{k}} = 2\pi\boldsymbol{k}/\lambda; \boldsymbol{k} \in \mathbb{Z}^d\}$ when constructing the Whittle likelihood. No justification is given for this discretisation. However, their observation has insight. We prove that this transformation is "optimal" for most estimators defined within the Fourier domain.

Motivated by the integrated Whittle likelihood, our aim is to consider estimators with the form $\int g_\theta(\boldsymbol{\omega})|J_n(\boldsymbol{\omega})|^2 d\boldsymbol{\omega}$. Such quantities have been widely studied in time series, dating as far back as (Parzen, 1957), but has received very little attention in the spatial literature. In practice this integral cannot be evaluated, and needs to be approximated by a Riemann sum

$$Q_{a,\Omega,\lambda}(g_\theta; 0) = \frac{1}{\Omega^d} \sum_{k_1,\ldots,k_d=-a}^{a} g_\theta(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})|J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})|^2 \tag{1.2}$$

where $\{\boldsymbol{\omega}_{\Omega,\boldsymbol{k}} = 2\pi\boldsymbol{k}/\Omega, \boldsymbol{k} = (k_1,\ldots,k_d), -a \le k_i \le a\}$ is the frequency grid over which the sum is evaluated. In terms of computation, evaluation of $\{J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}}); \boldsymbol{k} = (k_1,\ldots,k_d), k_j = -a,\ldots,a\}$ requires $O(a^d n)$ operations. However, once $\{J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})\}$ has been evaluated the evaluation of $Q_{a,\Omega,\lambda}(g_\theta; 0)$ only requires $O(a^d)$ operations.

As far as we are aware there exists no results on the sampling properties of the general quadratic form defined in (1.2). To derive the asymptotic sampling properties of $Q_{a,\Omega,\lambda}(g_\theta; 0)$ we will work under two asymptotic frameworks that are commonly used in spatial statistics. Our main focus will be the increasing domain framework, introduced in (Hall & Patil, 1994) (see also (Hall et al., 1994) and used in, for example, (Lahiri, 2003), (Matsuda & Yajima, 2009), (Bandyopadhyay & Lahiri, 2009), (Bandyopadhyay et al., 2015) and (Bandyopadhyay & Subba Rao, 2017)). This is where the number of observed locations $n \to \infty$ as the size of the spatial domain $\lambda \to \infty$ (we usually assume $\lambda^d/n \to 0$). We also analyze the sampling properties of $Q_{a,\Omega,\lambda}(g_\theta; 0)$ within the fixed-domain framework (where λ is kept fixed but the number of locations, n grows) considered in (Stein, 1999), (Stein, 1994), (Zhang, 2004) and (Zhang & Zimmerman, 2005). The sampling properties of $Q_{a,\Omega,\lambda}(g_\theta; 0)$ differ according to the framework used.

We show in Sections 3 and 4 that $Q_{a,\Omega,\lambda}(g_\theta; 0)$ is a consistent estimator of the functional $I(g_\theta; \frac{a}{\Omega})$ as $\lambda \to \infty$ and $\Omega \to \infty$, where

$$I\left(g_\theta; \frac{a}{\Omega}\right) = \frac{1}{(2\pi)^d} \int_{[-2\pi a/\Omega, 2\pi a/\Omega]^d} g_\theta(\boldsymbol{\omega}) f(\boldsymbol{\omega}) d\boldsymbol{\omega}. \tag{1.3}$$

However, the choice of frequency grid $\boldsymbol{\omega}_{\Omega,\boldsymbol{k}}$ plays a vital role in the rate of convergence. In particular, we show that

$$\mathrm{E}[Q_{a,\Omega,\lambda}(g_\theta;0)] = I\left(g_\theta;\frac{a}{\Omega}\right) + O\left(\frac{\log\lambda}{\lambda}+\frac{1}{\Omega}+\frac{1}{n}\right)$$

and

$$\mathrm{var}[Q_{a,\Omega,\lambda}(g_\theta;0)] = \begin{cases} O\left(\lambda^{-d}\right) & \Omega > \lambda \\ O\left(\Omega^{-d}\right) & \Omega \leq \lambda \end{cases}. \tag{1.4}$$

Therefore, under suitable conditions on g_θ, $Q_{a,\Omega,\lambda}(g_\theta;0) \overset{\mathcal{P}}{\to} I\left(g_\theta;\infty\right)$ if $a/\Omega \to \infty$ as $a \to \infty$ $n \to \infty$, $\lambda \to \infty$ and $\Omega \to \infty$.

To understand the influence the user chosen frequency grid has on the sampling properties, we show that the asymptotic limit of $\lambda^d \mathrm{var}[Q_{a,\Omega,\lambda}(g_\theta;0)]$ will always be the same for all $\Omega \geq \lambda$ as $\lambda \to \infty$. On the other hand, using a frequency grid which is coarser than $\{2\pi\boldsymbol{k}/\lambda;\boldsymbol{k}\in\mathbb{Z}^d\}$ leads to an estimator with a larger bias and variance. Thus balancing efficiency with computational burden, in general, $\{J_n(\boldsymbol{\omega}_{\lambda,\boldsymbol{k}})\}_{\boldsymbol{k}\in\mathbb{Z}^d}$ is the optimal transformation of the spatial data into the frequency domain. As mentioned above (Bandyopadhyay et al., 2015) also use the Fourier domain for spatial inference, however their objectives are very different to those in this paper. (Bandyopadhyay et al., 2015) show that the transformations $\{J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})\}_{\boldsymbol{k}}$ are asymptotically independent if $\Omega/\lambda \to 0$ as $\lambda\to\infty$ (this corresponds to a very coarse frequency grid). Based on this property they use $Q_{a,\Omega,\lambda}(g;0)$, where Ω is such that $\Omega/\lambda \to 0$ as $\lambda \to \infty$ and $\Omega \to \infty$, to construct the empirical likelihood. The justification for their construction is that the distribution of the resulting empirical likelihood is asymptotically pivotal as $\lambda \to \infty$. The sampling properties of $Q_{a,\Omega,\lambda}(g;0)$ are not derived in (Bandyopadhyay et al., 2015). However, it is clear from (1.4), that using a frequency grid where $\Omega << \lambda$ leads to an estimator that is not optimal in the mean squared sense.

Since $Q_{a,\Omega,\lambda}(g;0)$ is optimal when using the frequency grid $\Omega = \lambda$, in Section 4 we focus on deriving the sampling properties of $Q_{a,\lambda,\lambda}(g_\theta;0)$. We consider the slightly more general statistic

$$Q_{a,\lambda,\lambda}(g_\theta;\boldsymbol{r}) = \frac{1}{\lambda^d}\sum_{k_1,\ldots,k_d=-a}^{a} g_\theta(\boldsymbol{\omega}_{\lambda,\boldsymbol{k}}) J_n(\boldsymbol{\omega}_{\lambda,\boldsymbol{k}})\overline{J_n(\boldsymbol{\omega}_{\lambda,\boldsymbol{k}+\boldsymbol{r}})}, \quad \boldsymbol{r}\in\mathbb{Z}^d \tag{1.5}$$

and show asymptotic normality of $Q_{a,\lambda,\lambda}(g_\theta;\boldsymbol{r})$ when the random field is stationary and Gaussian and obtain the second order properties of $Q_{a,\lambda,\lambda}(g_\theta;\boldsymbol{r})$ when the random field is stationary (but not necessarily Gaussian). The sampling properties of $Q_{a,\lambda,\lambda}(g_\theta;0)$ when the domain is kept fixed are considered in Section 4.4. The variance of $Q_{a,\lambda,\lambda}(g_\theta;0)$ is usually difficult to directly estimate. However, in Section 5 we show that if the locations are independent, uniformly distributed random variables, then $\{Q_{a,\lambda,\lambda}(g_\theta;\boldsymbol{r})\}$ forms a 'near uncorrelated' sequence whose variance is asymptotically equivalent to $Q_{a,\lambda,\lambda}(g_\theta;0)$. More precisely, if $Q_{a,\lambda,\lambda}(g_\theta;0)$ is real we define the the studentized statistic

$$T_{\mathcal{S}} = \frac{Q_{a,\lambda,\lambda}(g_\theta;0) - I(g_\theta;\frac{a}{\lambda})}{\sqrt{\frac{1}{|\mathcal{S}|}\sum_{\boldsymbol{r}\in\mathcal{S}}|Q_{a,\lambda,\lambda}(g_\theta;\boldsymbol{r})|^2}},$$

for some fixed set $\mathcal{S} \subset \mathbb{Z}^d/\{0\}$. We show that $T_{\mathcal{S}} \stackrel{\mathcal{D}}{\to} t_{2|\mathcal{S}|}$ as $\lambda \to \infty$, where $t_{2|\mathcal{S}|}$ denotes a t-distribution with $2|\mathcal{S}|$ degrees of freedom and $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$.

We now summarize the paper. In Section 2 we state the assumptions required in this paper and the sampling properties of the Fourier transform $\{J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})\}$. In Section 2.3, we use these properties to motivate examples of estimators which have the form $Q_{a,\Omega,\lambda}(g_\theta;0)$. In Section 3 we summarize the sampling properties of $\{Q_{a,\Omega,\lambda}(g_\theta;0)\}$. In Section 4 we focus on $Q_{a,\lambda,\lambda}(g_\theta;\boldsymbol{r})$ and these results are used to study the sampling properties of $T_{\mathcal{S}}$ in Section 5. $Q_{a,\Omega,\lambda}(g_\theta;0)$ is a quadratic form of an irregular sampled spatial process and as far as we are aware there exists very few results on the moment and sampling properties of such quadratic forms. The purpose of the supplementary material, (Subba Rao, 2017b), is to take a few steps in this direction. Many of these results build on the work of (Kawata, 1959) and may be of independent interest. A simulation study to illustrate the performance of the nonparametric non-negative definite estimator of the spatial covariance is given in Appendix J, (Subba Rao, 2017b).

2 Assumptions and Examples

2.1 Assumptions and notation

In this section we state the required assumptions and notation. This section can be skipped on first reading.

We observe the spatial random field $\{Z(\boldsymbol{s});\boldsymbol{s} \in \mathbb{R}^d\}$ at the locations $\{\boldsymbol{s}_j\}_{j=1}^n$ where $\boldsymbol{s}_j \in [-\lambda/2,\lambda/2]^d$. Throughout this paper we will use the following assumptions on the spatial random field.

Assumption 2.1 (Spatial random field) *(i)* $\{Z(\boldsymbol{s});\boldsymbol{s} \in \mathbb{R}^d\}$ *is a second order stationary random field with mean zero and covariance function* $c(\boldsymbol{s}_1-\boldsymbol{s}_2) = \mathrm{cov}(Z(\boldsymbol{s}_1),Z(\boldsymbol{s}_2)|\boldsymbol{s}_1,\boldsymbol{s}_2)$. *We define the spectral density function as* $f(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} c(\boldsymbol{s})\exp(-i\boldsymbol{s}'\boldsymbol{\omega})d\boldsymbol{s}$ *(and* $c(\boldsymbol{s}) = (2\pi)^{-d}\int_{\mathbb{R}^d} f(\boldsymbol{\omega})\exp(i\boldsymbol{s}'\boldsymbol{\omega})d\boldsymbol{\omega}$*).*

(ii) $\{Z(\boldsymbol{s});\boldsymbol{s} \in \mathbb{R}^d\}$ *is a stationary Gaussian random field.*

We require the following definitions. For some finite $0 < C < \infty$ and $\delta > 0$, let

$$\beta_{\delta}(s) = \begin{cases} C & |s| \in [-1,1] \\ C|s|^{-\delta} & |s| > 1 \end{cases}. \tag{2.1}$$

Let $\beta_{\delta}(\boldsymbol{s}) = \prod_{j=1}^{d} \beta_{\delta}(s_j)$. For sequences, we define $\xi_{\eta}(j) = C[I(j=0) + I(j \neq 0)|j|^{-\eta}]$ (for some finite constant C and $I(\cdot)$ denotes the indicator function). To minimise notation we will often use $\sum_{\boldsymbol{k}=-a}^{a}$ to denote the multiple sum $\sum_{k_1=-a}^{a} \cdots \sum_{k_d=-a}^{a}$. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ denote the ℓ_1-norm and ℓ_2-norm of a vector, respectively. Let $\Re X$ and $\Im X$ denote the real and imaginary parts of X. We make heavy use of the sinc function which is defined as

$$\operatorname{sinc}(\omega) = \frac{\sin(\omega)}{\omega} \text{ and } \operatorname{Sinc}(\boldsymbol{\omega}) = \prod_{j=1}^{d} \operatorname{sinc}(\omega_j). \tag{2.2}$$

Define the triangle kernel, $T : \mathbb{R} \to \mathbb{R}$ where $T(u) = 1 - |u|$ for $u \in [-1,1]$ and zero elsewhere and the d-dimensional triangle kernel $T(\boldsymbol{u}) = \prod_{j=1}^{d} T(u_j)$. We use the notation $\{\boldsymbol{\omega}_{\Omega,\boldsymbol{k}} = 2\pi\boldsymbol{k}/\Omega; \boldsymbol{k} \in \mathbb{Z}^d\}$ for a general frequency grid. If $\Omega > \lambda$, we say the frequency grid is "fine". Conversely, if $\Omega < \lambda$, we say the frequency grid is "coarse". Further, as mentioned in the introduction using $\lambda = \Omega$ is optimal, therefore to reduce notation we let $\{\boldsymbol{\omega}_{\boldsymbol{k}} = 2\pi\boldsymbol{k}/\lambda; \boldsymbol{k} \in \mathbb{Z}^d\}$.

We adopt the assumptions of (Hall & Patil, 1994), (Matsuda & Yajima, 2009) and (Bandyopadhyay & Lahiri, 2009) and assume that $\{\boldsymbol{s}_j\}$ are iid random variables with density $\frac{1}{\lambda^d} h(\frac{\cdot}{\lambda})$, where $h : [-1/2, -1/2]^d \to \mathbb{R}$.

Assumption 2.2 (Non-uniform sampling) *The locations $\{\boldsymbol{s}_j\}$ are independent distributed random variables on $[-\lambda/2, \lambda/2]^d$, where the density of $\{\boldsymbol{s}_j\}$ is $\frac{1}{\lambda^d} h(\frac{\cdot}{\lambda})$, and $h(\cdot)$ admits the Fourier representation*

$$h(\boldsymbol{u}) = \sum_{\boldsymbol{j} \in \mathbb{Z}^d} \gamma_{\boldsymbol{j}} \exp(i2\pi \boldsymbol{j}'\boldsymbol{u}),$$

where $\sum_{\boldsymbol{j} \in \mathbb{Z}^d} |\gamma_{\boldsymbol{j}}| < \infty$ such that $|\gamma_{\boldsymbol{j}}| \leq C \prod_{i=1}^{d} \xi_{1+\delta}(j_i)$ (for some $\delta > 0$). This assumption is satisfied if the second derivative of h is bounded on the d-dimensional torus $[-1/2, 1/2]^d$.

Remark 2.1 *If h is such that $\sup_{\boldsymbol{s} \in [-1/2,1/2]^d} |\frac{\partial^{m_1+\ldots,m_d} h(s_1,\ldots,s_d)}{\partial s_1^{m_1} \ldots \partial s_d^{m_d}}| < \infty$ ($0 \leq m_i \leq 2$) but h is not continuous on the d-dimensional torus $[-1/2, 1/2]^d$ then $|\gamma_{\boldsymbol{j}}| \leq C \prod_{i=1}^{d} \xi_1(j_i)$ and the above condition will not be satisfied. However, this assumption can be induced by tapering the observations such that $Z(\boldsymbol{s}_j)$ is replaced with $\widetilde{Z}(\boldsymbol{s}_j)$, where $\widetilde{Z}(\boldsymbol{s}_j) = t(\boldsymbol{s}_j)Z(\boldsymbol{s}_j)$, $t(\boldsymbol{s}) = \prod_{i=1}^{d} t(s_i)$ and t is a weight function which has a bounded second derivative, $t(-1/2) = t(1/2) = 0$ and $t'(1/2) = t'(-1/2) = 0$. By using $\widetilde{Z}(\boldsymbol{s}_j)$ instead of $Z(\boldsymbol{s}_j)$, in all the derivations below we replace the density $h(\boldsymbol{s})$ with $t(\boldsymbol{s})h(\boldsymbol{s})$. This means the results now rely on the*

Fourier coefficients of $t(\boldsymbol{s})h(\boldsymbol{s})$, which decay at the rate $|\int_{[-1/2,1/2]^d} t(\boldsymbol{s})h(\boldsymbol{s})\exp(i2\pi \boldsymbol{j}'\boldsymbol{s})d\boldsymbol{s}| \leq C\prod_{i=1}^{d}\xi_2(j_i)$, and thus the above condition is satisfied. Note that (Matsuda & Yajima, 2009), Definition 2, uses a similar data-tapering scheme to induce a similar condition.

The case that the locations follow a uniform distribution is an example of a distribution which satisfies Assumption 2.2. It gives rise to several elegant simplifications. Thus we state the uniform case as a separate assumption.

Assumption 2.3 (Uniform sampling) *The locations $\{\boldsymbol{s}_j\}$ are independent uniformly distributed random variables on $[-\lambda/2, \lambda/2]^d$.*

Many of the results in this paper use that the locations follow a random design. This helps in understand the sampling properties of these complex estimators. However, it can "mask" the approximation errors when replacing sums by integrals and the role that the sample size n plays in these approximations. To get some idea of these approximations when the domain λ is kept fixed but $n \to \infty$ we will, on occassion, treat the locations as deterministic and make the following assumption.

Assumption 2.4 (Near lattice locations for $d = 1$) *Let $\{s_{n,j}; j = 1, \ldots, n\}$ denote the locations. The number of locations $n \to \infty$ in such a way that*

$$\begin{aligned} \sum_{j=1}^{n-1}\left|\frac{\lambda}{n} - \left(s_{n,(j+1)} - s_{n,(j)}\right)\right| &= O\left(\frac{\lambda}{n}\right), \\ \sum_{j=1}^{n-1}\left(s_{n,(j+1)} - s_{n,(j)}\right)^2 &= O\left(\frac{\lambda}{n}\right) \end{aligned}$$

where $\{s_{n,(j)}\}_j$ denotes the order statistics corresponding $\{s_{n,j}\}_j$.

The integrated periodogram $Q_{a,\Omega,\lambda}(g; 0)$ resembles the integrated periodogram estimator commonly used in time series (see for example, (Walker, 1964), (Hannan, 1971), (Dunsmuir, 1979), (Dahlhaus & Janas, 1996), (Can, Mikosch, & Samorodnitsky, 2010) and (Niebuhr & Kreiss, 2014)). However, there are some fundamental differences, between time series estimators and $Q_{a,\Omega,\lambda}(g; 0)$ which makes the analysis very different. Unlike regularly spaced or near regularly spaced data, 'truely' irregular sampling means that the DFT can estimate high frequencies, without the curse of aliasing (a phenomena which was noticed as early as (Shapiro & Silverman, 1960) and (Beutler, 1970)). In this case, if the function g_θ, in the definition of $Q_{a,\Omega,\lambda}(g_\theta; 0)$ is bounded, there's no need for the frequency grid to be bounded, and a can be magnitudes larger than λ. Below we state assumptions on the function g_θ and the frequency grid.

Assumption 2.5 (Assumptions on $g_\theta(\cdot)$ and the size of frequency grid) *Suppose*

$$Q_{a,\Omega,\lambda}(g_\theta;\boldsymbol{r}) = \frac{1}{\Omega^d}\sum_{\boldsymbol{k}=-a}^{a} g_\theta(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})\overline{J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}+\boldsymbol{r}})}$$

(i) If g_θ is not a bounded function over $\mathbb{R}^d$ but $\sup_{\boldsymbol{\omega}\in[-C,C]^d}|g_\theta(\boldsymbol{\omega})| < \infty$, then we must restrict the frequency grid $\{\boldsymbol{\omega}_{\Omega,\boldsymbol{k}}; -a \leq k_1,\ldots,k_d \leq a\}$ to lie in $[-C,C]^d$ (thus $a = C\Omega$). Further we assume for all $1 \leq j \leq d$, $\sup_{\boldsymbol{\omega}\in[-C,C]^d}|\frac{\partial g_\theta(\boldsymbol{\omega})}{\partial\omega_j}| < \infty$.

(ii) If $\sup_{\boldsymbol{\omega}\in\mathbb{R}^d}|g_\theta(\boldsymbol{\omega})| < \infty$, then the frequency grid can be unbounded (in the sense that $a/\Omega \to \infty$ as a and $\Omega \to \infty$). Further we assume for all $1 \leq j \leq d$, $\sup_{\boldsymbol{\omega}\in\mathbb{R}^d}|\frac{\partial g_\theta(\boldsymbol{\omega})}{\partial\omega_j}| < \infty$.

The same assumptions apply to the "bias" corrected version of $Q_{a,\Omega,\lambda}(g;0)$ which is defined in Section 3.

Assumption 2.6 (Conditions on the spatial process) *(a) $|c(\boldsymbol{s})| \leq \beta_{2+\delta}(\boldsymbol{s})$.*

Required for the bounded frequency grid, to obtain the covariance of $\{J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})\}_{\boldsymbol{k}}$.

(b) For some $\delta > 0$, $f(\boldsymbol{\omega}) \leq \beta_{1+\delta}(\boldsymbol{\omega})$.

Required for the unbounded frequency grid - using this assumption instead of (a) in the case of an bounded frequency grid leads to slightly larger errors bounds in the derivation of the mean and variance of $Q_{a,\Omega,\lambda}(g;\boldsymbol{r})$. This assumption is also used to obtain the CLT result for both the bounded and unbounded frequency grids.

(c) For all $1 \leq j \leq d$ and some $\delta > 0$, the partial derivatives satisfy $|\frac{\partial f(\boldsymbol{\omega})}{\partial\omega_j}| \leq \beta_{1+\delta}(\boldsymbol{\omega})$.

We use this condition to approximate sums with integral for both the bounded and unbounded frequency grids. It is also used to make a series of approximations to derive the limiting variance of $Q_{a,\Omega,\lambda}(g;\boldsymbol{r})$ in the case that the frequency grid is unbounded.

(d) For some $\delta > 0$, $|\frac{\partial^d f(\boldsymbol{\omega})}{\partial\omega_1,\ldots,\partial\omega_d}| \leq \beta_{1+\delta}(\boldsymbol{\omega})$.

Required only in the proof of Theorem 4.1(ii)(b).

(e) For some $\delta > 0$, $|f(\boldsymbol{\omega})| \leq \beta_{2+\delta}(\boldsymbol{\omega})$.

Required only for the fixed domain asymptotics.

Remark 2.2 *Assumption 2.6(a) is satisfied by a wide range of covariance functions. Examples include:*

(i) The Wendland covariance, since its covariance is bounded and has a compact support.

(ii) The Matern covariance, which for $\nu > 0$ is defined as $c_\nu(\|\boldsymbol{s}\|_2) = \|\boldsymbol{s}\|_2^\nu K_\nu(\|\boldsymbol{s}\|_2)$ (K_ν is the modified Bessel function of the second kind); see (Stein, 1999). To see why, we note that if $\nu > 0$ then $c_\nu(\boldsymbol{s})$ is a bounded function. Furthermore, for large $\|\boldsymbol{s}\|_2$, $c_\nu(\|\boldsymbol{s}\|_2) \sim C_\nu \|\boldsymbol{s}\|_2^{\nu-0.5} \exp(-\|\boldsymbol{s}\|_2)$ as $\|\boldsymbol{s}\|_2 \to \infty$ (where C_ν is a finite constant). Thus by using the inequality

$$d^{-1/2}(|s_1| + |s_2| + \ldots + |s_d|) \leq \sqrt{s_1^2 + s_2^2 + \ldots + s_d^2} \leq (|s_1| + |s_2| + \ldots + |s_d|)$$

we can show $|c_\nu(\boldsymbol{s})| \leq \beta_{2+\delta}(\boldsymbol{s})$ for any $\delta > 0$.

Remark 2.3 *Assumption 2.6(b,c,d) appears quite technical, but it is satisfied by a wide range of spatial covariance functions. For example, the spectral density of the Matern covariance defined in Remark 2.2 is $f_\nu(\boldsymbol{\omega}) = \frac{2^{\nu-1}\Gamma(\nu+\frac{d}{2})}{\pi^{d/2}(1+\|\boldsymbol{\omega}\|_2^2)^{(\nu+d/2)}}$ (see (Stein, 1999), page 49). It is straightforward to show that this spectral density satisfies Assumption 2.6(b,c,d), noting that the δ used to define $\beta_{1+\delta}$ will vary with ν, dimension d and order of derivative.*

If the spatial random field is non-Gaussian we require the following assumptions on the higher order cumulants.

Assumption 2.7 (Non-Gaussian random fields) *$\{Z(\boldsymbol{s}); \boldsymbol{s} \in \mathbb{R}^d\}$ is a fourth order stationary spatial random field, in the sense that $\mathrm{E}[Z(\boldsymbol{s})] = 0$, $\mathrm{cov}[Z(\boldsymbol{s}_1), Z(\boldsymbol{s}_2)] = c(\boldsymbol{s}_1 - \boldsymbol{s}_2)$, $\mathrm{cum}[Z(\boldsymbol{s}_1), Z(\boldsymbol{s}_2), Z(\boldsymbol{s}_3)] = \kappa_2(\boldsymbol{s}_1 - \boldsymbol{s}_2, \boldsymbol{s}_1 - \boldsymbol{s}_3)$ and $\mathrm{cum}[Z(\boldsymbol{s}_1), Z(\boldsymbol{s}_2), Z(\boldsymbol{s}_3), Z(\boldsymbol{s}_4)] = \kappa_4(\boldsymbol{s}_1 - \boldsymbol{s}_2, \boldsymbol{s}_1 - \boldsymbol{s}_3, \boldsymbol{s}_1 - \boldsymbol{s}_4)$, for some functions $\kappa_3(\cdot)$ and $\kappa_4(\cdot)$ and all $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_4 \in \mathbb{R}^d$. We define the fourth order spectral density as $f_4(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3) = \int_{\mathbb{R}^{3d}} \kappa_4(\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3) \exp(-i \sum_{j=1}^{3} \boldsymbol{s}_j' \boldsymbol{\omega}_j) d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2 d\boldsymbol{\omega}_3$. We assume that for some $\delta > 0$ the spatial tri-spectral density function is such that $|f_4(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3)| \leq \beta_{1+\delta}(\boldsymbol{\omega}_1)\beta_{1+\delta}(\boldsymbol{\omega}_2)\beta_{1+\delta}(\boldsymbol{\omega}_3)$ and $|\frac{\partial f_4(\omega_1,\ldots,\omega_{3d})}{\partial \omega_j}| \leq \beta_{1+\delta}(\boldsymbol{\omega}_1)\beta_{1+\delta}(\boldsymbol{\omega}_2)\beta_{1+\delta}(\boldsymbol{\omega}_3)$.*

2.2 Properties of Fourier transforms

In this section we briefly summarize some of the characteristics of the Fourier transforms $J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})$. These results will be used in the construction of several estimators.

Theorem 2.1 (Increasing domain asymptotics) *Let us suppose that $\{Z(\boldsymbol{s}); \boldsymbol{s} \in \mathbb{R}^d\}$ is a stationary spatial random field whose covariance function (defined in Assumption 2.1(i)) satisfies Assumption 2.6(a) for some $\delta > 0$. Furthermore, the locations $\{\boldsymbol{s}_j\}$ satisfy Assumption 2.2. Then we have*

$$\mathrm{cov}\left[J_n(\boldsymbol{\omega}_{\boldsymbol{k}_1}), J_n(\boldsymbol{\omega}_{\boldsymbol{k}_2})\right] = \langle \gamma, \gamma_{(\boldsymbol{k}_2-\boldsymbol{k}_1)} \rangle f\left(\boldsymbol{\omega}_{\boldsymbol{k}_1}\right) + \frac{c(0)\gamma_{\boldsymbol{k}_2-\boldsymbol{k}_1}\lambda^d}{n} + O\left(\frac{1}{\lambda}\right), \qquad (2.3)$$

where the bounds are uniform in $\boldsymbol{k}_1, \boldsymbol{k}_2 \in \mathbb{Z}^d$ and $\langle \gamma, \gamma_{\boldsymbol{r}} \rangle = \sum_{\boldsymbol{j} \in \mathbb{Z}^d} \gamma_{\boldsymbol{j}} \gamma_{\boldsymbol{r}-\boldsymbol{j}}$.

Further, suppose Assumption 2.6(c) holds and that $|\gamma_{\boldsymbol{j}}| \leq \prod_{i=1}^{d} \xi_{2+\delta}(j_i)$ *for some* $\delta > 0$. *If* $\Omega > \lambda$ *(fine frequency grid is used), then*

$$\begin{aligned}\mathrm{cov}\left[J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}_1}), J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}_2})\right] &= f\left(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}_1}\right) \sum_{\boldsymbol{j}_1,\boldsymbol{j}_2 \in \mathbb{Z}^d} \gamma_{\boldsymbol{j}_1}\gamma_{\boldsymbol{j}_2} \mathrm{Sinc}\left(\pi\left[(\boldsymbol{j}_1+\boldsymbol{j}_2) - \frac{\lambda}{\Omega}(\boldsymbol{k}_1-\boldsymbol{k}_2)\right]\right) \\ &\quad + O\left(\frac{\lambda^d}{n} + \frac{\log\lambda}{\lambda}\right). \end{aligned} \tag{2.4}$$

PROOF See Appendix A, (Subba Rao, 2017b). □

Comparing (2.3) with (2.4) we see that if $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are such that $\frac{\lambda}{2}\max|\boldsymbol{\omega}_{\Omega,\boldsymbol{k}_1} - \boldsymbol{\omega}_{\Omega,\boldsymbol{k}_2}| < 1$ then there is a high amount of correlation between the Fourier transforms. On the other hand if $\lambda \max|\boldsymbol{\omega}_{\Omega,\boldsymbol{k}_1} - \boldsymbol{\omega}_{\Omega,\boldsymbol{k}_2}| \to \infty$ as $\lambda \to \infty$, the correlation declines. This means if the frequency grid is very coarse, $\Omega << \lambda$ (as considered in (Bandyopadhyay & Lahiri, 2009) and (Bandyopadhyay et al., 2015)) the DFTs are almost uncorrelated. On the other hand, if the frequency grid is very fine, $\Omega >> \lambda$ (see (2.4)) frequencies which are close to each other are highly correlated. These observations suggest that estimators based on $J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})$ don't gain in efficiency when $\Omega > \lambda$ but there will be a loss in efficiency when $\Omega < \lambda$. We show this heuristic to be true in Section 3.

The results in the above theorem give the limit within the increasing domain framework. In Theorem G.1, (Subba Rao, 2017b), we obtain the properties of $J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})$ within the fixed domain framework, where λ is kept fixed but $n \to \infty$. The expressions are long, but we summarize the most relevant parts in the remark below.

Remark 2.4 (Fixed domain asymptotics) *(i) Let*

$$A_\lambda\left(\frac{\boldsymbol{k}}{\Omega}\right) = \int_{[-\lambda,\lambda]^d} T\left(\frac{\boldsymbol{u}}{\lambda}\right) c(\boldsymbol{u}) \exp\left(\frac{2i\pi\boldsymbol{k}'\boldsymbol{u}}{\Omega}\right) d\boldsymbol{u}, \tag{2.5}$$

where $T(\cdot)$ *is the d-dimensional triangle kernel. Using Theorem G.1(iii), (Subba Rao, 2017b), under Assumption 2.3 (locations are uniformly distributed) we have* $\mathrm{var}\left[J_n\left(\frac{2\pi\boldsymbol{k}}{\Omega}\right)\right] = A_\lambda(\frac{\boldsymbol{k}}{\Omega}) + \frac{\lambda^d c(0)}{n}$.

Therefore if the sampling frequency, Ω, *is chosen such that* $\Omega \geq 2\lambda$, *then* $\frac{1}{\Omega^d}A_\lambda(\frac{\boldsymbol{k}}{\Omega})$ *are the Fourier coefficients of* $T\left(\frac{\boldsymbol{u}}{\lambda}\right)c(\boldsymbol{u})$ *defined on the domain* $[-\Omega/2,\Omega/2]^d$. *In Section 2.3 we show that this fixed domain approximation can, in some cases, be used to obtain unbiased estimators.*

(ii) If the locations are not uniformly distributed, then by keeping λ *fixed and using Theorem G.1(i) we see that* $\mathrm{var}\left[J_n\left(\frac{2\pi\boldsymbol{k}}{\Omega}\right)\right]$ *is not a separable function of the spatial spectral density and the Fourier coefficients of the random design. Whereas the approximation*

of $\text{var}\left[J_n\left(\frac{2\pi\boldsymbol{k}}{\Omega}\right)\right]$ *as* $\lambda \to \infty$ *given in Theorem 2.1 is a separable function of the spatial spectral density and spatial design. Using the separable approximation as the basis of an estimation scheme is simpler than using the exact, but non-separable formula.*

Remark 2.5 *We observe from Theorems 2.1 that in the increasing domain framework* $|J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})|^2$ *is an estimator of the spectral density function,* $f(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})$ *whereas within the fixed domain framework* $|J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})|^2$ *is an estimator of the Fourier coefficient* $A_\lambda(\boldsymbol{k}/\Omega)$*. However, if* $\int_{\mathbb{R}^d}|f(\boldsymbol{\omega})|d\boldsymbol{\omega} < \infty$ *and the ratio* $\boldsymbol{k}/\Omega$ *is kept fixed then* $A_\lambda(\frac{\boldsymbol{k}}{\Omega}) \to f(\omega_{\Omega,\boldsymbol{k}})$ *as* $\lambda \to \infty$*.*

2.3 Examples of estimators defined within the Fourier domain

Many parameters or quantities of interest can be written as a linear functional involving the spectral density function f. In Theorem 2.1 and Remark 2.5 we showed that if λ is large and the design of locations uniform then $\text{E}[J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})] \approx f(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})$ (if the design is not uniform then there will be an additional multiplicative constant). Motivated by this observation, in this section we consider estimators (or criterions) which take the form (1.5). If the locations follow a uniform distribution then for some of the examples below it is possible to reduce the (fixed domain) bias in the estimator.

2.3.1 The Whittle likelihood

Suppose the stationary spatial process $\{Z(\boldsymbol{s}); \boldsymbol{s} \in \mathbb{R}^d\}$ has spectral density $f_{\theta_0}(\boldsymbol{\omega})$ (and corresponding covariance $c_{\theta_0}(\boldsymbol{s})$) where θ_0 is unknown but belongs to the compact parameter space Θ. (Matsuda & Yajima, 2009) propose using the integrated Whittle likelihood to estimate θ_0. More precisely, they define the Whittle likelihood as

$$\mathcal{L}_{I,n}(\theta,\eta^2) = \int_\Omega \left(\log\left[f_\theta(\boldsymbol{\omega}) + \eta^2\right] + \frac{|J_n(\boldsymbol{\omega})|^2}{[f_\theta(\boldsymbol{\omega}) + \eta^2]}\right) d\boldsymbol{\omega},$$

and use $(\widehat{\theta},\widehat{\eta}) \in \arg\min_{\theta,\eta}\mathcal{L}_n(\theta,\eta)$ as an estimator of θ and η (where η is an estimator of the "ridge effect"). Of course, this integral cannot be evaluated in practice and a Riemann sum approximation is necessary. Using $\{\boldsymbol{\omega}_{\boldsymbol{k}} = 2\pi\boldsymbol{k}/\lambda; \boldsymbol{k} = (k_1,\ldots,k_d), -C\lambda \leq k_j \leq C\lambda\}$ we approximate the integral with the sum

$$\mathcal{L}_{S,n}(\theta,\eta^2) = \frac{1}{\lambda^d}\sum_{\boldsymbol{k}=-a}^{a}\left(\log\left[f_\theta(\boldsymbol{\omega}_{\boldsymbol{k}}) + \eta^2\right] + \frac{|J_n(\boldsymbol{\omega}_{\boldsymbol{k}})|^2}{[f_\theta(\boldsymbol{\omega}_{\boldsymbol{k}}) + \eta^2]}\right).$$

A heuristic motivation for the above likelihood is that in the case the locations are uniformly distributed then $\{J_n(\boldsymbol{\omega}_{\boldsymbol{k}})\}$ are near uncorrelated random variables with asymptotic variance $f_{\theta_0}(\boldsymbol{\omega}_{\boldsymbol{k}})+\eta_0^2$. If $J_n(\boldsymbol{\omega}_{\boldsymbol{k}})$ were Gaussian, uncorrelated random variables with variance $f_{\theta_0}(\boldsymbol{\omega}_{\boldsymbol{k}})+\eta_0^2$ then $\mathcal{L}_{S,n}(\theta,\eta^2)$ would be the true likelihood. The choice of $a = C\lambda$ is necessary (where

C does not depend on λ), since $|f_\theta(\boldsymbol{\omega})| \to 0$ as $\|\boldsymbol{\omega}\| \to \infty$ (for any norm $\|\cdot\|$), thus the discretized Whittle likelihood is only well defined over a bounded frequency grid. The choice of C is tied to how fast the tails in the parametric class of spectral density functions $\{f_\theta; \theta \in \Theta\}$ decay to zero.

If either Assumption 2.3 or 2.4 is satisfied, then we observe from Remark 2.4 that $A_\lambda(\frac{\boldsymbol{k}}{\lambda}; \theta_0)$ is a better approximation of $\mathrm{var}[J_n(\boldsymbol{\omega})]$ than $f_{\theta_0}(\boldsymbol{\omega_k})$ (where $A_\lambda(\cdot)$ is defined in (2.5)). Therefore if Assumption 2.3 or 2.4 is satisfied, a better finite sample approximation can be obtained by using $\widehat{\theta} = \arg\min \mathcal{L}_{S,n}(\theta)$ as an estimator of θ, where

$$\mathcal{L}_{S,n}(\theta, \eta^2) = \frac{1}{\lambda^d} \sum_{\boldsymbol{k}=-a}^{a} \left(\log \left[A_\lambda \left(\frac{\boldsymbol{k}}{\lambda}; \theta \right) + \eta^2 \right] + \frac{|J_n(\boldsymbol{\omega_k})|^2}{\left[A_\lambda \left(\frac{\boldsymbol{k}}{\lambda}; \theta \right) + \eta^2 \right]} \right) \tag{2.6}$$

and $A_\lambda\left(\frac{\boldsymbol{k}}{\lambda}; \theta\right) = \int_{[-\lambda,\lambda]^d} T\left(\frac{\boldsymbol{u}}{\lambda}\right) c_\theta(\boldsymbol{u}) \exp\left(\frac{2i\pi \boldsymbol{k}'\boldsymbol{u}}{\lambda}\right) d\boldsymbol{u}$.

2.3.2 The spectral density estimator

We recall from Theorem 2.1 that $\mathrm{var}[J_n(\boldsymbol{\omega_k})] \approx \langle \gamma, \gamma_0 \rangle f(\boldsymbol{\omega_k})$. Since $f(\cdot)$ is locally constant in a neighbourhood of $\boldsymbol{\omega}$ and motivated by spectral methods in time series we use $\widehat{f}_{\lambda,n}(\boldsymbol{\omega})$ as a nonparametric estimator of f (or a constant multiple of it), where

$$\widehat{f}_{\lambda,n}(\boldsymbol{\omega}) = \sum_{\boldsymbol{k}=-\lambda/2}^{\lambda/2} W_b(\boldsymbol{\omega} - \boldsymbol{\omega_k}) |J_n(\boldsymbol{\omega_k})|^2 = \frac{1}{b^d} Q_{a,\lambda,\lambda}(W_b, 0),$$

$W_b(\boldsymbol{\omega}) = b^{-d} \prod_{j=1}^d W(\frac{\omega_j}{b})$ and $W : [-1/2, -1/2] \to \mathbb{R}$ is a spectral window. In this case we set the number of frequencies $a = \lambda/2$, and Assumption 2.5(i) is satisfied.

2.3.3 A nonparametric non-negative definite estimator of the spatial covariance

In this section we propose a nonparametric estimator of the covariance. The estimator is based on the representation

$$c(\boldsymbol{u}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(\boldsymbol{\omega}) \exp(i\boldsymbol{\omega}'\boldsymbol{u}) d\boldsymbol{\omega}.$$

Since the expectation of $|J_n(\boldsymbol{\omega_k})|^2$ is approximately $f(\boldsymbol{\omega_k})$, to estimate the spatial covariance we propose approximating the above integral with a sum and the spectral density with the absolute square of the Fourier transform. However, using the frequency grid $\boldsymbol{\omega_k} = \frac{2\pi \boldsymbol{k}}{\lambda}$ is problematic outside the region $[-\lambda/2, \lambda/2]^d$. Instead, we propose using a finer grid to estimate the covariance, namely

$$\widetilde{c}_{\Omega,n}(\boldsymbol{u}) = \frac{1}{\Omega^d} \sum_{\boldsymbol{k}=-a}^{a} \left| J_n\left(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}}\right) \right|^2 \exp(i\boldsymbol{u}'\boldsymbol{\omega}_{\Omega,\boldsymbol{k}}) \qquad \boldsymbol{u} \in \left[-\frac{\Omega}{2}, \frac{\Omega}{2} \right]^d$$

where $\Omega \geq 2\lambda$. In this case, $a = a(\Omega)$ can be chosen such that $a/\Omega \to \infty$ as $a \to \infty$ and $\Omega \to \infty$ (thus Assumption 2.5(ii) is satisfied).

A disadvantage with the above "raw estimator" of the covariance is that there is no guarantee that it yields a non-negative definite spatial auto-covariance function. However, this can easily be remedied by multiplication of $\widetilde{c}_{\Omega,n}(\boldsymbol{u})$ with the triangle kernel. More precisely, we define the estimator

$$\widehat{c}_{\Omega,n}(\boldsymbol{u}) \quad = \quad T\left(\frac{\boldsymbol{u}}{\widehat{\Omega}}\right)\widetilde{c}_{\Omega,n}(\boldsymbol{u})$$

where $T(\boldsymbol{u}) = \prod_{j=1}^{d} T(u_j)$ and $\widehat{\Omega} \leq \Omega$. This covariance estimator has the advantage that it is zero outside the region $[-\widehat{\Omega}, \widehat{\Omega}]^d$. Moreover, $\widehat{c}_{\Omega,n}(\boldsymbol{u})$ is a non-negative definite sequence. To show this result, we use that the Fourier transform of the triangle kernel, $T(u)$ is $\text{sinc}^2(\frac{\omega}{2})$. Thus the Fourier transform of $\widehat{c}_{\Omega,n}(\boldsymbol{u})$ is

$$\widehat{f}_{\Omega}(\boldsymbol{\omega}) \quad = \quad \int_{[-\widehat{\Omega},\widehat{\Omega}]^d} \widehat{c}_{\Omega,n}(\boldsymbol{u}) \exp(-i\boldsymbol{\omega}'\boldsymbol{u}) d\boldsymbol{u} = \frac{\widehat{\Omega}^d}{\Omega^d} \sum_{\boldsymbol{k}=-a}^{a} |J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})|^2 \text{Sinc}^2 \left[\frac{\widehat{\Omega}}{2}(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}} - \boldsymbol{\omega})\right].$$

Clearly, $\widehat{f}_{\Omega}(\boldsymbol{\omega}) \geq 0$, therefore, the estimator $\{\widehat{c}_{\Omega,n}(\boldsymbol{u})\}$ is a non-negative definite function and thus a valid covariance function.

In Appendix J, (Subba Rao, 2017b), we illustrate the performance of the nonparametric non-negative definite estimator of the spatial covariance with some simulations.

2.3.4 A nonlinear least squares estimator of a parametric covariance function

We recall that the Whittle likelihood can only be defined on a bounded frequency grid. This can be an issue if the observed locations are dense on the spatial domain and thus contain a large amount of high frequency information which would be missed by the Whittle likelihood. An alternative method for parameter estimation of a spatial process is to use a different loss function. Motivated by (Rice, 1979), the discussion on the Whittle estimator in Section 2.3.1 and Theorem 2.1 we define the quadratic loss function

$$L_n(\theta) = \frac{1}{\lambda^d} \sum_{\boldsymbol{k}=-a}^{a} \left(|J_n(\boldsymbol{\omega_k})|^2 - \langle \gamma, \gamma_0 \rangle f_\theta(\boldsymbol{\omega_k})\right)^2,$$

and let $\widehat{\theta}_n = \arg\min_{\theta \in \Theta} L_n(\theta)$ or equivalently solve $\nabla_\theta L_n(\theta) = 0$, where

$$\begin{aligned} \nabla_\theta L_n(\theta) \quad &= \quad -\frac{2\langle \gamma, \gamma_0 \rangle}{\lambda^d} \sum_{\boldsymbol{k}=-a}^{a} \nabla_\theta f_\theta(\boldsymbol{\omega_k}) \left\{|J_n(\boldsymbol{\omega_k})|^2 - \langle \gamma, \gamma_0 \rangle f_\theta(\boldsymbol{\omega_k})\right\} \\ &= \quad -\langle \gamma, \gamma_0 \rangle \left[Q_{a,\lambda,\lambda}(2\nabla_\theta f_\theta(\cdot); 0) - \langle \gamma, \gamma_0 \rangle \frac{2}{\lambda^d} \sum_{\boldsymbol{k}=-a}^{a} f_\theta(\boldsymbol{\omega_k}) \nabla_\theta f_\theta(\boldsymbol{\omega_k})\right]. \end{aligned}$$

It is well known that the distributional properties of a quadratic loss function are determined by its first derivative. In particular, the asymptotic sampling properties of $\widehat{\theta}_n$ are determined by $Q_{a,\lambda,\lambda}(2\nabla_\theta f(\cdot;\theta);0)$. In this case a can be such that $a/\lambda \to \infty$ as $\lambda \to \infty$ and Assumption 2.5(ii) is satisfied. An estimator of $\langle \gamma, \gamma_0 \rangle$ is given in Remark 4.1. Note that in the definition of $L_n(\theta)$, $\langle \gamma, \gamma_0 \rangle$ can be replaced with σ^2, in which case one is estimating a multiple of $f_{\theta_0}(\boldsymbol{\omega})$.

If either Assumption 2.3 or 2.4 is satisfied then we can replace $f_\theta(\boldsymbol{\omega}_{\boldsymbol{k}})$ with $A_\lambda(\frac{\boldsymbol{k}}{\lambda};\theta)$, to obtain a better fixed domain approximation.

3 A summary of the sampling properties of $Q_{a,\Omega,\lambda}(g;0)$

In this section we consider the sampling properties of $Q_{a,\Omega,\lambda}(g;0)$ for the general frequency grid $\{\boldsymbol{\omega}_{\Omega,\boldsymbol{k}} = \frac{2\pi\boldsymbol{k}}{\Omega}\}$. The proof and more general results can be found in Appendix D, (Subba Rao, 2017b). To simplify notation in this section we mainly consider the case that the locations are uniformly distributed.

Lemma 3.1 *Suppose Assumptions 2.1(i), 2.3 and 2.6(a,c) or (b,c) hold. Let $I(g;\frac{a}{\Omega})$ and $A_\lambda(\cdot)$ be defined as in (1.3) and (2.5) respectively. Then*

$$\mathrm{E}\left[Q_{a,\Omega,\lambda}(g;0)\right] = \frac{c_2}{\Omega^d}\sum_{\boldsymbol{k}=-a}^{a} g\left(\frac{2\pi\boldsymbol{k}}{\Omega}\right) A_\lambda\left(\frac{\boldsymbol{k}}{\Omega}\right) + \frac{c(0)\lambda^d}{n\Omega^d}\sum_{\boldsymbol{k}=-a}^{a} g(\omega_{\Omega,\boldsymbol{k}}) \tag{3.1}$$

where $c_2 = n(n-1)/2$. If we let $\lambda \to \infty$, then

$$\mathrm{E}\left[Q_{a,\Omega,\lambda}(g;0)\right] = c_2 I\left(g;\frac{a}{\Omega}\right) + \frac{c(0)\lambda^d}{n\Omega^d}\sum_{\boldsymbol{k}=-a}^{a} g(\omega_{\Omega,\boldsymbol{k}}) + O\left(\frac{\log\lambda}{\lambda} + \frac{1}{\Omega}\right). \tag{3.2}$$

PROOF See Appendix A, (Subba Rao, 2017b). □

We observe an exact expression for the expectation of $Q_{a,\Omega,\lambda}(g;0)$ is in terms of the Fourier coefficients $A_\lambda(\boldsymbol{k}/\Omega)$. However, an approximation of the expection of $Q_{a,\Omega,\lambda}(g;0)$ (within the increasing domain framework) is in terms of an integral of the spectral density function.

We apply the above results to some of the examples considered in the previous section. The results are given in the general case that the locations are random variables but not necessarily uniformly distributed (see Theorem 4.2 and Lemma A.2 for the details).

Example 3.1 *(i) The Whittle likelihood Under Assumption 2.2, using Theorems 2.1 and*

4.2, within the increasing domain asymptotics framework we have

$$\mathrm{E}\left[\mathcal{L}_{S,n}(\theta,\eta^2)\right] = \frac{1}{(2\pi)^d}\int_{[-a/\lambda,a/\lambda]^d}\left(\log\left[f_\theta(\boldsymbol{\omega})+\eta^2\right] + \frac{\langle\gamma,\gamma_0\rangle f_{\theta_0}(\boldsymbol{\omega})+\gamma_0\eta_0^2}{f_\theta(\boldsymbol{\omega})+\eta^2}\right)d\omega + O\left(\frac{1}{\lambda}\right),$$

where $f_{\theta_0}(\cdot)$ denotes the true spectral density, $\eta_0^2 = \lambda^d n^{-1}c(0;\theta_0)$ (note that $\eta_0^2 = O(\lambda^d/n)$) with $c(\boldsymbol{s};\theta_0)$ the corresponding spatial covariance.

Assuming Assumption 2.3 holds, within the fixed domain framework the expectation of (2.6) is

$$\mathrm{E}[\mathcal{L}_{S,n}(\theta,\eta^2)] = \frac{1}{\lambda^d}\sum_{\boldsymbol{k}=-a}^{a}\left(\log\left[A_\lambda\left(\frac{\boldsymbol{k}}{\lambda};\theta\right)+\eta^2\right] + \frac{A_\lambda(\frac{\boldsymbol{k}}{\lambda};\theta_0)}{\left[A_\lambda\left(\frac{\boldsymbol{k}}{\lambda};\theta\right)+\eta^2\right]}\right) + O\left(\frac{1}{n}\left(\frac{a}{\lambda}\right)^{2d}\right),$$

where in the above error bound we use that the tails of $A_\lambda(\boldsymbol{k}/\lambda;\theta)$ decay at the rate $\prod_{i=1}^d \xi_2(k_i)$.

(ii) The nonparametric covariance Within the increasing domain framework and using Lemma A.2(ii), (Subba Rao, 2017b), for $\boldsymbol{u}\in[-\min(\lambda,\Omega/2),\min(\lambda,\Omega/2)]^d$ we have

$$\mathrm{E}[\widetilde{c}_{\Omega,n}(\boldsymbol{u})] = \langle\gamma,\gamma_0\rangle c(\boldsymbol{u}) + O\left(\frac{\log\lambda}{\lambda}+\frac{1}{\Omega}\right),$$

and $\mathrm{E}[\widehat{c}_{\Omega,n}(\boldsymbol{u})] = \langle\gamma,\gamma_0\rangle T(\frac{\boldsymbol{u}}{\Omega})c(\boldsymbol{u}) + O\left(T(\frac{\boldsymbol{u}}{\Omega})[\frac{\log\lambda}{\lambda}+\frac{1}{\Omega}]\right)$ where $\langle\gamma,\gamma_0\rangle$ is defined in Theorem 2.1.

In order to understand the properties of $\widetilde{c}_{\Omega,n}(\boldsymbol{u})$ within the fixed domain framework we assume that Assumption 2.3 holds. If $\Omega\geq 2\lambda$ and $\boldsymbol{u}\in[-\lambda,\lambda]$ then by using Lemma 3.1 we have

$$\begin{aligned}\mathrm{E}\left[\widetilde{c}_{\Omega,n}(\boldsymbol{u})\right] &= \frac{1}{\Omega^d}\sum_{\boldsymbol{k}=-\infty}^{\infty}A_\lambda\left(\frac{\boldsymbol{k}}{\Omega}\right)\exp\left(i\boldsymbol{u}'\boldsymbol{\omega}_{\Omega,\boldsymbol{k}}\right) + O\left(\frac{1}{n}+\frac{1}{a}\right)\\ &= T\left(\frac{\boldsymbol{u}}{\lambda}\right)c(\boldsymbol{u}) + O\left(\frac{1}{n}+\frac{1}{a}\right),\end{aligned}$$

where we recall $T(\cdot)$ denotes the triangle kernel.

In Lemma 3.1 we observe the bias $\frac{c(0)\lambda^d}{n\Omega^d}\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})$. It can be removed by using a bias corrected version of $Q_{a,\Omega,\lambda}(g;0)$

$$\widetilde{Q}_{a,\Omega,\lambda}(g;0) = \frac{1}{\Omega^d}\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})\left|J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})\right|^2 - \frac{\lambda^d}{\Omega^d n}\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})\frac{1}{n}\sum_{j=1}^{n} Z(\boldsymbol{s}_j)^2. \tag{3.3}$$

For the remainder of this section we focus on this bias corrected estimator. The analogous result for $Q_{a,\Omega,\lambda}(g;0)$ can be found in Appendix H, (Subba Rao, 2017b).

We show in Appendix D, (Subba Rao, 2017b) that in the case that $\{Z(\boldsymbol{u});\boldsymbol{u}\in\mathbb{R}^d\}$ is a Gaussian stationary spatial process

$$\mathrm{var}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;0)\right]\approx C_1\left(\frac{a}{\Omega}\right)\frac{1}{\Omega^d}\sum_{k_1,\ldots,k_d=-2a}^{2a}\mathrm{Sinc}^2\left(\frac{\lambda}{\Omega}\boldsymbol{k}\pi\right)$$

where

$$C_1\left(\frac{a}{\Omega}\right)=\frac{1}{(2\pi)^d}\int_{-[2\pi a/\Omega,2\pi a/\Omega]^d}f(\boldsymbol{\omega})^2\left[g(\boldsymbol{\omega})|^2+g(\boldsymbol{\omega})\overline{g(-\boldsymbol{\omega})}\right]d\boldsymbol{\omega}. \tag{3.4}$$

Observe that the rate of convergence of var $\left[\widetilde{Q}_{a,\Omega,\lambda}(g;0)\right]$ is determined by

$$\sum_{k_1,\ldots,k_d=-2a}^{2a}\mathrm{Sinc}^2\left(\frac{\lambda}{\Omega}\boldsymbol{k}\pi\right)=\prod_{i=1}^{d}\sum_{k_i=-2a}^{2a}\mathrm{sinc}^2\left(\frac{\lambda}{\Omega}k_i\pi\right).$$

It is this term along with the following result which gives the crucial insight into the rate of convergence for different frequency grids $\{\boldsymbol{\omega}_{\Omega,\boldsymbol{k}}\}$. If $a\to\infty$ then

$$\frac{1}{\Omega}\sum_{k=-\infty}^{\infty}\mathrm{sinc}^2\left(\frac{\lambda}{\Omega}k\pi\right)=\begin{cases}\frac{1}{\lambda} & \frac{\lambda}{\Omega}<1\\ \frac{1}{\Omega} & \frac{\lambda}{\Omega}\in\mathbb{Z}\\ O\left(\frac{1}{\Omega}\right) & \frac{\lambda}{\Omega}>1 \text{ and } \frac{\lambda}{\Omega}\notin\mathbb{Z},\end{cases}$$

further, if $\lambda/\Omega\to\infty$ then $\sum_{k=-\infty}^{\infty}\mathrm{sinc}^2\left(\frac{\lambda}{\Omega}k\pi\right)\to 1$ (see Appendix D, (Subba Rao, 2017b) for the proof). This result implies that

$$\mathrm{var}[\widetilde{Q}_{a,\Omega,\lambda}(g;0)]=\begin{cases}O\left(\frac{1}{\lambda^d}\right) & \lambda<\Omega\\ O\left(\frac{1}{\Omega^d}\right) & \lambda\geq\Omega\end{cases}.$$

In other words, the frequency grid $\boldsymbol{\omega}_{\lambda,\boldsymbol{k}}=\frac{2\pi\boldsymbol{k}}{\lambda}$ or finer will yield a rate of convergence of $O(\lambda^{-d})$ and $\mathrm{var}[\widetilde{Q}_{a,\Omega,\lambda}(g;0)]\approx\lambda^{-d}C_1(\frac{a}{\Omega})$. However a coarse frequency grid $\boldsymbol{\omega}_{\Omega,\boldsymbol{k}}=\frac{2\pi\boldsymbol{k}}{\Omega}$ where $\Omega<\lambda$ will yield a slower rate of convergence of $O(\Omega^{-d})$. In the theorem below we make this precise.

For the following theorem we consider general stationary spatial random fields, this requires the following definition

$$D_1\left(\frac{a}{\Omega}\right)=\frac{1}{(2\pi)^{2d}}\int_{[-2\pi a/\Omega,2\pi a/\Omega]^{2d}}g(\boldsymbol{\omega}_1)\overline{g(\boldsymbol{\omega}_2)}f_4\left(-\boldsymbol{\omega}_1,-\boldsymbol{\omega}_2,\boldsymbol{\omega}_2\right)d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2. \tag{3.5}$$

This term arises if the spatial random field is non-Gaussian.

Theorem 3.1 *Suppose Assumptions 2.1(i), 2.2, 2.5(i) or (ii), 2.6(b,c) and 2.7 hold. Let $C_1(\cdot)$ and $D_1(\cdot)$ be defined as in (3.4) and (3.5) respectively.*

(i) If a fine frequency grid is used ($\frac{\lambda}{\Omega} < 1$) then

$$\lambda^d \mathrm{var}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;0)\right] = C_1\left(\frac{a}{\Omega}\right)\left[\frac{\lambda^d}{\Omega^d}\sum_{\boldsymbol{k}=-2a}^{2a}\mathrm{Sinc}^2\left(\frac{\lambda}{\Omega}\boldsymbol{k}\pi\right)\right] + D_1\left(\frac{a}{\Omega}\right) + O\left(\ell^{(2)}_{a,\Omega,\lambda}\right)$$

(ii) If a coarse frequency grid is used ($\frac{\lambda}{\Omega} \geq 1$) then

$$\Omega^d \mathrm{var}[\widetilde{Q}_{a,\Omega,\lambda}(g;0)] = C_1\left(\frac{a}{\Omega}\right)\left[\sum_{\boldsymbol{k}=-2a}^{2a}\mathrm{Sinc}^2\left(\frac{\lambda}{\Omega}\boldsymbol{k}\pi\right)\right] + \left(\frac{\Omega}{\lambda}\right)^d D_1\left(\frac{a}{\Omega}\right) + O\left(\widetilde{\ell}^{(2)}_{a,\Omega,\lambda}\right)$$

where $\widetilde{\ell}^{(2)}_{a,\Omega,\lambda}$ is defined in (D.16), Appendix D, (Subba Rao, 2017b).

PROOF See Appendix D, (Subba Rao, 2017b). □

As mentioned above, one important implication of the above result is that the rate of convergence depends on whether the frequency grid is coarser or finer than $1/\lambda$, where λ is the length of the spatial domain. In terms of the asymptotic sampling properties (see Lemma 3.1 and Theorem 3.1) there seems to be little benefit using a very fine frequency grid, as it does not reduce the bias or variance (but is computationally costly).

We observe that if the spatial process is non-Gaussian then an additional term, $D_1(a/\Omega)$, involving the fourth order cumulant of the spatial process, arises. However, if the frequency grid is extremely coarse in the sense that $\lambda/\Omega \to \infty$ as $\lambda \to \infty$ and $\Omega \to \infty$, then $D_1(a/\Omega)$ is asymptotic negligible compared with the leading term which is a function of the spectral density. For example, if $\lambda/\Omega \in \mathbb{Z}^+$ then

$$\Omega^d \mathrm{var}[\widetilde{Q}_{a,\Omega,\lambda}(g;0)] = C_1\left(\frac{a}{\Omega}\right) + O\left(\ell^{(2)}_{a,\Omega,\lambda} + \frac{\Omega^d}{\lambda^d}\right).$$

Thus if $\lambda/\Omega \to \infty$ as $\lambda \to \infty$ and $\Omega \to \infty$ (and $\overline{g(\boldsymbol{\omega})} = g(-\boldsymbol{\omega})$) we have verified condition (C.4) in (Bandyopadhyay et al., 2015);

$$\frac{\mathrm{var}\left[\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})|J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})|^2\right]}{2\sum_{\boldsymbol{k}=-a}^{a}|g(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})|^2 f(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})^2} \xrightarrow{\mathcal{P}} 1,$$

which is required for their proposed spatial spectral empirical likelihood methodology. Therefore a very coarse grid has the advantage that the term $D_1(\cdot)$ is negligible. However, we see from Lemma 3.1 and Theorem 3.1 that the disadvantage is that there is a substantial increase in both variance and bias.

Since the grid, $\boldsymbol{\omega}_{\boldsymbol{k}} = 2\pi\boldsymbol{k}/\lambda$, yields optimal samping properties in Section 4 we focus on deriving sampling properties of $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$, where

$$\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}) = \frac{1}{\lambda^d}\sum_{k_1,\ldots,k_d=-a}^{a} g(\boldsymbol{\omega}_{\boldsymbol{k}})J_n(\boldsymbol{\omega}_{\boldsymbol{k}})\overline{J_n(\boldsymbol{\omega}_{\boldsymbol{k}+\boldsymbol{r}})} - \frac{1}{n}\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega}_{\boldsymbol{k}})\frac{1}{n}\sum_{j=1}^{n} Z(\boldsymbol{s}_j)^2 e^{-i\boldsymbol{s}_j'\boldsymbol{\omega}_{\boldsymbol{r}}}. \tag{3.6}$$

4 Sampling properties of $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$

In this section we show that under the increasing domain framework $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$ (defined (3.6)) is a consistent estimator of $I(g;\frac{a}{\lambda})$ (or some multiple of it), where $I(g;\frac{a}{\lambda})$ is defined in (1.3). The sampling properties in the fixed domain framework are given in Section 4.4.

4.1 The expectation of $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$

We start with the expectation of $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$. We show if $\sup_{\boldsymbol{\omega}\in\mathbb{R}^d}|g(\boldsymbol{\omega})|<\infty$, the choice of a does not play a significant role in the asymptotic properties of $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$. If $a>>\lambda$, the analysis of $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$ requires more delicate techniques. We start by stating some pertinent features in the analysis of $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$, which gives a flavour of our approach. By writing $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$ as a quadratic form it is straightforward to show that

$$\begin{aligned} \mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\right] &= c_2\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega_k})\frac{1}{\lambda^d}\int_{[-\lambda/2,\lambda/2]^{2d}} c(\boldsymbol{s}_1-\boldsymbol{s}_2)\exp(i\boldsymbol{\omega}_k'(\boldsymbol{s}_1-\boldsymbol{s}_2)-i\boldsymbol{s}_2'\boldsymbol{\omega_r})\times \\ & \quad h\left(\frac{\boldsymbol{s}_1}{\lambda}\right)h\left(\frac{\boldsymbol{s}_2}{\lambda}\right)d\boldsymbol{s}_1 d\boldsymbol{s}_2, \end{aligned} \tag{4.1}$$

where $c_2=n(n-1)/n^2$. The proof of Theorem 2.1 is based on making a change of variables $v=\boldsymbol{s}_1-\boldsymbol{s}_2$ and then systematically changing the limits of the integral. This method can be applied to the above, if a is such that the ratio a/λ is fixed for all λ. However, if the frequency grid $[-a/\lambda,a/\lambda]^d$ is allowed to grow with λ, applying this brute force method to $\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\right]$ has the disadvantage that it aggregrates the errors within the sum of $\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\right]$. Instead, to further the analysis, we replace $c(\boldsymbol{s}_1-\boldsymbol{s}_2)$ by its spectral representation $c(\boldsymbol{s}_1-\boldsymbol{s}_2)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}f(\boldsymbol{\omega})\exp(i\boldsymbol{\omega}'(\boldsymbol{s}_1-\boldsymbol{s}_2))d\boldsymbol{\omega}$ and focus on the case that the sampling design is uniform; $h(\boldsymbol{s}/\lambda)=\lambda^{-d}I_{[-\lambda/2,\lambda/2]}(\boldsymbol{s})$ (later we consider general sampling densities). This reduces the first term in $\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\right]$ to the Fourier transforms of step functions, which is the product of sinc functions. Specifically, we obtain

$$\begin{aligned} \mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\right] &= \frac{c_2}{(2\pi)^d}\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega_k})\int_{\mathbb{R}^d} f(\boldsymbol{\omega})\mathrm{Sinc}\left(\frac{\lambda\boldsymbol{\omega}}{2}+\boldsymbol{k}\pi\right)\mathrm{Sinc}\left(\frac{\lambda\boldsymbol{\omega}}{2}+(\boldsymbol{k}+\boldsymbol{r})\pi\right)d\boldsymbol{\omega} \\ &= \frac{c_2}{\pi^d}\int_{\mathbb{R}^d}\mathrm{Sinc}(\boldsymbol{y})\mathrm{Sinc}(\boldsymbol{y}+\boldsymbol{r}\pi)\left[\frac{1}{\lambda^d}\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega_k})f(\frac{2\boldsymbol{y}}{\lambda}-\boldsymbol{\omega_k})\right]d\boldsymbol{y}, \end{aligned}$$

where the last line above is due to a change of variables $\boldsymbol{y}=\frac{\lambda\boldsymbol{\omega}}{2}+\boldsymbol{k}\pi$. Since the spectral density function is absolutely integrable it is clear that $\left[\frac{1}{\lambda^d}\sum_{\boldsymbol{k}=-a}^{a}g(\boldsymbol{\omega_k})f(\frac{2\boldsymbol{y}}{\lambda}-\boldsymbol{\omega_k})\right]$ is uniformly bounded over $\boldsymbol{y}$ and that $\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\right]$ is finite for all λ. Furthermore, if $f(\frac{2\boldsymbol{y}}{\lambda}-\boldsymbol{\omega_k})$ were replaced with $f(-\boldsymbol{\omega_k})$, then what remains in the integral are two shifted sinc functions, which is zero if $\boldsymbol{r} \in \mathbb{Z}^d/\{0\}$, i.e.

$$\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\right] = \frac{c_2}{\pi^d}\int_{\mathbb{R}^d} \mathrm{Sinc}(\boldsymbol{y})\mathrm{Sinc}(\boldsymbol{y}+\boldsymbol{r}\pi)\left[\frac{1}{\lambda^d}\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega_k})f(-\boldsymbol{\omega_k})\right]d\boldsymbol{y} + R,$$

where

$$R = \frac{c_2}{\pi^d}\int_{\mathbb{R}^d} \mathrm{Sinc}(\boldsymbol{y})\mathrm{Sinc}(\boldsymbol{y}+\boldsymbol{r}\pi)\left[\frac{1}{\lambda^d}\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega_k})\left(f(\frac{2\boldsymbol{y}}{\lambda}-\boldsymbol{\omega_k})-f(-\boldsymbol{\omega_k})\right)\right]d\boldsymbol{y}.$$

In the following theorem we show that under certain conditions on f, R is asymptotically negligible.

Theorem 4.1 *Let $I(g;\cdot)$ be defined as in (1.3). Throughout the theorem we suppose Assumptions 2.1(i) and 2.3 hold. Let $b(\boldsymbol{r})$ denote the number of zero values in $\boldsymbol{r}$.*

(i) If Assumptions 2.5(i) and 2.6(a,c) hold, then we have

$$\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\right] = \begin{cases} O(\frac{1}{\lambda^{d-b(\boldsymbol{r})}}) & \boldsymbol{r} \in \mathbf{Z}^d/\{0\} \\ I(g;C) + O(\frac{1}{\lambda}) & \boldsymbol{r} = \mathbf{0} \end{cases} \tag{4.2}$$

(ii) Suppose Assumptions 2.5(ii) holds and

(a) Assumption2.6(b) holds, then $\sup_a \left|\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\right]\right| < \infty$.

(b) Assumption 2.6(b,c,d) holds, then we have

$$\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\right] = \begin{cases} O\left(\frac{1}{\lambda^{d-b(\boldsymbol{r})}}\prod_{j=1}^{d-b(\boldsymbol{r})}(\log\lambda + \log|m_j|)\right) & \boldsymbol{r} \in \mathbf{Z}^d/\{0\} \\ I\left(g;\frac{a}{\lambda}\right) + O\left(\frac{\log\lambda}{\lambda} + \frac{1}{n}\right) & \boldsymbol{r} = \mathbf{0} \end{cases}.$$

where $\{m_1,\ldots,m_{d-b(\boldsymbol{r})}\}$ is the subset of non-zero values in $\boldsymbol{r} = (r_1,\ldots,r_d)$.

(c) If only Assumption 2.6(b,c) holds, then the $O\left(\frac{1}{\lambda^{d-b(\boldsymbol{r})}}\prod_{j=1}^{d-b(\boldsymbol{r})}(\log\lambda + \log|m_j|)\right)$ term in (b) is replaced with the slower rate $O\left(\frac{1}{\lambda}(\log\lambda + \log[1+\|\boldsymbol{r}\|_1])\right)$.

Note that the above bounds for (b) and (c) are uniform in a.

PROOF See Appendix A, (Subba Rao, 2017b). □

We observe that if $\boldsymbol{r} \neq \mathbf{0}$, then $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$ is estimating zero as $\lambda \to \infty$. It would appear that $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$ when $\boldsymbol{r} \neq 0$ does not contain any useful information, however in Section 5 we show how these terms can be used to estimate nuisance parameters.

In order to analyze $\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\right]$ in the case that the locations are not from a uniform distribution we return to (4.1) and replace $c(\boldsymbol{s}_1-\boldsymbol{s}_2)$ and $h(\cdot)$ by their Fourier representations

$$\begin{aligned}&\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\right]\\ =\;& \frac{c_2}{\pi^d}\sum_{\boldsymbol{j}_1,\boldsymbol{j}_2\in\mathbb{Z}}\gamma_{\boldsymbol{j}_1}\gamma_{\boldsymbol{j}_2}\frac{1}{\lambda^d}\sum_{\boldsymbol{k}=-a}^{a}g(\boldsymbol{\omega}_{\boldsymbol{k}})\int_{-\infty}^{\infty}f\left(\frac{2\boldsymbol{y}}{\lambda}-\boldsymbol{\omega}_{\boldsymbol{k}}\right)\mathrm{Sinc}(\boldsymbol{y})\mathrm{Sinc}(\boldsymbol{y}+(\boldsymbol{r}-\boldsymbol{j}_1-\boldsymbol{j}_2)\pi)d\boldsymbol{y}.\end{aligned}$$

This representation allows us to use similar techniques to those used in the uniform sampling case to prove the following result.

Theorem 4.2 *Let $I(g;\cdot)$ be defined as in (1.3). Suppose Assumptions 2.1(i) and 2.2 hold.*

(i) If in addition Assumptions 2.5(i) and 2.6(a,c) hold, then we have

$$\mathrm{E}\big[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\big]=\langle\gamma,\gamma_{\boldsymbol{r}}\rangle I\left(g;\frac{a}{\lambda}\right)+O(\lambda^{-1}),$$

$O(\lambda^{-1})$ is uniform over $\boldsymbol{r}\in\mathbb{Z}^d$.

(ii) If in addition Assumptions 2.5(ii) and 2.6(b,c) hold, then we have

$$\mathrm{E}\big[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\big]=\langle\gamma,\gamma_{-\boldsymbol{r}}\rangle I\left(g;\frac{a}{\lambda}\right)+O\left(\frac{\log\lambda+\log(1+\|\boldsymbol{r}\|_1)}{\lambda}\right).$$

PROOF See Appendix A, (Subba Rao, 2017b). □

We observe that by applying Theorem 4.2 to the case that h is uniform (using that $\gamma_0=1$ else $\gamma_{\boldsymbol{j}}=0$) gives $\mathrm{E}\big[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\big]=O(\lambda^{-1})$ for $\boldsymbol{r}\neq 0$. Hence, in the case that the sampling is uniform, Theorems 4.1 and 4.2 give similar results, though the bounds in Theorem 4.1 are sharper.

Remark 4.1 (Estimation of $\sum_{\boldsymbol{j}\in\mathbb{Z}^d}|\gamma_{\boldsymbol{j}}|^2$) *The above lemma implies that $\mathrm{E}\big[\widetilde{Q}_{a,\lambda}(g;0)\big]=\langle\gamma,\gamma_{\boldsymbol{0}}\rangle I\left(g;\frac{a}{\lambda}\right)$. Therefore, to estimate $I\left(g;\frac{a}{\lambda}\right)$ we require an estimator of $\langle\gamma,\gamma_{\boldsymbol{0}}\rangle$. To do this, we recall that*

$$\langle\gamma,\gamma_{\boldsymbol{0}}\rangle=\sum_{\boldsymbol{j}\in\mathbb{Z}}|\gamma_{\boldsymbol{j}}|^2=\frac{1}{\lambda^d}\int_{[-\lambda/2,\lambda/2]^d}h_\lambda(\boldsymbol{\omega})^2d\boldsymbol{\omega}.$$

Therefore one method for estimating the above integral is to define a grid on $[-\lambda/2,\lambda/2]^d$ and estimate h_λ at each point, then to take the average squared over the grid (see Remark 1, (Matsuda & Yajima, 2009)). An alternative, computationally simpler method, is to use the method proposed in (Gine & Nickl, 2008). That is, use

$$\widehat{\langle\gamma,\gamma_{\boldsymbol{0}}\rangle}=\frac{2}{n(n-1)b}\sum_{1\leq j_1<j_2\leq n}K\left(\frac{\boldsymbol{s}_{j_1}-\boldsymbol{s}_{j_2}}{b}\right)^2,$$

as an estimator of $\langle\gamma,\gamma_{\mathbf{0}}\rangle$*, where* $K : [-1/2,1/2]^d \to \mathbb{R}$ *is a kernel function. Note that multiplying the above kernel with* $\exp(-i\boldsymbol{\omega}_{\boldsymbol{r}}'\boldsymbol{s}_{j_2})$ *results in an estimator of* $\langle\gamma,\gamma_{\boldsymbol{r}}\rangle$*. In the case* $d=1$ *and under certain regularity conditions, (Gine & Nickl, 2008) show if the bandwidth* b *is selected in an appropriate way then* $\widehat{\langle\gamma,\gamma_{\mathbf{0}}\rangle}$ *attains the classical* $O(n^{-1/2})$ *rate under suitable regularity conditions (see, also, (Bickel & Ritov, 1988) and (Laurent, 1996)). It seems plausible a similar result holds for* $d>1$ *(though we do not prove it here). Therefore, an estimator of* $I\left(g;\frac{a}{\lambda}\right)$ *is* $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})/\widehat{\langle\gamma,\gamma_{\mathbf{0}}\rangle}$*.*

4.2 The covariance and asymptotic normality

In the previous section we showed that the expectation of $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$ depends only on the number of frequencies a through the limit of the integral $I(g;\frac{a}{\lambda})$ (if $\sup_{\boldsymbol{\omega}\mathbb{R}^d}|g(\boldsymbol{\omega})|<\infty$). In this section, we show that a plays a mild role in the higher order properties of $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$. We focus on the case that the random field is Gaussian and later describe how the results differ in the case that the random field is non-Gaussian.

Theorem 4.3 *Suppose Assumptions 2.1, 2.2 hold. Let* $U_1(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{\omega}_{\boldsymbol{r}_1},\boldsymbol{\omega}_{\boldsymbol{r}_2})$ *and* $U_2(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{\omega}_{\boldsymbol{r}_1},\boldsymbol{\omega}_{\boldsymbol{r}_2})$ *be defined as in equation (C.1), (Subba Rao, 2017b).*

(i) If Assumption 2.5(i) and 2.6(a,c) also hold. Then uniformly for all $0\le\|\boldsymbol{r}_1\|_1,\|\boldsymbol{r}_2\|_1\le C|\lambda|$*, we have*

$$\lambda^d\mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1),\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)\right]=U_1(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{\omega}_{\boldsymbol{r}_1},\boldsymbol{\omega}_{\boldsymbol{r}_2})+O\left(\frac{1}{\lambda}+\frac{\lambda^d}{n}\right)$$

and

$$\lambda^d\mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1),\overline{\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)}\right]=U_2(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{\omega}_{\boldsymbol{r}_1},\boldsymbol{\omega}_{\boldsymbol{r}_2})+O\left(\frac{1}{\lambda}+\frac{\lambda^d}{n}\right)$$

(ii) If Assumption 2.5(ii) and 2.6(b) also hold. Then

$$\lambda^d\sup_{a,\boldsymbol{r}}\mathrm{var}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\right]<\infty$$

with $\lambda^d/n\to c$ *(where* $0\le c<\infty$*) as* $\lambda\to\infty$ *and* $n\to\infty$*.*

(iii) If Assumption 2.5(ii) and 2.6(b,c) also hold. Then uniformly for all $0\le\|\boldsymbol{r}_1\|_1,\|\boldsymbol{r}_2\|_1\le C|a|$ *(for some finite constant* C*) we have*

$$\lambda^d\mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1),\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)\right]=U_1(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{\omega}_{\boldsymbol{r}_1},\boldsymbol{\omega}_{\boldsymbol{r}_2})+O(\ell_{\lambda,a,n})$$

$$\lambda^d\mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1),\overline{\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)}\right]=U_2(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{\omega}_{\boldsymbol{r}_1},\boldsymbol{\omega}_{\boldsymbol{r}_2})+O(\ell_{\lambda,a,n}),$$

where

$$\ell_{\lambda,a,n} = \log^2(a)\left[\frac{\log a + \log \lambda}{\lambda}\right] + \frac{\lambda^d}{n}. \quad (4.3)$$

PROOF See Appendix C, (Subba Rao, 2017b). □

We now briefly discuss the above results. From Theorem 4.3(ii) we see that $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$ is a mean squared consistent estimator of $\langle \gamma, \gamma_{\boldsymbol{r}}\rangle I(g;\frac{a}{\lambda})$, i.e. $\mathrm{E}[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}) - \langle \gamma, \gamma_{\boldsymbol{r}}\rangle I(g;\frac{a}{\lambda})]^2 = O(\lambda^{-d} + (\frac{\log \lambda}{\lambda} + \frac{1}{n})^2)$ as $a \to \infty$ and $\lambda \to \infty$.

In order to obtain an explicit expression for the variance additional conditions are required. In particular, Theorem 4.3(iii) states that if the frequency grid is unbounded we require some additional conditions on the spectral density function and some mild constraints on the rate of growth of the frequency domain a. More precisely, a should be such that $a = O(\lambda^k)$ for some $1 \leq k < \infty$.

Remark 4.2 (Selecting a in practice) *The above gives theoretical guidelines. In practice, if* $\sup_{\boldsymbol{\omega}\in\mathbb{R}^d} |g(\boldsymbol{\omega})| < \infty$ *we suggest plotting* $|J_n(\boldsymbol{\omega}_{\boldsymbol{k}})|^2$ *against* $\boldsymbol{\omega}_{\boldsymbol{k}}$*.* $|J_n(\boldsymbol{\omega}_{\boldsymbol{k}})|^2$ *will drop close to zero for large* $\|\boldsymbol{\omega}_{\boldsymbol{k}}\|_1$ *(see Figure 1, Section J, (Subba Rao, 2017b)). Thus a should be chosen such that it lies after this point. The precise value does not matter too much as the results are not too sensitive to the choice of a.*

The expressions for $\mathrm{cov}[\lambda^d \mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1), \overline{\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)}\right]]$ (see equation (C.1), (Subba Rao, 2017b)) are unwieldy, however, some simplifications can be made if $\|\boldsymbol{r}_1\|_1 << \lambda$ and $\|\boldsymbol{r}_2\|_1 << \lambda$.

Corollary 4.1 *Suppose Assumptions 2.2, 2.5 and 2.6(a,c) or 2.6(b,c) hold. Then*

$$\begin{aligned} U_1(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{\omega}_{\boldsymbol{r}_1},\omega_{\boldsymbol{r}_2}) &= \Gamma_{\boldsymbol{r}_1-\boldsymbol{r}_2} C_1 + O\left(\varepsilon_{\boldsymbol{r}_1,\boldsymbol{r}_2}(\lambda)\right) \\ U_2(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{\omega}_{\boldsymbol{r}_1},\omega_{\boldsymbol{r}_2}) &= \Gamma_{\boldsymbol{r}_1+\boldsymbol{r}_2} C_2 + O\left(\varepsilon_{\boldsymbol{r}_1,\boldsymbol{r}_2}(\lambda)\right) \end{aligned} \quad (4.4)$$

where $\Gamma_{\boldsymbol{r}} = \sum_{\boldsymbol{j}_1+\boldsymbol{j}_2+\boldsymbol{j}_3+\boldsymbol{j}_4=\boldsymbol{r}} \gamma_{\boldsymbol{j}_1}\gamma_{\boldsymbol{j}_2}\gamma_{\boldsymbol{j}_3}\gamma_{\boldsymbol{j}_4}$, $\varepsilon_{\boldsymbol{r}_1,\boldsymbol{r}_2}(\lambda) = \frac{\|\boldsymbol{r}_1\|_1+\|\boldsymbol{r}_2\|_1}{\lambda}$ *and*

$$\begin{aligned} C_1 &= \frac{1}{(2\pi)^d}\int_{2\pi[-a/\lambda,a/\lambda]^d} f(\boldsymbol{\omega})^2\left[|g(\boldsymbol{\omega})|^2 + g(\boldsymbol{\omega})\overline{g(-\boldsymbol{\omega})}\right] d\boldsymbol{\omega} \\ C_2 &= \frac{1}{(2\pi)^d}\int_{2\pi[-a/\lambda,a/\lambda]^d} f(\boldsymbol{\omega})^2\left[g(\boldsymbol{\omega})g(-\boldsymbol{\omega}) + g(\boldsymbol{\omega})g(\boldsymbol{\omega})\right] d\boldsymbol{\omega}. \end{aligned}$$

Recall that $C_1 = C_1(a/\lambda)$ *(where* $C_1(\cdot)$ *is defined in (3.4)).*

In the following theorem we derive bounds for the cumulants of $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$, which are subsequently used to show asymptotical normality of $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$.

Theorem 4.4 *Suppose Assumptions 2.1, 2.2, 2.5 and 2.6(b) hold. Then for all $q \geq 3$ and uniform in $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_q \in \mathbb{Z}^d$ we have*

$$\mathrm{cum}_q\left[\widetilde{Q}_{a,\lambda}(g,\boldsymbol{r}_1), \ldots, \widetilde{Q}_{a,\lambda}(g,\boldsymbol{r}_q)\right] = O\left(\frac{\log^{2d(q-2)}(a)}{\lambda^{d(q-1)}}\right) \quad (4.5)$$

if $\frac{\lambda^d}{n\log^{2d}(a)} \to 0$ *as* $n \to \infty$, $a \to \infty$ *and* $\lambda \to \infty$.

PROOF See Section E, (Subba Rao, 2017b). □

From the above theorem we see that if $\frac{\lambda^d}{n\log^{2d}(a)} \to 0$ and $\log^2(a)/\lambda^{1/2} \to 0$ as $\lambda \to \infty$, $n \to \infty$ and $a \to \infty$, then we have $\lambda^{dq/2}\mathrm{cum}_q(\widetilde{Q}_{a,\lambda}(g,\boldsymbol{r}_1), \ldots, \widetilde{Q}_{a,\lambda}(g,\boldsymbol{r}_q)) \to 0$ for all $q \geq 3$. Using this result we show asymptotic normality of $\widetilde{Q}_{a,\lambda}(g,\boldsymbol{r})$.

Theorem 4.5 *Suppose Assumptions 2.1, 2.2, 2.5 and 2.6(b,c) hold. Let C_1 and C_2, be defined as in Corollary 4.1. Under these conditions we have*

$$\lambda^{d/2}\Delta^{-1/2}\begin{pmatrix} \Re\left(\widetilde{Q}_{a,\lambda}(g,\boldsymbol{r}) - \langle\gamma,\gamma_{-\boldsymbol{r}}\rangle I(g;\frac{a}{\lambda})\right) \\ \Im\left(\widetilde{Q}_{a,\lambda}(g,\boldsymbol{r}) - \langle\gamma,\gamma_{-\boldsymbol{r}}\rangle I(g;\frac{a}{\lambda})\right) \end{pmatrix} \overset{\mathcal{D}}{\to} \mathcal{N}(0, I_2),$$

where

$$\Delta = \frac{1}{2}\begin{pmatrix} \Re(\Gamma_0 C_1 + \Gamma_{2\boldsymbol{r}}C_2) & -\Im(\Gamma_{2\boldsymbol{r}}C_2) \\ -\Im(\Gamma_{2\boldsymbol{r}}C_2) & \Re(\Gamma_0 C_1 - \Gamma_{2\boldsymbol{r}}C_2) \end{pmatrix}$$

with $\frac{\log^2(a)}{\lambda^{1/2}} \to 0$ *and* $\lambda^d/n \to 0$ *as* $\lambda \to \infty$, $n \to \infty$ *and* $a \to \infty$.

PROOF See Appendix E, (Subba Rao, 2017b). □

It is likely that the above result also holds when the assumption of Gaussianity of the spatial random field is relaxed and replaced with the conditions stated in Theorem 4.6 (below) together with some mixing-type assumptions. We leave this for future work. However, in the following theorem, we obtain an expression for the variance of $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$ for non-Gaussian random fields.

Theorem 4.6 *Let us suppose that $\{Z(\boldsymbol{s}); \boldsymbol{s} \in \mathbb{R}^d\}$ is a fourth order stationary spatial random field that satisfies Assumption 2.1(i), 2.2, 2.5, 2.7 and 2.6(a,c) or 2.6(b,c) are satisfied.*

If $\|\boldsymbol{r}\|_1, \|\boldsymbol{r}\|_2 << \lambda$, then we have

$$\lambda^d \mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1), \widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)\right] = \Gamma_{\boldsymbol{r}_1-\boldsymbol{r}_2}(C_1 + D_1) + O\left(\ell_{\lambda,a,n} + \frac{(a\lambda)^d}{n^2} + \varepsilon_{\boldsymbol{r}_1,\boldsymbol{r}_2}(\lambda)\right)$$

$$\lambda^d \mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1), \overline{\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)}\right] = \Gamma_{\boldsymbol{r}_1+\boldsymbol{r}_2}(C_2 + D_2) + O\left(\ell_{\lambda,a,n} + \frac{(a\lambda)^d}{n^2} + \varepsilon_{\boldsymbol{r}_1,\boldsymbol{r}_2}(\lambda)\right),$$

where C_1 and C_2 are defined as in Corollary 4.1 and

$$D_1 = \frac{1}{(2\pi)^{2d}} \int_{2\pi[-a/\lambda,a/\lambda]^{2d}} g(\boldsymbol{\omega}_1)\overline{g(\boldsymbol{\omega}_2)} f_4(-\boldsymbol{\omega}_1, -\boldsymbol{\omega}_2, \boldsymbol{\omega}_2) d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2$$
$$D_2 = \frac{1}{(2\pi)^{2d}} \int_{2\pi[-a/\lambda,a/\lambda]^{2d}} g(\boldsymbol{\omega}_1) g(\boldsymbol{\omega}_2) f_4(-\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, -\boldsymbol{\omega}_2) d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2.$$

PROOF See Appendix C, (Subba Rao, 2017b). □

We observe that to ensure the term $\frac{(a\lambda)^d}{n^2} \to 0$ we need to choose a such that $a^d = o(n^2/\lambda^d)$. In contrast for Gaussian random fields $a = O(\lambda^k)$ for some $1 \leq k < \infty$ was sufficient for obtaining an expression for the variance and asymptotic normality.

4.3 Mixed Domain verses Pure Increasing Domain asymptotics

The asymptotics in this paper are mainly done using mixed domain asymptotics, that is, as the domain $\lambda \to \infty$, the number of locations observed grows at a faster rate than λ, in other words $\lambda^d/n \to 0$ as $n \to \infty$. However, as rightly pointed out by a referee, for a given application it may be difficult to disambiguate Mixed Domain (MD) from the Pure Increasing Domain (PID) set-up, where $\lambda^d/n \to c$ $(0 < c < \infty)$. We briefly discuss how the results change under PID asymptotics and the implications of this. We find that the results point to a rather intriguing difference for spatial processes that are Gaussian and non-Gaussian.

In the case that spatial process is Gaussian, using both MD and PID asymptotics we have $\lambda^d \mathrm{var}[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})] = O(1)$ (see Theorem 4.3(i)). Furthermore, an asymptotic expression for the variance is

$$\lambda^d \mathrm{var}[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})] = \Gamma_0[C_1 + E_1] + O\left(\ell_{\lambda,a,n} + \varepsilon_{\boldsymbol{r},\boldsymbol{r}}(\lambda)\right)$$

where $E_1 = O(\lambda^d/n)$ is a function of the spectral density; this term is not asymptotically negligible under PID asymptotics. From the above we see that if we choose a such that $a = O(\lambda^k)$ for some $1 < k < \infty$ then similar results as those stated in Sections 4.1 and 4.2 hold under PID asymptotics. In the case that the process is non-Gaussian, using Theorem 4.6 we have

$$\lambda^d \mathrm{var}[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})] = \Gamma_0[C_1 + D_1 + E_1 + F_1] + O\left(\ell_{\lambda,a,n} + \frac{(a\lambda)^d}{n^2} + \varepsilon_{\boldsymbol{r},\boldsymbol{r}}(\lambda)\right),$$

where $F_1 = O(\lambda^d/n)$ is a function of the fourth order spectral density function. However, there arises an additional term $O((a\lambda)^d/n^2)$. From the proof of Theorem 4.6, we see if $\frac{(a\lambda)^d}{n^2} \to \infty$ as $a, \lambda, n \to \infty$, then $\lambda^d \mathrm{var}[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})]$ is not bounded. Thus, the number of frequencies, a, should be such that $(a\lambda)^d/n^2 \to 0$. In the case of MD asymptotics, we choose

a such that $(a\lambda)^d/n^2 \to 0$ and $\log^3(a)/\lambda \to 0$. Under these two conditions the frequency grid can be unbounded and grow at the rate a/λ as $\lambda \to \infty$. However, under PID asymptotics (where $\lambda = O(n^{1/d})$) in order to ensure that $(a\lambda)^d/n^2 = O(1)$ we require $a = O(n^{1/d}) = O(\lambda)$. This constrains the frequency grid to be bounded. To summarize, in the case that the spatial process is non-Gaussian and $n = O(\lambda^d)$ in order that $\mathrm{var}[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})] \to 0$ as $\lambda \to \infty$, the frequency grid must be bounded or a coarser frequency grid $\boldsymbol{\omega}_{\Omega,\boldsymbol{k}}$ (where $\lambda > \Omega$) used (see Section 3).

4.4 Fixed domain asymptotics

We now turn our attention to asymptotic sampling properties of $\widetilde{Q}_{a,\lambda}(g;0)$ when the domain λ is kept fixed but the number of sampling locations $n \to \infty$. In order to simplify notation we consider the case $d = 1$. We will assume the locations, $\{s_{n,j}\}$, lie close to a lattice and satisfy Assumption 2.4. It is clear that as $n \to \infty$ the Fourier transform $J_n(\omega_k)$ can be approximated by the Fourier transform over the continuum

$$\mathcal{J}_\lambda\left(\frac{k}{\lambda}\right) = \frac{1}{\lambda^{1/2}}\int_{-\lambda/2}^{\lambda/2} Z(s)\exp\left(\frac{2\pi iks}{\lambda}\right)ds.$$

Thus asymptotic expressions for the mean and variance of $\widetilde{Q}_{a,\lambda}(g;0)$, where λ is fixed but $n \to \infty$ are in terms of the covariances of $\{\mathcal{J}_\lambda\left(\frac{k}{\lambda}\right)\}_{k\in\mathbb{Z}}$. In Appendix G, (Subba Rao, 2017b) we show that

$$\mathrm{var}\left[\mathcal{J}_\lambda\left(\frac{k}{\lambda}\right)\right] = A_\lambda\left(\frac{k}{\lambda}\right) \text{ and } \mathrm{cov}\left[\mathcal{J}_\lambda\left(\frac{k_1}{\lambda}\right), \mathcal{J}_\lambda\left(\frac{k_2}{\lambda}\right)\right] = \frac{(-1)^{k_1-k_2+1}}{\pi(k_1-k_2)}\left[B_\lambda\left(\frac{k_1}{\lambda}\right) - B_\lambda\left(\frac{k_2}{\lambda}\right)\right]$$

where

$$A_\lambda\left(\frac{k}{\lambda}\right) = \int_{-\lambda}^{\lambda}\left(1-\frac{|u|}{\lambda}\right)c(u)e^{2\pi iu/\lambda}du \text{ and } B_\lambda\left(\frac{k}{\lambda}\right) = \int_0^{\lambda} c(u)\sin\left(\frac{2\pi ku}{\lambda}\right)du \tag{4.6}$$

These expression are used to prove the following result.

Theorem 4.7 *Suppose Assumptions 2.1, 2.4 and 2.6(e) hold. Then keeping λ fixed but letting $n \to \infty$ we have*

$$\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;0)\right] = \frac{1}{\lambda}\sum_{k=-a}^{a} g(\omega_k) A_\lambda\left(\frac{k}{\lambda}\right) + O\left(\frac{1}{n}\left[\sum_{k=-a}^{a}(|k|+1)|g(\omega_k)|\right]\right) \tag{4.7}$$

and if $\sup_{\omega\in\mathbb{R}}|g(\omega)|<\infty$

$$\mathrm{var}\left[\widetilde{Q}_{a,\lambda}(g;0)\right] = \frac{1}{\lambda^2}\sum_{k_1,k_2=-\infty}^{\infty} g\left(\omega_k\right)\overline{g\left(\omega_{k_1}\right)}B_\lambda(k_1,k_2) + O\left(\frac{a^4}{n^2}+\frac{a\log a}{n}+\frac{1}{a}\right) \tag{4.8}$$

where

$$B_\lambda(k_1,k_2) = \begin{cases} 2I(k=0)A_\lambda(0)^2 + I(k\neq 0)[A_\lambda\left(\frac{k}{\lambda}\right)^2 + \frac{1}{\pi^2k^2}B_\lambda(\frac{k}{\lambda})^2] & k_1=k_2(=k) \\ \frac{1}{\pi^2(k_1-k_2)^2}\left[B_\lambda(\frac{k_1}{\lambda})-B_\lambda(\frac{k_2}{\lambda})\right]^2 + \frac{1}{\pi^2(k_1+k_2)^2}\left[B_\lambda(\frac{k_1}{\lambda})+B_\lambda(\frac{k_2}{\lambda})\right]^2 & k_1\neq k_2 \end{cases}$$

and $A_\lambda(\cdot)$ *and* $B_\lambda(\cdot)$ *are defined in (4.6).*

PROOF See Appendix G, (Subba Rao, 2017b). □

From the above result we see that if $\sup_{\omega\in\mathbb{R}}|g(\omega)|<\infty$ and the number of terms in the definition of $\widetilde{Q}_{a,\lambda}(g;0)$, grows at a sufficiently slow rate as $n\to\infty$ then $\widetilde{Q}_{a,\lambda}(g;0)$ is asymptotically an unbiased estimator of $\sum_{k\in\mathbb{Z}}g(\omega_k)A_\lambda(\frac{k}{\lambda})$ and the variance bounded (which fits with the conclusions of Theorem 4.3(ii)).

We now consider a special example of an estimator defined within the frequency domain and later compare it with the Gaussian maximum likelihood estimator for the same problem. More precisely, we consider the case that the covariance of a spatial Gaussian process is $\sigma^2c(u)$ where $c(u)$ is known but σ^2 is unknown and our aim is to estimate σ^2. Let $A_\lambda(\frac{k}{\lambda}) = \int_{-\lambda}^{\lambda}c(u)(1-|u|/\lambda)\exp(2\pi iku/\lambda)du$. Since $\mathrm{E}[|J_n(\omega_k)|^2]=\sigma^2A_\lambda(\frac{k}{\lambda})+O(n^{-1}(\lambda+|k|))$ it seems natural to use $\widehat{\sigma}^2$ as an estimator of σ^2, where

$$\widehat{\sigma}^2 = \frac{1}{a}\sum_{k=1}^{a}\frac{|J_n(\omega_k)|^2}{A_\lambda(\frac{k}{\lambda})}.$$

Note $\widehat{\sigma}^2$ corresponds to the Whittle likelihood estimator of σ^2 when $c(\cdot)$ is known. Using Theorem 4.7, equation (4.7) we have

$$\mathrm{E}[\widehat{\sigma}^2] = \sigma^2 + O\left(\frac{1}{n}\left[\frac{1}{a}\sum_{k=1}^{a}(|k|+1)|k|^2\right]\right).$$

Hence, if a is chosen such that the error above goes to zero as $n\to\infty$ then $\widehat{\sigma}^2$ is asymptotically an unbiased estimator of σ^2. Now we evaluate its variance. By using Corollary G.1, (Subba Rao, 2017b) it can be shown that

$$\mathrm{var}[\widehat{\sigma}^2] = \frac{1}{a^2}\sum_{k=1}^{a}\frac{B_\lambda(k,k)}{A_\lambda(\frac{k}{\lambda})^2} + \frac{2}{a^2}\sum_{k_1=1}^{a}\sum_{k_2=k_1+1}^{a}\frac{B_\lambda(k_1,k_2)}{A_\lambda(\frac{k_1}{\lambda})A_\lambda(\frac{k_2}{\lambda})} + O\left(\frac{a^6}{n^2}+\frac{a^3}{n}\right). \tag{4.9}$$

But here we run into a problem. One would expect that if a is chosen such that $a^3/n \to 0$ as $a \to \infty$ and $n \to \infty$, then $\mathrm{var}[\widehat{\sigma}^2] = O(a^{-1})$. This is true for the first term on the right hand side of the above, *but* it is not necessarily true for the second term, which for most covariances will remain of order $O(1)$ for all a. The problem is that $A_\lambda(\frac{k}{\lambda})$ is a continuous function on the torus $[-1,1]$, thus for large k $A_\lambda(\frac{k}{\lambda})$ decays at the rate k^{-2}. On the other hand $B_\lambda(k_1/\lambda, k_2/\lambda)$ is a function of

$$B_\lambda\left(\frac{k_1}{\lambda}\right) - B_\lambda\left(\frac{k_2}{\lambda}\right) = \frac{1}{2}\int_0^\lambda c(u)\sin\left(\frac{k_1-k_2}{\lambda}\pi u\right)\cos\left(\frac{k_1+k_2}{\lambda}\pi u\right)du.$$

For most covariances $c(0) \neq c(\lambda)$, consequently, using integration by parts we see that $|B_\lambda(\frac{k_1}{\lambda}) - B_\lambda(\frac{k_2}{\lambda})| \sim |k_1-k_2|^{-1}$ (a faster rate of convergence is not possible). Applying these bounds to (4.9) gives $\mathrm{var}[\widehat{\sigma}^2] = O(1)$. Thus even as $n \to \infty$ and $a \to \infty$, $\widehat{\sigma}^2$ is *not* a mean squared consistent estimator of σ^2. One exception is when $c(u) = c(\lambda - u)$ for all $u \in [0, \lambda/2]$, in this case $B_\lambda(\frac{k}{\lambda}) = 0$ for all k and $\mathrm{var}[\widehat{\sigma}^2] \approx a^{-1}$.

Therefore, in general, it seems that we cannot consistently estimate σ^2 using a Fourier domain approach. We conjecture that the only transformation of the data that will consistently estimate σ^2 is a transformation with the eigenfunctions associated with the covariance operator c. In contrast, (Zhang, 2004) and (Zhang & Zimmerman, 2005) showed that if the maximum likelihood were used to estimate σ^2 in a Gaussian random field with covariance $\sigma^2 c(\cdot)$ where $c(\cdot)$ is a known Matern covariance function, then even within the fixed domain framework σ^2 can be consistently estimated. This demonstrates that there exists situations where there are clear gains by working within the likelihood framework (if the correct distribution is specified). However, if the true covariance is $c(u) = c(u;\theta)$ and θ is also unknown, then even within the Gaussian likelihood framework one cannot consistently estimate σ^2 and θ.

5 A studentized $\widetilde{Q}_{a,\lambda}(g;0)$-statistic

The expression for the variance $\widetilde{Q}_{a,\lambda}(g;0)$ given in the examples above, is rather unwieldy and difficult to estimate directly. In this section we describe a simple method for estimating the variance of $\widetilde{Q}_{a,\lambda}(g;0)$ under the assumption the locations are uniformly distributed. This estimator is used to obtain a simple studentized statistic for $\widetilde{Q}_{a,\lambda}(g;0)$. We assume in this section that $\widetilde{Q}_{a,\lambda}(g;0)$ is a real random variable. Our approach is motivated by the method of orthogonal samples for time series proposed in (Subba Rao, 2017a), where the idea is to define a sample which by construction shares some of the properties as the estimator of interest. In this section we show that $\{\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}); \boldsymbol{r} \neq 0\}$ is an orthogonal sample associated with $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$.

We will assume, for ease of presentation that the spatial random field is Gaussian. Using Theorem 4.1 we have $\mathrm{E}[\widetilde{Q}_{a,\lambda}(g;0)] = I\left(g;\frac{a}{\lambda}\right) + o(1)$. Furthermore, noting that the Fourier coefficients of $h_{\lambda}(\cdot)$ for uniformly sampled locations are $\gamma_{\mathbf{0}} = 1$ and $\gamma_{\boldsymbol{j}} = 0$ if $\boldsymbol{j} \neq \mathbf{0}$, using Theorem 4.3 we have

$$\lambda^{d}\mathrm{var}[\widetilde{Q}_{a,\lambda}(g;0)] \quad = \quad C_1 + O(\ell_{\lambda,a,n}).$$

In contrast, we observe that if no elements of the vector $\boldsymbol{r}$ are zero, then by Theorem 4.1 $\mathrm{E}[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})] = O(\prod_{i=1}^{d}[\log\lambda+\log|r_i|]/\lambda^{d})$ (slightly slower rates are obtained when $\boldsymbol{r}$ contains zeros). In other words, $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$ is (asymptotically) estimating zero. On the other hand, using Theorem 4.3 we have

$$\lambda^{d}\mathrm{cov}\left[\Re\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1),\Re\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)\right] \quad = \quad \begin{cases} \frac{1}{2}C_1 + O\left(\ell_{\lambda,a,n} + \frac{\|\boldsymbol{r}\|_1}{\lambda}\right) & \boldsymbol{r}_1 = \boldsymbol{r}_2 (= \boldsymbol{r}) \\ O\left(\ell_{\lambda,a,n}\right) & \boldsymbol{r}_1 \neq \boldsymbol{r}_2, \boldsymbol{r}_1 \neq -\boldsymbol{r}_2 \end{cases}.$$

A similar result holds for $\{\Im\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\}$, furthermore we have $\lambda^{d}\mathrm{cov}[\Re\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1),\Im\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)] = O(\ell_{\lambda,a,n})$.

In summary, if $\|\boldsymbol{r}\|_1$ is not too large, then $\{\Re\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}),\Im\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\}$ are 'near uncorrelated' random variables whose variance is approximately the same as $\widetilde{Q}_{a,\lambda}(g;0)/\sqrt{2}$. This suggests we use $\{\Re\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}),\Im\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r});\boldsymbol{r}\in\mathcal{S}\}$ to estimate $\mathrm{var}[\widetilde{Q}_{a,\lambda}(g;0)]$, where the set $\mathcal{S}$ is defined as

$$\mathcal{S} = \{\boldsymbol{r};\|\boldsymbol{r}\|_1 \leq M, \boldsymbol{r}_1 \neq \boldsymbol{r}_2 \text{ and all elements of } \boldsymbol{r} \text{ are non-zero}\}. \tag{5.1}$$

This leads to the following estimator

$$\widetilde{V}_{\mathcal{S}} = \frac{\lambda^{d}}{2|\mathcal{S}|}\sum_{\boldsymbol{r}\in\mathcal{S}}\left(2|\Re\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})|^2 + 2|\Im\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})|^2\right) = \frac{\lambda^{d}}{|\mathcal{S}|}\sum_{\boldsymbol{r}\in\mathcal{S}}|\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})|^2, \tag{5.2}$$

where $|\mathcal{S}|$ denotes the cardinality of the set $\mathcal{S}$. Note that we specifically select the set $\mathcal{S}$ such that no element $\boldsymbol{r}$ contains zero, this is to ensure that $\mathrm{E}[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})]$ is small and does not induce a large bias in $\widetilde{V}_{\mathcal{S}}$

In the following theorem we obtain a mean squared bound for $\widetilde{V}_{\mathcal{S}}$.

Theorem 5.1 *Let $\widetilde{V}_{\mathcal{S}}$ be defined as in (5.2), where $\mathcal{S}$ is defined in (5.1). Suppose Assumptions 2.1, 2.3 and 2.6(a,b,c) hold and either Assumption 2.5(i) or (ii) holds. Then we have*

$$\mathrm{E}\left(\widetilde{V}_{\mathcal{S}} - \lambda^{d}\mathrm{var}[\widetilde{Q}_{a,\lambda}(g;0)]\right)^2 = O(|\mathcal{S}|^{-1} + |M|\lambda^{-1} + \ell_{\lambda,a,n} + \lambda^{-d}\log^{4d}(a))$$

as $\lambda\to\infty$, $a\to\infty$ and $n\to\infty$ (where $\ell_{a,\lambda,n}$ is defined in (B.4)).

PROOF. See Appendix I, (Subba Rao, 2017b). □

Thus it follows from the above result that if the set $\mathcal{S}$ grows at a rate such that $|M|\lambda^{-1} \to 0$ as $\lambda \to \infty$, then $\widetilde{V}_{\mathcal{S}}$ is a mean square consistent estimator of $\lambda^d \mathrm{var}[\widetilde{Q}_{a,\lambda}(g;0)]$. We use this result to define an asymptotically pivotal statistic. Let

$$T_{\mathcal{S}} = \frac{\lambda^{d/2}[\widetilde{Q}_{a,\lambda}(g;0) - I(g;\frac{a}{\lambda})]}{\sqrt{\widetilde{V}_{\mathcal{S}}}}.$$

By using Theorem E.1, (Subba Rao, 2017b), we can immediately show that for fixed $\mathcal{S}$, $T_{\mathcal{S}} \stackrel{\mathcal{D}}{\to} t_{2|\mathcal{S}|}$ as $\lambda \to \infty$. Therefore $T_{\mathcal{S}}$ is asymptotically pivotal and can be used to construct confidence intervals and test hypothesis about the parameter $I(g;\frac{a}{\lambda})$.

We note that the same approach and studentisation can be used in the case that the random field is non-Gaussian. However, it is trickier to relax the assumption that the locations are uniformly distributed. This is because in the case of a non-uniform design $\mathrm{E}[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})]$ ($\boldsymbol{r} \neq 0$) will not, necessarily, be estimating zero.

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Supplementary material for Statistical inference for spatial statistics defined in the Fourier domain

In this paper we give the proofs for all the results in this paper. And in addition some related results. We summarize the contents below.

- In Appendix A give the expectation calculations. These results are used to prove Lemma 3.1 and Theorems 2.1 4.1 and Theorem 4.2 (expectation $\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\right]$ in the case of uniform and non-uniform sampling).
- In Appendix B we derive an asymptotic expression for $\mathrm{cov}[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1),\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)]$ in the case that the locations are uniformly sampled.
- In Appendix C we use the results in Appendix B to generalize the approximation for $\mathrm{cov}[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1),\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)]$ to general random sampling of the locations.
- In Appendix D we use the results in Appendix B to derive an asymptotic expression for $\mathrm{cov}[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_1),\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_2)]$ (a general frequency grid).
- The central limit theorem using cumulants in proved in Appendix E. Furthermore, Lemmas E.1 and E.2 are used to prove that some terms in the expansion of $\mathrm{cov}[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1),\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)]$ are asymptotically negligible.
- In Appendix F the technical lemmas that are used throughout the appendix are stated and proved.
- In Appendix G results in the fixed domain framework are given. These results are used to prove Remark 2.4, Lemma 3.1 and the results in Section 4.4.
- In Appendix H the sampling properties of $\mathrm{cov}[Q_{a,\Omega,\lambda}(g;\boldsymbol{r}_1),Q_{a,\Omega,\lambda}(g;\boldsymbol{r}_2)]$ are derived.
- In Appendix I additional lemmas are given.
- In Appendix J simulations for the non-parametric spatial covariance estimator defined in Section 2.3.3 are given.

A The first moment of the quadratic form

In this section we mainly focus on the expectation calculations which are used to prove Lemma 3.1(i).

It is clear from Section 3 and the motivation at the start of Section 4.1 that the sinc function plays an important role in the analysis of $\widetilde{Q}_{a,\Omega,\lambda}(g;0)$ and $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$. Therefore we now summarise some of its properties. It is well known that $\frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2}\exp(ix\omega)dx = \mathrm{sinc}(\frac{\lambda\omega}{2})$ and

$$\int_{-\infty}^{\infty}\mathrm{sinc}(u)du = \pi \text{ and } \int_{-\infty}^{\infty}\mathrm{sinc}^2(u)du = \pi. \tag{A.1}$$

We state a well known result that is an important component of the proofs in this paper.

Lemma A.1 *[Orthogonality of the* sinc *function] For all $x \in \mathbb{R}$*

$$\int_{-\infty}^{\infty}\mathrm{sinc}(u)\mathrm{sinc}(u+x)du = \pi\mathrm{sinc}(x) \tag{A.2}$$

and if $s \in \mathbb{Z}/\{0\}$ then

$$\int_{-\infty}^{\infty}\mathrm{sinc}(u)\mathrm{sinc}(u+s\pi)du = 0. \tag{A.3}$$

PROOF In Appendix F, (Subba Rao, 2017b). □

We define $\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r})$ where

$$\begin{aligned}\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}) &= \frac{1}{\Omega^d}\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})\overline{J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}+\boldsymbol{r}})} - \\ &\quad \frac{\lambda^d}{\Omega^d}\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})\frac{1}{n^2}\sum_{j=1}^{n}Z(\boldsymbol{s}_j)^2\exp(-i\boldsymbol{s}_j'\boldsymbol{\omega}_{\Omega,\boldsymbol{r}}).\end{aligned} \tag{A.4}$$

All the bias corrected estimators considered in this paper can be written in the form $\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r})$. To study $\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r})$ we expand the above

$$\begin{aligned}&\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}) \\ = &\ \frac{\lambda^d}{\Omega^d}\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})\frac{1}{n^2}\sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}}^{n} Z(\boldsymbol{s}_{j_2})Z(\boldsymbol{s}_{j_2})\exp(i\boldsymbol{s}_{j_1}'\boldsymbol{\omega}_{\Omega,\boldsymbol{k}} - i\boldsymbol{s}_{j_1}'\boldsymbol{\omega}_{\Omega,\boldsymbol{k}+\boldsymbol{r}}).\end{aligned} \tag{A.5}$$

In the lemma below we apply the ideas outlined at the start of Section 4.1 and the lemma above to study $\mathrm{E}[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r})]$.

Lemma A.2 *Suppose Assumptions 2.1(i), Assumptions 2.5(i) or (ii) and 2.6(b,c) hold. Let $I(g;\cdot)$ and $\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r})$ be defined as in (1.3) and (A.4)*

(i) If Assumption 2.3 (locations follow a uniform distribution) holds then for all $\boldsymbol{r} \in \mathbb{Z}^d$

$$\mathrm{E}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r})\right] = I\left(g;\frac{a}{\Omega}\right)\mathrm{Sinc}\left(\frac{\lambda\pi\boldsymbol{r}}{\Omega}\right) + O\left(\frac{\log\lambda + \log\left(1+\frac{\lambda\|\boldsymbol{r}\|_1}{\Omega}\right)}{\lambda} + \frac{1}{\Omega} + \frac{1}{n}\right).$$

(ii) If Assumption 2.2 (locations follow any general distribution) holds then

$$
\begin{aligned}
& \mathrm{E}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r})\right] \\
= & \sum_{\boldsymbol{j}_1,\boldsymbol{j}_2\in\mathbb{Z}^d} \gamma_{\boldsymbol{j}_1}\gamma_{\boldsymbol{j}_2}\mathrm{Sinc}\left(\frac{\pi\lambda\boldsymbol{r}}{\Omega}-(\boldsymbol{j}_1+\boldsymbol{j}_2)\pi\right)\frac{1}{(2\pi)^d}\int_{[-2\pi a/\Omega,2\pi a/\Omega]^d} g(\boldsymbol{\omega})f\left(-\boldsymbol{\omega}-\boldsymbol{\omega}_{j_1}\right)d\boldsymbol{\omega} \\
& +O\left(\frac{1}{\Omega}+\frac{\log\lambda+\log\left(1+\frac{\lambda\|\boldsymbol{r}\|_1}{\Omega}\right)}{\lambda}+\frac{1}{n}\right).
\end{aligned}
$$

(iii) If Assumption 2.2 holds then

$$
\mathrm{E}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;0)\right] = I\left(g;\frac{a}{\Omega}\right)\sum_{\boldsymbol{j}\in\mathbb{Z}^d}|\gamma_{\boldsymbol{j}}|^2+O\left(\frac{\log\lambda}{\lambda}+\frac{1}{\Omega}+\frac{1}{n}\right).
$$

PROOF. To simplify notation, we let $d=1$. To prove (i) we take expectation of (A.5), replace the spatial covariance with its Fourier (spectral) representation

$$
\mathrm{E}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;r)\right] = \frac{c_2\lambda}{2\pi\Omega}\sum_{\boldsymbol{k}=-a}^{a} g(\omega_{\Omega,k})\int_{\mathbb{R}} f(\omega)\mathrm{sinc}\left(\frac{\lambda}{2}\left[\omega+\omega_{\Omega,k}\right]\right)\mathrm{Sinc}\left(\frac{\lambda}{2}\left[\omega+\omega_{\Omega,k+r}\right]\right)d\omega.
$$

We make a change of variables $y=\frac{\lambda}{2}\left[\omega+\omega_{\Omega,k}\right]$

$$
\mathrm{E}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;r)\right] = \frac{c_2}{\pi}\int_{\mathbb{R}}\mathrm{sinc}(y)\mathrm{sinc}\left(y+\frac{\lambda\pi r}{\Omega}\right)\left[\frac{1}{\Omega}\sum_{k=-a}^{a} g(\omega_{\Omega,k})f\left(\frac{2y}{\lambda}-\omega_{\Omega,k}\right)\right]dy.
$$

Replacing summand with integral and $f(\frac{2y}{\lambda}-\omega_{\Omega,k})$ with $f(-\omega_{\Omega,k})$ (and using Lemma F.2, equation (F.5)) we have

$$
\begin{aligned}
\mathrm{E}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;r)\right] &= \frac{2c_2}{\pi}\int_{\mathbb{R}}\mathrm{sinc}(y)\mathrm{sinc}\left(y+\frac{\lambda\pi r}{\Omega}\right)\int_{-2\pi a/\Omega}^{2\pi a/\Omega} g(\omega)f\left(\frac{2y}{\lambda}-\omega\right)d\omega dy+O\left(\frac{1}{\Omega}\right). \\
&= \frac{c_2}{\pi}\int_{\mathbb{R}^d}\mathrm{sinc}(y)\mathrm{sinc}\left(y+\frac{\lambda\pi r}{\Omega}\right)dy\int_{-2\pi a/\Omega}^{2\pi a/\Omega} g(\omega)f(\omega)d\omega \\
&\quad +O\left(\frac{\log\lambda+\log(1+|\lambda r/\Omega|)}{\lambda}+\frac{1}{\Omega}\right).
\end{aligned}
$$

Finally by using Lemma A.1 and replacing $c_2=n(n-1)/n^2$ with one, we have the expression in (i).

We now prove (ii), which is similar to the proof of (i) but uses the Fourier expansion of the location density $h(\cdot)$. Taking expectation

$$
\begin{aligned}
& \mathrm{E}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;r)\right] \\
= & \frac{c_2\lambda}{2\pi\Omega}\sum_{j_1,j_2=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\sum_{k=-a}^{a} g(\omega_{\Omega,k})\int_{-\infty}^{\infty} f(\omega)\mathrm{sinc}\left(\frac{\lambda}{2}\left(\omega+\omega_{\Omega,k}+\omega_{\lambda,j_1}\right)\right)\times \\
& \mathrm{sinc}\left(\frac{\lambda}{2}\left(\omega+\omega_{\Omega,k}-\omega_{\lambda,j_1}\right)\right)d\omega.
\end{aligned}
$$

Using the change of variables $y = \frac{\lambda}{2}(\omega + \omega_{\Omega,k} + \omega_{\lambda,j_1})$ we obtain

$$\begin{aligned}
& \mathrm{E}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;r)\right] \\
= & \frac{c_2}{\pi\Omega} \sum_{j_1,j_2=-\infty}^{\infty} \gamma_{j_1}\gamma_{j_2} \sum_{k=-a}^{a} g(\omega_{\Omega,k}) \int_{-\infty}^{\infty} f\left(\frac{2y}{\lambda} - \omega_{\lambda,j_1} - \omega_{\Omega,k}\right) \times \\
& \mathrm{sinc}(y)\mathrm{sinc}\left(y + \frac{\lambda\pi r}{\Omega} - (j_1+j_2)\pi\right) dy.
\end{aligned}$$

Replacing sum with an integral and using Lemma I.1(ii) gives

$$\begin{aligned}
& \mathrm{E}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;r)\right] \\
= & \frac{c_2}{2\pi^2} \sum_{j_1,j_2=-\infty}^{\infty} \gamma_{j_1}\gamma_{j_2} \int_{-2\pi a/\Omega}^{2\pi a/\Omega} g(\omega) \int_{-\infty}^{\infty} f\left(\frac{2y}{\lambda} - \omega_{\lambda,j_1} - \omega\right) \times \\
& \mathrm{sinc}(y)\mathrm{sinc}\left(y + \frac{\lambda\pi r}{\Omega} - (j_1+j_2)\pi\right) dy d\omega + O\left(\frac{1}{\Omega}\right).
\end{aligned}$$

Next, replacing $f(\frac{2y}{\lambda} - \omega - \omega_{\lambda,j_1})$ with $f(-\omega - \omega_{\lambda,j_1})$ gives

$$\begin{aligned}
& \mathrm{E}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;r)\right] \\
= & \frac{c_2}{2\pi^2} \sum_{j_1,j_2=-\infty}^{\infty} \gamma_{j_1}\gamma_{j_2} \int_{-2\pi a/\Omega}^{2\pi a/\Omega} g(\omega) f\left(-\omega - \omega_{\lambda,j_1}\right) d\omega \int_{-\infty}^{\infty} \mathrm{sinc}(y)\mathrm{sinc}\left(y + \frac{\pi\lambda r}{\Omega} - (j_1+j_2)\pi\right) dy \\
& + R_n + O\left(\frac{1}{\Omega}\right),
\end{aligned}$$

where

$$\begin{aligned}
R_n = & \frac{c_2}{2\pi^2} \sum_{j_1,j_2=-\infty}^{\infty} \gamma_{j_1}\gamma_{j_2} \int_{-2\pi a/\Omega}^{2\pi a/\Omega} g(\omega) \left[f\left(\frac{2y}{\lambda} - \omega - \omega_{\lambda,j_1}\right) - f(-\omega - \omega_{\lambda,j_1})\right] \\
& \times \int_{-\infty}^{\infty} \mathrm{sinc}(y)\mathrm{sinc}\left(y + \frac{\pi\lambda r}{\Omega} - (j_1+j_2)\pi\right) dy d\omega.
\end{aligned}$$

By using Lemma F.2 we have

$$\begin{aligned}
|R_n| \leq & C \sum_{j_1,j_2=-\infty}^{\infty} |\gamma_{j_1}\gamma_{j_2}| \frac{\log\lambda + \log(1 + |\lambda r/\Omega| + \log|j_1| + \log|j_2|)}{\lambda} \\
= & O\left(\frac{\log\lambda + \log(1 + |\lambda r/\Omega|)}{\lambda}\right).
\end{aligned}$$

Using Lemma A.1 we have

$$
\begin{aligned}
& \mathrm{E}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;r)\right] \\
= & \frac{c_2}{2\pi^2}\sum_{j_1,j_2=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\int_{-2\pi a/\Omega}^{2\pi a/\Omega} g(\omega)f\left(-\omega-\omega_{j_1}\right)d\omega\int_{-\infty}^{\infty}\mathrm{sinc}(y)\mathrm{sinc}\left(y+\frac{\pi\lambda r}{\Omega}-(j_1+j_2)\pi\right)dy \\
& +O\left(\frac{1}{\Omega}+\frac{\log\lambda+\log(1+|\lambda r/\Omega|)}{\lambda}\right) \\
= & \frac{1}{2\pi}\sum_{j_1,j_2=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\mathrm{sinc}\left(\frac{\pi\lambda r}{\Omega}-(j_1+j_2)\pi\right)\int_{-2\pi a/\Omega}^{2\pi a/\Omega} g(\omega)f\left(-\omega-\omega_{j_1}\right)d\omega + \\
& O\left(\frac{1}{\Omega}+\frac{\log\lambda+\log(1+|\lambda r/\Omega|)}{\lambda}+\frac{1}{n}\right).
\end{aligned}
$$

This proves (ii). To prove (iii) we note that when $r=0$ (using Lemma A.1) $\mathrm{sinc}\left(\frac{\pi\lambda r}{\Omega}-(j_1+j_2)\pi\right)=0$ unless $j_1=-j_2$. This means that (ii) reduces to

$$
\begin{aligned}
\mathrm{E}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;0)\right] &= \frac{1}{2\pi}\sum_{j=-\infty}^{\infty}|\gamma_j|^2\int_{-2\pi a/\Omega}^{2\pi a/\Omega} g(\omega)f(\omega+\omega_j)d\omega+O\left(\frac{\log\lambda}{\lambda}+\frac{1}{\Omega}+\frac{1}{n}\right) \\
&= \frac{1}{2\pi}\sum_{j=-\infty}^{\infty}|\gamma_j|^2\int_{-2\pi a/\Omega}^{2\pi a/\Omega} g(\omega)f(\omega)d\omega+O\left(\frac{\log\lambda}{\lambda}+\frac{1}{\Omega}+\frac{1}{n}\right),
\end{aligned}
$$

where in the last bound of the above we use that $f(\cdot)$ is Lipschitz continuous and $\sum_{j\in\mathbb{Z}}|j||\gamma_j|^2<\infty$. Thus we obtain (iii). □

PROOF of Theorem 2.1 We prove the result for $d=1$. First we prove (2.3). By expanding $\mathrm{cov}\left[J_n(\omega_{k_1}),J_n(\omega_{k_2})\right]$ (and assuming that $\mathrm{E}[Z(s)|s]=0$) we have

$$
\begin{aligned}
& \mathrm{cov}\left[J_n(\omega_{k_1}),J_n(\omega_{k_2})\right] \\
= & c_2\lambda\mathrm{E}\left[Z(s_1)Z(s_2)\exp(is_1\omega_{k_1}-is_2\omega_2)\right]+n^{-1}\lambda\mathrm{E}\left[Z(s)^2\exp(is(\omega_{k_1}-\omega_{k_2}))\right] \\
= & \frac{c_2}{\lambda}\int_{[-\lambda/2,\lambda/2]^2} c(s_1-s_2)e^{is(\omega_{k_1}-\omega_{k_2}}h\left(\frac{s_1}{\lambda}\right)h\left(\frac{s_2}{\lambda}\right)ds_1ds_2 \\
& +\frac{c(0)}{n}\int_{-\lambda/2}^{\lambda/2}h\left(\frac{s}{\lambda}\right)e^{is(\omega_{k_1}-\omega_{k_2})}ds,
\end{aligned}
\tag{A.6}
$$

where $c_2=n(n-1)/2$. Replacing $h(s/\lambda)$ with its Fourier representation $h(s/\lambda)=\sum_{j=-\infty}^{\infty}\gamma_j e^{i2\pi js/\lambda}$

$$
\begin{aligned}
& \mathrm{cov}\left[J_n(\omega_{k_1}),J_n(\omega_{k_2})\right] \\
= & \frac{c_2}{\lambda}\sum_{j_1,j_2=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\int_{-\lambda/2}^{\lambda/2}\int_{-\lambda/2}^{\lambda/2}c(s_1-s_2)e^{is_1\omega_{j_1}}e^{is_2\omega_{j_2}}e^{is_1\omega_{k_1}-is_2\omega_{k_2}}ds_1ds_2 \\
& +\frac{c(0)\gamma_{k_2-k_1}\lambda}{n}.
\end{aligned}
$$

Changing variables in the integral (using $t = s_1 - s_2$ and $s = s_2$), then the limits of the integral gives

$$\begin{aligned}\mathrm{cov}\left[J_n(\omega_{k_1}), J_n(\omega_{k_2})\right] &= \frac{c_2}{\lambda}\sum_{j_1,j_2=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\left(\int_{-\lambda/2}^{\lambda/2} e^{-is(\omega_{k_2}-\omega_{j_2}-\omega_{j_1}-\omega_{k_1})}ds\right) \\ &\quad\times\left(\int_{-\lambda/2}^{\lambda/2} c(t)e^{it(\omega_{j_1}+\omega_{k_1})}dt\right) \\ &\quad+\frac{c(0)\gamma_{k_2-k_1}\lambda}{n} + O\left(\frac{1}{\lambda}\right),\end{aligned}$$

where to obtain the remainder $O(\frac{1}{\lambda})$ we use that $\sum_{j=-\infty}^{\infty}|\gamma_j| < \infty$ (note that a similar change in the limits of an integral is given in the proof of Theorem 2.1, (Bandyopadhyay & Subba Rao, 2017)). Next, by using the identity

$$\int_{-\lambda/2}^{\lambda/2} e^{-is(\omega_{k_2}-\omega_{j_2}-\omega_{j_1}-\omega_{k_1})}ds = \begin{cases} 0 & k_2 - j_2 \neq k_1 + j_1 \\ \lambda & k_2 - j_2 = k_1 + j_1 \end{cases}$$

we have

$$\begin{aligned}\mathrm{cov}\left[J_n(\omega_{k_1}), J_n(\omega_{k_2})\right] &= c_2\sum_{j=-\infty}^{\infty}\gamma_j\gamma_{k_2-k_1-j}\int_{-\lambda/2}^{\lambda/2} c(t)e^{it(\omega_{j_1}+\omega_{k_1})}dtds + \frac{c(0)\gamma_{k_1-k_2}\lambda}{n} + O(\lambda^{-1}) \\ &= \sum_{j=-\infty}^{\infty}\gamma_j\gamma_{k_2-k_1-j}f(\omega_{k_1+j}) + \frac{c(0)\gamma_{k_1-k_2}\lambda}{n} + O\left(\frac{1}{\lambda}+\frac{1}{n}\right).\end{aligned}$$

Finally, we replace $f(\omega_{k_1+j})$ with $f(\omega_{k_1})$ and use the Lipschitz continiuity of $f(\cdot)$ to give

$$\left|\sum_{j=-\infty}^{\infty}\gamma_j\gamma_{k_2-k_1-j}\left[f(\omega_{k_1}) - f(\omega_{k_1+j})\right]\right| \leq \frac{C}{\lambda}\sum_{j=-\infty}^{\infty}|j|\cdot|\gamma_j\gamma_{k_2-k_1-j}| = O\left(\frac{1}{\lambda}\right),$$

where the last line follows from $|\gamma_j| \leq C|j|^{-(1+\delta)}I(j \neq 0)$. Altogether, this gives

$$\mathrm{cov}\left[J_n(\omega_{k_1}), J_n(\omega_{k_2})\right] = f(\omega_{k_1})\sum_{j=-\infty}^{\infty}\gamma_j\gamma_{k_2-k_1-j} + \frac{c(0)\gamma_{k_2-k_1}\lambda}{n} + O\left(\frac{1}{\lambda}\right).$$

This completes the proof for $d = 1$, the proof for $d > 1$ is the same.

To prove (2.4) we use a similar expansion to that given in (A.6)

$$\mathrm{cov}\left[J_n(\omega_{\Omega,k_1}), J_n(\omega_{\Omega,k_2})\right] = A_1 + A_2$$

where

$$\begin{aligned}A_1 &= \frac{c_2}{\lambda}\sum_{j_1,j_2=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\int_{-\lambda/2}^{\lambda/2}\int_{-\lambda/2}^{\lambda/2} c(s_1-s_2)e^{is_1\omega_{j_1}}e^{is_2\omega_{j_2}}e^{is_1\omega_{\Omega,k_1}-is_2\omega_{\Omega,k_2}}ds_1ds_2 \\ A_2 &= \frac{c(0)}{n}\int_{-\lambda/2}^{\lambda/2} h\left(\frac{s}{\lambda}\right)e^{is(\omega_{\Omega,k_1-k_2})}ds.\end{aligned}$$

We see that the second term in (2.4) is A_2, which is of order $O(\frac{\lambda}{n})$. We now show that the first term in (2.4) is an approximation of A_1. Representing the spatial covariance and density in terms of their Fourier representations we have

$$
\begin{aligned}
A_1 &= \frac{\lambda c_2}{(2\pi)} \sum_{j_1,j_2=-\infty}^{\infty} \gamma_{j_1}\gamma_{j_2} \int_{\mathbb{R}} f(\omega) \frac{1}{\lambda^2} \int_{-\lambda/2}^{\lambda/2}\int_{-\lambda/2}^{\lambda/2} e^{is_1(\omega+\omega_{j_1}+\omega_{\Omega,k_1})} e^{is_2(\omega_{j_2}-\omega-\omega_{\Omega,k_2})} ds_1 ds_2 \\
&= \frac{\lambda c_2}{(2\pi)} \sum_{j_1,j_2=-\infty}^{\infty} \gamma_{j_1}\gamma_{j_2} \int_{\mathbb{R}} f(\omega) \mathrm{sinc}\left(\frac{\lambda}{2}(\omega+\omega_{j_1}+\omega_{\Omega,k_1})\right) \mathrm{sinc}\left(\frac{\lambda}{2}(\omega-\omega_{j_2}+\omega_{\Omega,k_2})\right) d\omega \\
&= \frac{1}{\pi} \sum_{j_1,j_2=-\infty}^{\infty} \gamma_{j_1}\gamma_{j_2} \int_{\mathbb{R}} f\left(\frac{2u}{\lambda}-\omega_{\Omega,k_1}-\omega_{j_1}\right) \mathrm{sinc}(u)\, \mathrm{sinc}\left(u-(j_1+j_2)\pi+\frac{\lambda}{\Omega}(k_2-k_1)\pi\right) du.
\end{aligned}
$$

We replace in the above integral $f\left(\frac{2u}{\lambda}-\omega_{\Omega,k_1}-\omega_{j_1}\right)$ with $f(-\omega_{\Omega,k_1}-\omega_{j_1}) = f(\omega_{\Omega,k_1}+\omega_{j_1})$. By using Lemma F.2 the remainder is bounded by $O(\log\lambda/\lambda)$, this gives

$$
\begin{aligned}
A_1 &= \frac{1}{\pi} \sum_{j_1,j_2=-\infty}^{\infty} \gamma_{j_1}\gamma_{j_2} f(\omega_{\Omega,k_1}+\omega_{j_1}) \int_{\mathbb{R}} \mathrm{sinc}(u)\, \mathrm{sinc}\left(u-(j_1+j_2)\pi+\frac{\lambda}{\Omega}(k_2-k_1)\pi\right) du \\
&\quad + O\left(\frac{\log\lambda}{\lambda}\right) \\
&= \sum_{j_1,j_2=-\infty}^{\infty} \gamma_{j_1}\gamma_{j_2} f(\omega_{\Omega,k_1}+\omega_{j_1})\, \mathrm{sinc}\left((j_1+j_2)\pi+\frac{\lambda}{\Omega}(k_1-k_2)\pi\right) + O\left(\frac{\log\lambda}{\lambda}\right).
\end{aligned}
$$

Finally, under Assumption 2.6(c) (which gives the Lipschitz continuity of f) we replace $f(\omega_{\Omega,k_1}+\omega_{j_1})$ with $f(\omega_{\Omega,k_1})$ to give

$$
\begin{aligned}
\mathrm{cov}\left[J_n(\omega_{\Omega,k_1}), J_n(\omega_{\Omega,k_2})\right] &= f(\omega_{\Omega,k_1}) \sum_{j_1,j_2=-\infty}^{\infty} \gamma_{j_1}\gamma_{j_2} \mathrm{sinc}\left((j_1+j_2)\pi+\frac{\lambda}{\Omega}(k_1-k_2)\pi\right) \\
&\quad + \frac{c(0)}{n} \int_{-\lambda/2}^{\lambda/2} h\left(\frac{s}{\lambda}\right) e^{is(\omega_{\Omega,k_1-k_2})} ds + O\left(\frac{\log\lambda}{\lambda}\right),
\end{aligned}
$$

thus giving the required result. □

PROOF of Lemma 3.1 Taking expectation of (1.2) gives

$$
\mathrm{E}[Q_{a,\Omega,\lambda}(g_\theta;0)] = \frac{1}{\Omega^d} \sum_{\boldsymbol{k}=-a}^{a} g_\theta(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}}) \mathrm{E}[|J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})|^2]. \tag{A.7}
$$

To prove equation (3.1) we substitute Theorem G.1(iii), (Subba Rao, 2017b), into the above which immediately gives (3.1). To prove equation (3.2) we note that

$$
\mathrm{E}[Q_{a,\Omega,\lambda}(g_\theta;0)] = \mathrm{E}[\widetilde{Q}_{a,\Omega,\lambda}(g_\theta;0)] + \frac{\lambda^d}{\Omega^d n} \sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}}) \frac{1}{n} \sum_{j=1}^{n} \underbrace{\mathrm{E}[Z(\boldsymbol{s}_j)^2]}_{=c(0)}.
$$

An expression for $\mathrm{E}[\widetilde{Q}_{a,\Omega,\lambda}(g_\theta;0)]$ is given in Lemma A.2(iii), (Subba Rao, 2017b), where we use that $\gamma_0 = 1$ and $\gamma_{\boldsymbol{k}} = 0$ for $\boldsymbol{k} \neq 0$. Substituting this expression into the above immediately gives equation (3.2) . $\square$

PROOF of Theorem 4.1. We first prove (i). By using the same proof used to prove (Bandyopadhyay & Subba Rao, 2017), Theorem 1 we can show that

$$\mathrm{E}\left[J_n(\boldsymbol{\omega}_{\boldsymbol{k}_1})\overline{J_n(\boldsymbol{\omega}_{\boldsymbol{k}_2})} - \frac{\lambda^d}{n^2}\sum_{j=1}^{n} Z(\boldsymbol{s}_j)^2 e^{i\boldsymbol{s}_j'(\boldsymbol{\omega}_{\boldsymbol{k}_1}-\boldsymbol{\omega}_{\boldsymbol{k}_2})}\right] = \begin{cases} f(\boldsymbol{\omega}_k) + O(\frac{1}{\lambda}) & \boldsymbol{k}_1 = \boldsymbol{k}_2 (= \boldsymbol{k}), \\ O(\frac{1}{\lambda^{d-b}}) & \boldsymbol{k}_1 - \boldsymbol{k}_2 \neq 0. \end{cases},$$

where $b = b(\boldsymbol{k}_1 - \boldsymbol{k}_2)$ denotes the number of zero elements in the vector $\boldsymbol{r}$. Therefore, since $a = O(\lambda)$, by taking expectations of $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$ we use the above to give

$$\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\right] = \begin{cases} \frac{1}{\lambda^d}\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega}_{\boldsymbol{k}})f(\boldsymbol{\omega}_{\boldsymbol{k}}) + O(\frac{1}{\lambda}) & \boldsymbol{r} = 0 \\ O(\frac{1}{\lambda^{d-b}}) & \boldsymbol{r} \neq 0 \end{cases}.$$

Therefore, by replacing the summand with the integral we obtain (4.2).

The above method cannot be used to prove (ii) since $a/\lambda \to \infty$, this leads to bounds which may not converge. Therefore, as discussed in Section 4.1 we consider an alternative approach. To do this we expand $\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$ as a quadratic form to give

$$\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}) = \frac{1}{n^2}\sum_{\substack{j_1,j_2=1 \\ j_1\neq j_2}}^{n}\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega}_{\boldsymbol{k}})Z(\boldsymbol{s}_{j_1})Z(\boldsymbol{s}_{j_2})\exp(i\boldsymbol{\omega}_k'(\boldsymbol{s}_{j_1}-\boldsymbol{s}_{j_2}))\exp(-i\boldsymbol{\omega}_{\boldsymbol{r}}'\boldsymbol{s}_{j_2}).$$

Taking expectation gives

$$\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\right] = c_2\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega}_{\boldsymbol{k}})\mathrm{E}\left[c(\boldsymbol{s}_1-\boldsymbol{s}_2)\exp(i\boldsymbol{\omega}_k'(\boldsymbol{s}_1-\boldsymbol{s}_2) - i\boldsymbol{s}_2'\boldsymbol{\omega}_{\boldsymbol{r}})\right]$$

where $c_2 = n(n-1)/2$. In the case that $d = 1$ the above reduces to

$$\begin{aligned}\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;r)\right] &= c_2\sum_{k=-a}^{a} g(\omega_k)\mathrm{E}\left[c(s_1-s_2)\exp(i\omega_k(s_1-s_2) - is_2\omega_r)\right] \\ &= \frac{c_2}{\lambda^2}\sum_{k=-a}^{a} g(\omega_k)\int_{-\lambda/2}^{\lambda/2}\int_{-\lambda/2}^{\lambda/2} c(s_1-s_2)\exp(i\omega_k(s_1-s_2) - is_2\omega_r)ds_1ds_2.\end{aligned} \tag{A.8}$$

Replacing $c(s_1 - s_2)$ with the Fourier representation of the covariance function gives

$$\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;r)\right] = \frac{c_2}{2\pi}\sum_{k=-a}^{a} g(\omega_k)\int_{-\infty}^{\infty} f(\omega)\mathrm{sinc}\left(\frac{\lambda\omega}{2} + k\pi\right)\mathrm{sinc}\left(\frac{\lambda\omega}{2} + (k+r)\pi\right)d\omega.$$

By a change of variables $y = \lambda\omega/2 + k\pi$ and replacing the sum with an integral we have

$$\begin{aligned}
\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;r)\right] &= \frac{c_2}{\pi\lambda}\sum_{k=-a}^{a} g(\omega_k)\int_{-\infty}^{\infty} f(\frac{2y}{\lambda}-\omega_k)\mathrm{sinc}(y)\mathrm{sinc}(y+r\pi)dy \\
&= \frac{c_2}{\pi}\int_{-\infty}^{\infty}\mathrm{sinc}(y)\mathrm{sinc}(y+r\pi)\left(\frac{1}{\lambda}\sum_{k=-a}^{a} g(\omega_k)f(\frac{2y}{\lambda}-\omega_k)\right)dy \\
&= \frac{c_2}{2\pi^2}\int_{-\infty}^{\infty}\mathrm{sinc}(y)\mathrm{sinc}(y+r\pi)\int_{-2\pi a/\lambda}^{2\pi a/\lambda} g(u)f(\frac{2y}{\lambda}-u)dudy + O\left(\frac{1}{\lambda}\right),
\end{aligned}$$

where the $O(\lambda^{-1})$ comes from Lemma I.1(ii) in (Subba Rao, 2017b). Replacing $f(\frac{2y}{\lambda}-u)$ with $f(-u)$ gives

$$\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;r)\right] = \frac{c_2}{2\pi^2}\int_{-\infty}^{\infty}\mathrm{sinc}(y)\mathrm{sinc}(y+r\pi)\int_{-2\pi a/\lambda}^{2\pi a/\lambda} g(u)f(-u)dudy + R_n + O(\lambda^{-1}),$$

where

$$R_n = \frac{c_2}{2\pi^2}\int_{-\infty}^{\infty}\mathrm{sinc}(y)\mathrm{sinc}(y+r\pi)\left(\int_{-2\pi a/\lambda}^{2\pi a/\lambda} g(u)\left(f(\frac{2y}{\lambda}-u)-f(-u)\right)dudy.\right.$$

By using Lemma F.2 in (Subba Rao, 2017b), we have $|R_n| = O(\frac{\log\lambda+\log(1+|r|)}{\lambda})$. Therefore, by using Lemma A.1, replacing c_2 with one (which leads to the error $O(n^{-1})$) and (A.1) we have

$$\begin{aligned}
&\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;r)\right] \\
&= \frac{c_2}{2\pi^2}\int_{-\infty}^{\infty}\mathrm{sinc}(y)\mathrm{sinc}(y+r\pi)\int_{-2\pi a/\lambda}^{2\pi a/\lambda} g(u)f(-u)dudy + O\left(\frac{\log\lambda+\log(1+|r|)}{\lambda}\right) \\
&= \frac{I(r=0)}{2\pi}\int_{-2\pi a/\lambda}^{2\pi a/\lambda} g(u)f(u)du + O\left(\frac{1}{\lambda}+\frac{\log\lambda+\log(1+|r|)}{\lambda}\right),
\end{aligned}$$

which gives (4.3) in the case $d = 1$.

To prove the result for $d > 1$, we will only consider the case $d = 2$, as it highlights the difference from the $d = 1$ case. By substituting the spectral representation into (4.1) we have

$$\begin{aligned}
&\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;(r_1,r_2))\right] \\
&= \frac{c_2}{\pi^2\lambda^2}\sum_{k_1,k_2=-a}^{a} g(\omega_{k_1},\omega_{k_2})\int_{\mathbb{R}^2} f\left(\frac{2u_1}{\lambda}-\omega_{k_1},\frac{2u_2}{\lambda}-\omega_{k_2}\right)\times \\
&\quad \mathrm{sinc}(u_1)\mathrm{sinc}(u_1+r_1\pi)\mathrm{sinc}(u_2)\mathrm{sinc}(u_2+r_2\pi)du_1du_2.
\end{aligned}$$

In the case that either $r_1 = 0$ or $r_2 = 0$ we use the same proof given in the case $d = 1$ to give

$$\begin{aligned} \mathrm{E}[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})] &= \frac{1}{(2\pi)^2\pi^2\lambda^2}\int_{[-a/\lambda,a/\lambda]^2} g(\omega_1,\omega_2)f(\omega_1,\omega_2) \\ &\quad \int_{\mathbb{R}^d}\mathrm{sinc}(u_1)\mathrm{sinc}(u_1+r_1\pi)\mathrm{sinc}(u_2)\mathrm{sinc}(u_2+r_2\pi)du_1du_2 + R_n, \end{aligned}$$

where $|R_n| = O(\frac{\log\lambda+\log(1+\|\boldsymbol{r}\|_1)}{\lambda} + n^{-1})$, which gives the desired result. However, in the case that both $r_1 \neq 0$ and $r_2 \neq 0$, we can use Lemma F.2, equation (F.8), (Subba Rao, 2017b) to obtain

$$\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;(r_1,r_2))\right] = O\left(\frac{\prod_{i=1}^{2}[\log\lambda + \log|r_i|]}{\lambda^2}\right)$$

thus we obtain the faster rate of convergence.

It is straightforward to generalize these arguments to $d > 2$. □

PROOF of Theorem 4.2 We prove the result for the case $d = 1$. Using the same method used to prove Theorem 4.1 (see the arguments at the start of Section 4.1) we obtain

$$\begin{aligned} &\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;r)\right] \\ &= \frac{c_2}{2\pi}\sum_{j_1,j_2=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\sum_{k=-a}^{a}g(\omega_k)\int_{-\infty}^{\infty}f(\omega)\mathrm{sinc}\left(\frac{\lambda\omega}{2}+(k+j_1)\pi\right)\mathrm{sinc}\left(\frac{\lambda\omega}{2}+(k+r-j_2)\pi\right)d\omega. \end{aligned}$$

By the change of variables $y = \frac{\lambda\omega}{2} + (k+j_1)\pi$ we obtain

$$\begin{aligned} &\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;r)\right] \\ &= \frac{c_2}{\lambda\pi}\sum_{j_1,j_2=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\sum_{k=-a}^{a}g(\omega_k)\int_{-\infty}^{\infty}f\left(\frac{2y}{\lambda}-\omega_{k+j_1}\right)\mathrm{sinc}(y)\mathrm{sinc}(y+(r-j_1-j_2)\pi)dy. \end{aligned}$$

Replacing sum with an integral and using Lemma I.1(ii) gives

$$\begin{aligned} &\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;r)\right] \\ &= \frac{c_2}{2\pi^2}\sum_{j_1,j_2=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\int_{-2\pi a/\lambda}^{2\pi a/\lambda}g(\omega)f\left(\frac{2y}{\lambda}-\omega-\omega_{j_1}\right)d\omega\int_{-\infty}^{\infty}\mathrm{sinc}(y)\mathrm{sinc}(y+(r-j_1-j_2)\pi)dy \\ &\quad +O\left(\frac{1}{\lambda}\right). \end{aligned}$$

Next, replacing $f(\frac{2y}{\lambda}-\omega-\omega_{j_1})$ with $f(-\omega-\omega_{j_1})$ we have

$$\begin{aligned} &\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;r)\right] \\ &= \frac{c_2}{2\pi^2}\sum_{j_1,j_2=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\int_{-2\pi a/\lambda}^{2\pi a/\lambda}g(\omega)f(-\omega-\omega_{j_1})d\omega \\ &\quad \times\int_{-\infty}^{\infty}\mathrm{sinc}(y)\mathrm{sinc}(y+(r-j_1-j_2)\pi)dy + R_n + O\left(\frac{1}{\lambda}\right), \end{aligned}$$

where

$$R_n = \frac{c_2}{2\pi^2}\sum_{j_1,j_2=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\int_{-2\pi a/\lambda}^{2\pi a/\lambda} g(\omega)\left[f\left(\frac{2y}{\lambda}-\omega-\omega_{j_1}\right)-f(-\omega-\omega_{j_1})\right] \times \int_{-\infty}^{\infty}\mathrm{sinc}(y)\mathrm{sinc}(y+(r-j_1-j_2)\pi)dyd\omega.$$

By using Lemma F.2 we have

$$\begin{aligned}|R_n| &\leq C\sum_{j_1,j_2=-\infty}^{\infty}|\gamma_{j_1}|\cdot|\gamma_{j_2}|\frac{\log\lambda+\log(1+|r-j_1-j_2|)}{\lambda}\\ &\leq C\sum_{j_1,j_2=-\infty}^{\infty}|\gamma_{j_1}|\cdot|\gamma_{j_2}|\frac{\log\lambda+\log(1+|r|)+\log(1+|j_1|)+\log(1+|j_2|)}{\lambda},\end{aligned}$$

noting that C is a generic constant that changes between inequalities. This gives

$$\begin{aligned}&\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;r)\right]\\ &= \frac{c_2}{2\pi^2}\sum_{j_1,j_2=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\int_{-2\pi a/\lambda}^{2\pi a/\lambda}g(\omega)f(-\omega-\omega_{j_1})d\omega\int_{-\infty}^{\infty}\mathrm{sinc}(y)\mathrm{sinc}(y+(r-j_1-j_2)\pi)dy+\\ &\quad O\left(\frac{\log\lambda+\log(1+|r|)}{\lambda}\right).\end{aligned}$$

Finally, by the orthogonality of the sinc function at integer shifts (and $f(-\omega-\omega_j)=f(\omega+\omega_j)$) we have

$$\begin{aligned}\mathrm{E}\left[\widetilde{Q}_{a,\lambda}(g;r)\right] &= \frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\gamma_j\gamma_{r-j}\int_{-2\pi a/\lambda}^{2\pi a/\lambda}g(\omega)f(\omega+\omega_j)d\omega+O\left(\frac{\log\lambda+\log(1+|r|)}{\lambda}+\frac{1}{n}\right)\\ &= \frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\gamma_j\gamma_{r-j}\int_{-2\pi a/\lambda}^{2\pi a/\lambda}g(\omega)f(\omega)d\omega+O\left(\frac{\log\lambda+\log(1+|r|)}{\lambda}+\frac{1}{n}\right)\end{aligned}$$

thus we obtain the desired result. □

B Uniformly sampled locations with frequency grid $\boldsymbol{\omega}_{\lambda,\boldsymbol{k}}$

We start by proving the results in Section 4.2 for the case that the locations are uniformly distributed. The proof here forms the building blocks for the proof of a uniformly sampled locations with arbitrary frequency grid $\boldsymbol{\omega}_{\Omega,\boldsymbol{k}}$ (see Appendix D) and for non-uniform sampled locations (see Appendix C).

We now obtain some approximations.

Lemma B.1 *Suppose Assumptions 2.1, 2.3 and*

(i) Assumptions 2.5(i) and 2.6(a,c) hold. Then we have

$$\lambda^d \mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1), \widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)\right] = \begin{cases} C_1(\boldsymbol{\omega_r}) + O(\frac{1}{\lambda} + \frac{\lambda^d}{n}) & \boldsymbol{r}_1 = \boldsymbol{r}_2 (= \boldsymbol{r}) \\ O(\frac{1}{\lambda} + \frac{\lambda^d}{n}) & \boldsymbol{r}_1 \neq \boldsymbol{r}_2 \end{cases}$$

and

$$\lambda^d \mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1), \overline{\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)}\right] = \begin{cases} C_2(\boldsymbol{\omega_r}) + O(\frac{1}{\lambda} + \frac{\lambda^d}{n}) & \boldsymbol{r}_1 = -\boldsymbol{r}_2 (= \boldsymbol{r}) \\ O(\frac{1}{\lambda} + \frac{\lambda^d}{n}) & \boldsymbol{r}_1 \neq -\boldsymbol{r}_2 \end{cases}$$

where

$$C_1(\boldsymbol{\omega_r}) = C_{1,1}(\boldsymbol{\omega_r}) + C_{1,2}(\boldsymbol{\omega_r}) \text{ and } C_2(\boldsymbol{\omega_r}) = C_{2,1}(\boldsymbol{\omega_r}) + C_{2,2}(\boldsymbol{\omega_r}), \tag{B.1}$$

with

$$\begin{aligned} C_{1,1}(\boldsymbol{\omega_r}) &= \frac{1}{(2\pi)^d}\int_{2\pi[-a/\lambda,a/\lambda]^d} f(\boldsymbol{\omega})f(\boldsymbol{\omega}+\boldsymbol{\omega_r})|g(\boldsymbol{\omega})|^2 d\boldsymbol{\omega}, \\ C_{1,2}(\boldsymbol{\omega_r}) &= \frac{1}{(2\pi)^d}\int_{\mathcal{D}_{\boldsymbol{r}}} f(\boldsymbol{\omega})f(\boldsymbol{\omega}+\boldsymbol{\omega_r})g(\boldsymbol{\omega})\overline{g(-\boldsymbol{\omega}-\boldsymbol{\omega_r})}d\boldsymbol{\omega}, \\ C_{2,1}(\boldsymbol{\omega_r}) &= \frac{1}{(2\pi)^d}\int_{2\pi[-a/\lambda,a/\lambda]^d} f(\boldsymbol{\omega})f(\boldsymbol{\omega}+\boldsymbol{\omega_r})g(\boldsymbol{\omega})g(-\boldsymbol{\omega})d\boldsymbol{\omega}, \\ C_{2,2}(\boldsymbol{\omega_r}) &= \frac{1}{(2\pi)^d}\int_{\mathcal{D}_{\boldsymbol{r}}} f(\boldsymbol{\omega})f(\boldsymbol{\omega}+\boldsymbol{\omega_r})g(\boldsymbol{\omega})g(\boldsymbol{\omega}+\boldsymbol{\omega_r})d\boldsymbol{\omega}, \end{aligned} \tag{B.2}$$

where the integral is defined as $\int_{\mathcal{D}_{\boldsymbol{r}}} = \int_{2\pi\max(-a,-a-r_1)/\lambda}^{2\pi\min(a,a-r_1)/\lambda} \cdots \int_{2\pi\max(-a,-a-r_d)/\lambda}^{2\pi\min(a,a-r_d)/\lambda}$ *(note that* $C_{1,1}(\boldsymbol{\omega_r})$ *and* $C_{1,2}(\boldsymbol{\omega_r})$ *are real).*

(ii) Assumptions 2.5(ii) and 2.6(b) hold. Then

$$\lambda^d \mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1), \widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)\right] = A_1(\boldsymbol{r}_1,\boldsymbol{r}_2) + A_2(\boldsymbol{r}_1,\boldsymbol{r}_2) + O(\frac{\lambda^d}{n}),$$

$$\lambda^d \mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1), \overline{\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)}\right] = A_3(\boldsymbol{r}_1,\boldsymbol{r}_2) + A_4(\boldsymbol{r}_1,\boldsymbol{r}_2) + O(\frac{\lambda^d}{n}),$$

where

$$\begin{aligned} A_1(\boldsymbol{r}_1,\boldsymbol{r}_2) = \; & \frac{1}{\pi^{2d}\lambda^d}\sum_{\boldsymbol{m}=-2a}^{2a}\sum_{\boldsymbol{k}=\max(-a,-a+\boldsymbol{m})}^{\min(a,a+\boldsymbol{m})}\int_{\mathbb{R}^{2d}} f(\frac{2\boldsymbol{u}}{\lambda}-\boldsymbol{\omega_k})f(\frac{2\boldsymbol{v}}{\lambda}+\boldsymbol{\omega_k}+\boldsymbol{\omega}_{\boldsymbol{r}_1}) \\ & g(\boldsymbol{\omega_k})\overline{g(\boldsymbol{\omega_k}-\boldsymbol{\omega_m})} \times \\ & \mathrm{Sinc}(\boldsymbol{u}-\boldsymbol{m}\pi)\mathrm{Sinc}(\boldsymbol{v}+(\boldsymbol{m}+\boldsymbol{r}_1-\boldsymbol{r}_2)\pi)\mathrm{Sinc}(\boldsymbol{u})\mathrm{Sinc}(\boldsymbol{v})d\boldsymbol{u}d\boldsymbol{v} \end{aligned}$$

$$A_2(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{1}{\pi^{2d}\lambda^d}\sum_{\boldsymbol{m}=-2a}^{2a}\sum_{\boldsymbol{k}=\max(-a,-a+\boldsymbol{m})}^{\min(a,a+\boldsymbol{m})}\int_{\mathbb{R}^{2d}} f(\frac{2\boldsymbol{u}}{\lambda}-\boldsymbol{\omega_k})f(\frac{2\boldsymbol{v}}{\lambda}+\boldsymbol{\omega_k}+\boldsymbol{\omega_{r_1}})\times$$
$$g(\boldsymbol{\omega_k})\overline{g(\boldsymbol{\omega_m}-\boldsymbol{\omega_k})}\times$$
$$\mathrm{Sinc}(\boldsymbol{u}-(\boldsymbol{m}+\boldsymbol{r}_2)\pi)\mathrm{Sinc}(\boldsymbol{v}+(\boldsymbol{m}+\boldsymbol{r}_1)\pi)\mathrm{Sinc}(\boldsymbol{u})\mathrm{Sinc}(\boldsymbol{v})d\boldsymbol{u}d\boldsymbol{v}$$

$$A_3(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{1}{\pi^{2d}\lambda^d}\sum_{\boldsymbol{m}=-2a}^{2a}\sum_{\boldsymbol{k}=\max(-a,-a+\boldsymbol{m})}^{\min(a,a+\boldsymbol{m})}\int_{\mathbb{R}^{2d}} f(\frac{2\boldsymbol{u}}{\lambda}-\boldsymbol{\omega_k})f(\frac{2\boldsymbol{v}}{\lambda}+\boldsymbol{\omega_k}+\boldsymbol{\omega_{r_1}})\times$$
$$g(\boldsymbol{\omega_k})g(\boldsymbol{\omega_m}-\boldsymbol{\omega_k})\times$$
$$\mathrm{Sinc}(\boldsymbol{u}+\boldsymbol{m}\pi)\mathrm{Sinc}(\boldsymbol{v}+(\boldsymbol{m}+\boldsymbol{r}_2+\boldsymbol{r}_1)\pi)\mathrm{Sinc}(\boldsymbol{u})\mathrm{Sinc}(\boldsymbol{v})d\boldsymbol{u}d\boldsymbol{v}$$

$$A_4(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{1}{\pi^{2d}\lambda^d}\sum_{\boldsymbol{m}=-2a}^{2a}\sum_{\boldsymbol{k}=\max(-a,-a+\boldsymbol{m})}^{\min(a,a+\boldsymbol{m})}\int_{\mathbb{R}^{2d}} f(\frac{2\boldsymbol{u}}{\lambda}-\boldsymbol{\omega_k})f(\frac{2\boldsymbol{v}}{\lambda}+\boldsymbol{\omega_k}+\boldsymbol{\omega_{r_1}})\times$$
$$g(\boldsymbol{\omega_k})g(\boldsymbol{\omega_k}-\boldsymbol{\omega_m})\times$$
$$\mathrm{Sinc}(\boldsymbol{u}-(\boldsymbol{m}-\boldsymbol{r}_2)\pi)\mathrm{Sinc}(\boldsymbol{v}+(\boldsymbol{m}+\boldsymbol{r}_1)\pi)\mathrm{Sinc}(\boldsymbol{u})\mathrm{Sinc}(\boldsymbol{v})d\boldsymbol{u}d\boldsymbol{v}$$

where $\boldsymbol{k} = \max(-a,-a-\boldsymbol{m}) = \{k_1 = \max(-a,-a-m_1),\ldots,k_d = \max(-a,-a-m_d)\}$, $\boldsymbol{k} = \min(-a+\boldsymbol{m}) = \{k_1 = \max(-a,-a+m_1),\ldots,k_d = \min(-a+m_d)\}$.

(iii) Assumptions 2.5(ii) and 2.6(b) hold. Then we have

$$\lambda^d \sup_a \left|\mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}),\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\right]\right| < \infty \quad \textit{and} \quad \lambda^d \sup_a \left|\mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}),\overline{\widetilde{Q}_{a,\lambda}(g;-\boldsymbol{r})}\right]\right| < \infty,$$

if $\lambda^d/n \to c$ $(0 \le c < \infty)$ *as* $\lambda\to\infty$ *and* $n\to\infty$.

PROOF We prove the result in the case $d = 1$ (the proof for $d > 1$ is identical). We first prove (i). By using indecomposable partitions, Theorem 2.1 and Lemma E.1 and noting that the fourth order cumulant is of order $O(1/n)$, it is straightforward to show that

$$\begin{aligned}
&\lambda\mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g;r_1),\widetilde{Q}_{a,\lambda}(g;r_2)\right]\\
=\ & \frac{1}{\lambda}\sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})}\Big[\mathrm{cov}\Big(J_n(\omega_{k_1})\overline{J_n(\omega_{k_1+r_1})}-\frac{\lambda}{n^2}\sum_{j=1}^n Z(s_j)^2e^{-is_j\omega_r},\\
& J_n(\omega_{k_2})\overline{J_n(\omega_{k_2+r_2})}-\frac{\lambda}{n^2}\sum_{j=1}^n Z(s_j)^2e^{-is_j\omega_r}\Big)\Big]\\
=\ & \begin{cases} \frac{1}{\lambda}\sum_{k=-a}^{a} f(\omega_k)f(\omega_k+\omega_r)g(\omega_k)\overline{g(\omega_k)} \\ +\frac{1}{\lambda}\sum_{k=\max(-a,-a-r)}^{\min(a,a-r)} f(\omega_k)f(\omega_k+\omega_r)g(\omega_k)\overline{g(-\omega_{k+r})} + O(\frac{a}{\lambda^2}+\frac{\lambda}{n}) & r_1 = r_2 \\ O(\frac{a}{\lambda^2}+\frac{\lambda}{n}) & r_1\neq r_2 \end{cases}
\end{aligned}$$

Since $a = C\lambda$ we have $O(\frac{a}{\lambda^2}) = O(\frac{1}{\lambda})$ and by replacing sum with integral (using Lemma I.1(i)) we have

$$\lambda \mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g;r_1), \widetilde{Q}_{a,\lambda}(g;r_2)\right] = \begin{cases} C_1(\omega_r) + O(\frac{1}{\lambda} + \frac{\lambda}{n}) & r_1 = r_2 (= r) \\ O(\frac{1}{\lambda} + \frac{\lambda}{n}) & r_1 \neq r_2 \end{cases}$$

where

$$C_1(\omega_r) = C_{11}(\omega_r) + C_{12}(\omega_r) + O\left(\frac{1}{\lambda}\right),$$

with

$$\begin{aligned} C_{11}(\omega_r) &= \frac{1}{2\pi}\int_{-2\pi a/\lambda}^{2\pi a/\lambda} f(\omega)f(\omega+\omega_r)g(\omega)\overline{g(\omega)}d\omega \\ C_{12}(\omega_r) &= \frac{1}{2\pi}\int_{2\pi\max(-a,-a-r)/\lambda}^{2\pi\min(a,a-r)/\lambda} f(\omega)f(\omega+\omega_r)g(\omega)\overline{g(-\omega-\omega_r)}d\omega. \end{aligned}$$

We now show that $C_1(\omega_r) = C_{11}(\omega_r) + C_{12}(\omega_r)$ is real. It is clear that $C_{11}(\omega_r)$ is real. Thus we focus on $C_{12}(\omega_r)$. To do this, we write $g(\omega) = g_1(\omega) + ig_2(\omega)$, therefore

$$\Im g(\omega)\overline{g(-\omega-\omega_r)} = [g_2(\omega)g_1(-\omega-\omega_r) - g_1(\omega)g_2(-\omega-\omega_r)].$$

Substituting the above into $\Im C_{12}(\omega_r)$ gives us $\Im C_{12}(r) = [C_{121}(r) + C_{122}(r)]$ where

$$\begin{aligned} C_{121}(r) &= \frac{1}{2\pi}\int_{2\pi\max(-a,-a-r)/\lambda}^{2\pi\min(a,a-r)/\lambda} f(\omega)f(\omega+\omega_r)g_2(\omega)g_1(-\omega-\omega_r)d\omega \\ C_{122}(r) &= -\frac{1}{2\pi}\int_{2\pi\max(-a,-a-r)/\lambda}^{2\pi\min(a,a-r)/\lambda} f(\omega)f(\omega+\omega_r)g_1(\omega)g_2(-\omega-\omega_r)d\omega \end{aligned}$$

We will show that $C_{122}(r) = -C_{121}(r)$. Focusing on C_{122} and making the change of variables $u = -\omega - \omega_r$ gives us

$$C_{122} = \frac{1}{2\pi}\int_{2\pi\min(a,a-r)/\lambda}^{2\pi\max(-a,-a-r)/\lambda} f(u+\omega_r)f(-u)g_1(-u-\omega_r)g_2(u)du,$$

noting that the spectral density function is symmetric with $f(-u) = f(u)$, and that $\int_{2\pi\min(a,a-r)/\lambda}^{2\pi\max(-a,-a-r)/\lambda} = -\int_{2\pi\max(-a,-a-r)/\lambda}^{2\pi\min(a,a-r)/\lambda}$. Thus we have $\Im C_{12}(r) = 0$, which shows that $C_1(\omega_r)$ is real. The proof of $\lambda^d \mathrm{cov}[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1), \overline{\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)}]$ is the same. Thus, we have proven (i).

To prove (ii) we first expand $\mathrm{cov}[\widetilde{Q}_{a,\lambda}(g;r_1), \widetilde{Q}_{a,\lambda}(g;r_2)]$ to give

$$\begin{aligned} &\lambda \mathrm{cov}[\widetilde{Q}_{a,\lambda}(g;r_1), \widetilde{Q}_{a,\lambda}(g;r_2)] \\ = \;& \lambda \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})} \sum_{j_1,\ldots,j_4=1}^{n} \Big(\mathrm{E}\big[c(s_{j_1}-s_{j_3})e^{is_{j_1}\omega_{k_1} - is_{j_3}\omega_{k_2}}\big] \times \\ & \mathrm{E}\big[c(s_{j_2}-s_{j_4})e^{-is_{j_2}\omega_{k_1+r_1} + is_{j_4}\omega_{k_2+r_2}}\big] + \\ & \mathrm{E}\big[c(s_{j_1}-s_{j_4})e^{is_{j_1}\omega_{k_1} + is_{j_4}\omega_{k_2+r_2}}\big]\mathrm{E}\big[c(s_{j_2}-s_{j_3})e^{-is_{j_2}\omega_{k_1+r_1} - is_{j_3}\omega_{k_2}}\big] + \\ & \mathrm{cum}\big[Z(s_{j_1})e^{i\omega_{k_1}s_{j_1}}, Z(s_{j_2})e^{-is_{j_2}(\omega_{k_1}+\omega_{r_2})}, Z(s_{j_3})e^{-i\omega_{k_2}s_{j_3}}, Z(s_{j_4})e^{is_{j_4}(\omega_{k_2}+\omega_{r_2})}\big]\Big). \end{aligned}$$

A "full" expansion of the above is given in (E.3). Using this expansion and Lemma E.1 we can show that

$$\lambda \mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(r_1), \widetilde{Q}_{a,\lambda}(g; r_2)\right] = A_1(r_1,r_2) + A_2(r_1,r_2) + O\left(\frac{\lambda}{n}\right) \tag{B.3}$$

where

$$\begin{aligned}
A_1(r_1,r_2) &= \lambda \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})}\mathrm{E}\big[c(s_1-s_3)\exp(is_1\omega_{k_1} - is_3\omega_{k_2})\big] \times \\
&\quad \mathrm{E}\big[c(s_2-s_4)\exp(-is_2\omega_{k_1+r_1} + is_4\omega_{k_2+r_2})\big] \\
A_2(r_1,r_2) &= \lambda \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})}\mathrm{E}\big[c(s_1-s_4)\exp(is_1\omega_{k_1} + is_4\omega_{k_2+r_2})\big] \times \\
&\quad \mathrm{E}\big[c(s_2-s_3)\exp(-is_2\omega_{k_1+r_1} - is_3\omega_{k_2})\big].
\end{aligned}$$

Note that the $O(\lambda/n)$ term includes the error $n^{-1}[A_1(r_1,r_2)] + A_2(r_1,r_2)]$ (we show below that $A_1(r_1,r_2)$ and $A_2(r_1,r_2)$ are both bounded over λ and a). To write $A_1(r_1,r_2)$ in the form stated in the lemma we integrate over s_1, s_2, s_3 and s_4 to give

$$\begin{aligned}
&A_1(r_1,r_2) \\
&= \lambda \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})}\frac{1}{\lambda^4}\int_{[-\lambda/2,\lambda/2]^4} c(s_1-s_3)c(s_2-s_4) \\
&\quad e^{is_1\omega_{k_1}-is_3\omega_{k_2}}e^{-is_2\omega_{k_1+r_1}+is_4\omega_{k_2+r_2}}ds_1ds_2ds_3ds_4.
\end{aligned}$$

By using the spectral representation theorem and integrating out $s_1,\ldots,s_4$ we can write the above as

$$\begin{aligned}
&A_1(r_1,r_2) \\
&= \frac{\lambda}{(2\pi)^2}\sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)f(y)\mathrm{sinc}\left(\frac{\lambda x}{2}+k_1\pi\right) \\
&\quad \mathrm{sinc}\left(\frac{\lambda y}{2}-(r_1+k_1)\pi\right)\mathrm{sinc}\left(\frac{\lambda x}{2}+k_2\pi\right)\mathrm{sinc}\left(\frac{\lambda y}{2}-(r_2+k_2)\pi\right)dxdy \\
&= \frac{1}{\pi^2\lambda}\sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\frac{2u}{\lambda}-\omega_{k_1})f(\frac{2v}{\lambda}+\omega_{k_1}+\omega_{r_1}) \\
&\quad \times\mathrm{sinc}(u)\mathrm{sinc}(u+(k_2-k_1)\pi)\mathrm{sinc}(v)\mathrm{sinc}(v+(k_1-k_2+r_1-r_2)\pi)dudv,
\end{aligned}$$

where the second equality is due to the change of variables $u = \frac{\lambda x}{2}+k_1\pi$ and $v = \frac{\lambda y}{2}-(r_1+k_1)\pi$. Finally, by making a change of variables $k=k_1$ and $m=k_1-k_2$ ($k_1=-k_2+m$) we obtain the expression for $A_1(r_1,r_2)$ given in Lemma B.1.

A similar method can be used to obtain the expression for $A_2(r_1, r_2)$

$$\begin{aligned}
&A_2(r_1, r_2) \\
= &\frac{\lambda}{(2\pi)^2} \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)f(y)\text{sinc}\left(\frac{\lambda x}{2} + k_1\pi\right) \\
&\text{sinc}\left(\frac{\lambda y}{2} - (r_1 + k_1)\pi\right) \text{sinc}\left(\frac{\lambda y}{2} + k_2\pi\right) \text{sinc}\left(\frac{\lambda x}{2} - (r_2 + k_2)\pi\right) dxdy \\
= &\frac{1}{\pi^2\lambda} \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\frac{2u}{\lambda} - \omega_{k_1})f(\frac{2v}{\lambda} + \omega_{k_1} + \omega_{r_1}) \\
&\times\text{sinc}(u)\text{sinc}(u - (k_2 + r_2 + k_1)\pi)\text{sinc}(v)\text{sinc}(v + (k_2 + k_1 + r_1)\pi)dudv.
\end{aligned}$$

By making the change of variables $k = k_1$ and $m = k_1 + k_2$ ($k_1 = -k_2 + m$) we obtain the stated expression for $A_2(r_1, r_2)$.

Finally following the same steps as those above we obtain

$$\lambda\text{cov}(\widetilde{Q}_{a,\lambda}(g; r_1), \overline{\widetilde{Q}_{a,\lambda}(g; r_2)}) = A_3(r_1, r_2) + A_4(r_1, r_2) + O(\frac{\lambda}{n}),$$

where

$$\begin{aligned}
A_3(r_1, r_2) &= \frac{1}{\lambda^3} \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})g(\omega_{k_2}) \int_{[-\lambda/2,\lambda/2]^4} e^{i(s_1\omega_{k_1}+s_3\omega_{k_2})}e^{-is_2\omega_{k_1+r_1}-is_4\omega_{k_2+r_2}} \\
&\times c(s_1 - s_3)c(s_2 - s_4)ds_1ds_2ds_3ds_4 \\
A_4(r_1, r_2) &= \frac{1}{\lambda^3} \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})g(\omega_{k_2}) \int_{[-\lambda/2,\lambda/2]^4} e^{i(s_1\omega_{k_1}+s_3\omega_{k_2})}e^{-is_2\omega_{k_1+r_1}-is_4\omega_{k_2+r_2}} \\
&\times c(s_1 - s_4)c(s_2 - s_3)ds_1ds_2ds_3ds_4.
\end{aligned}$$

Again by replacing the covariances in $A_3(r_1, r_2)$ and $A_4(r_1, r_4)$ with their spectral representation gives (ii) for $d = 1$. The result for $d > 1$ is identical.

It is clear that (iii) is true under Assumption 2.5(a,b). To prove (iii) under Assumption 2.6(a,b) we will show that for $1 \leq j \leq 4$, $\sup_a |A_j(\boldsymbol{r}_1, \boldsymbol{r}_2)| < \infty$. To do this, we first note that by the Cauchy Schwarz inequality we have

$$\begin{aligned}
&\sup_{a,\lambda} \frac{1}{\pi^{2d}} \left| \frac{1}{\lambda^d} \sum_{\boldsymbol{k}=\max(-a,-a+\boldsymbol{m})}^{\min(-a+\boldsymbol{m})} g(\omega_{\boldsymbol{k}})\overline{g(\omega_{\boldsymbol{k}} - \boldsymbol{\omega}_m)} f(\frac{2\boldsymbol{u}}{\lambda} - \boldsymbol{\omega}_{\boldsymbol{k}})f(\frac{2\boldsymbol{v}}{\lambda} + \boldsymbol{\omega}_{\boldsymbol{k}} + \boldsymbol{\omega}_{\boldsymbol{r}_1}) \right| \\
\leq\ & C \sup_{\boldsymbol{\omega}} |g(\boldsymbol{\omega})|^2 \|f\|_2^2,
\end{aligned}$$

where $\|f\|_2$ is the L_2 norm of the spectral density function and C is a finite constant. Thus by taking absolutes of $A_1(\boldsymbol{r}_1, \boldsymbol{r}_2)$ we have

$$\begin{aligned}
&|A_1(\boldsymbol{r}_1, \boldsymbol{r}_2)| \leq \\
&C \sup_{\boldsymbol{\omega}} |g(\boldsymbol{\omega})|^2 \|f\|_2^2 \sum_{\boldsymbol{m}=-\infty}^{\infty} \int_{\mathbb{R}^{2d}} |\text{Sinc}(\boldsymbol{u} - \boldsymbol{m}\pi)\text{Sinc}(\boldsymbol{v} + (\boldsymbol{m} + \boldsymbol{r}_1 - \boldsymbol{r}_2)\pi)\text{Sinc}(\boldsymbol{u})\text{Sinc}(\boldsymbol{v})| d\boldsymbol{u}d\boldsymbol{v}.
\end{aligned}$$

Finally, by using Lemma F.1(iii) we have that $\sup_a |A_1(\boldsymbol{r},\boldsymbol{r})| < \infty$. By using the same method we can show that $\sup_a |A_2(\boldsymbol{r},\boldsymbol{r})|, A_3(\boldsymbol{r},-\boldsymbol{r}), \sup_a |A_4(\boldsymbol{r},\boldsymbol{r})| < \infty$. This completes the proof. $\square$

For the remainder of this section to simplify notation we mainly restrict ourselves to the case $d = 1$. We now obtain simplified expressions for the terms $A_1(r_1,r_2),\ldots,A_4(r_1,r_2)$ under the slightly stronger condition that Assumption 2.6(c) also holds.

Lemma B.2 *Suppose Assumptions 2.5(ii) and 2.6(b,c) hold. Then for* $0 \leq |r_1|,|r_2| \leq C|a|$ *(where* C *is some finite constant) we have*

$$A_1(r_1,r_2) = \begin{cases} C_{1,1}(\omega_r) + O(\ell_{\lambda,a,n}) & r_1 = r_2 (= r) \\ O(\ell_{\lambda,a,n}) & r_1 \neq r_2 \end{cases}$$

$$A_2(r_1,r_2) = \begin{cases} C_{1,2}(\omega_r) + O(\ell_{\lambda,a,n}) & r_1 = r_2 (= r) \\ O(\ell_{\lambda,a,n}) & r_1 \neq r_2 \end{cases}$$

$$A_3(r_1,r_2) = \begin{cases} C_{2,1}(\omega_r) + O(\ell_{\lambda,a,n}) & r_1 = -r_2 (= r) \\ O(\ell_{\lambda,a,n}) & r_1 \neq -r_2 \end{cases}$$

and

$$A_4(r_1,r_2) = \begin{cases} C_{2,2}(\omega_r) + O(\ell_{\lambda,a,n}) & r_1 = -r_2 (= r) \\ O(\ell_{\lambda,a,n}) & r_1 \neq -r_2 \end{cases},$$

where $C_{1,1}(\omega_r),\ldots,C_{2,2}(\omega_r)$ *(using* $d=1$*) and* $\ell_{\lambda,a,n}$ *are defined in Lemma B.1.*

PROOF. We first consider $A_1(r_1,r_2)$ and write it as

$$A_1(r_1,r_2) = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{m=-2a}^{2a}\mathrm{sinc}(u)\mathrm{sinc}(v)\mathrm{sinc}(u-m\pi)\mathrm{sinc}(v+(m+r_1-r_2)\pi)H_{m,\lambda}\left(\frac{2u}{\lambda},\frac{2v}{\lambda};r_1\right)dudv,$$

where

$$H_{m,\lambda}\left(\frac{2u}{\lambda},\frac{2v}{\lambda};r_1\right) = \frac{1}{\lambda}\sum_{k=\max(-a,-a+m)}^{\min(a,a+m)} f\left(-\frac{2u}{\lambda}+\omega_k\right) f\left(\frac{2v}{\lambda}+\omega_k+\omega_r\right) g(\omega_k)\overline{g(\omega_k-\omega_m)}$$

noting that $f(\frac{2u}{\lambda}-\omega) = f(\omega-\frac{2u}{\lambda})$.

If $f(-\frac{2u}{\lambda}+\omega_k)$ and $f(\frac{2v}{\lambda}+\omega_k+\omega_r)$ are replaced with $f(\omega_k)$ and $f(\omega_k+\omega_r)$ respectively, then we can exploit the orthogonality property of the sinc functions. This requires the following series of approximations.

(i) We start by defining a similar version of $A_1(r_1, r_2)$ but with the sum replaced with an integral. Let

$$
\begin{aligned}
& B_1(r_1 - r_2; r_1) \\
= \; & \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{m=-2a}^{2a} \mathrm{sinc}(u)\mathrm{sinc}(u - m\pi)\mathrm{sinc}(v)\mathrm{sinc}(v + (m + r_1 - r_2)\pi) \times \\
& H_m\left(\frac{2u}{\lambda}, \frac{2v}{\lambda}; r_1\right) du dv,
\end{aligned}
$$

where

$$
H_m\left(\frac{2u}{\lambda}, \frac{2v}{\lambda}; r_1\right) = \frac{1}{2\pi} \int_{2\pi \max(-a,-a+m)/\lambda}^{2\pi \min(a,a+m)/\lambda} f(\omega - \frac{2u}{\lambda}) f(\frac{2v}{\lambda} + \omega + \omega_{r_1}) g(\omega) \overline{g(\omega - \omega_m)} d\omega.
$$

By using Lemma B.3, we have

$$
|A_1(r_1, r_2) - B_1(r_1 - r_2; r_1)| = O\left(\frac{\log^2 a}{\lambda}\right).
$$

(ii) Define the quantity

$$
\begin{aligned}
& C_1(r_1 - r_2; r_1) \\
= \; & \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{m=-2a}^{2a} \mathrm{sinc}(u)\mathrm{sinc}(u - m\pi)\mathrm{sinc}(v)\mathrm{sinc}(v + (m + r_1 - r_2)\pi) H_m(0, 0; r_1) du dv.
\end{aligned}
$$

By using Lemma B.4, we can replace $B_1(r_1 - r_2; r_1)$ with $C_1(r_1 - r_2; r_1)$ to give the replacement error

$$
|B_1(r_1 - r_2; r_1) - C_1(r_1 - r_2; r_1)| = O\bigg(\log^2(a) \left[\frac{(\log a + \log \lambda)}{\lambda}\right] \bigg).
$$

(iii) Finally, we analyze $C_1(r_1 - r_2; r_1)$. Since $H_m(0, 0; r_1)$ does not depend on u or v we take it out of the integral to give

$$
\begin{aligned}
& C_1(r_1 - r_2; r_1) \\
= \; & \frac{1}{\pi^2} \sum_{m=-2a}^{2a} H_m(0, 0; r_1) \left(\int_{\mathbb{R}} \mathrm{sinc}(u)\mathrm{sinc}(u - m\pi) du \right) \\
& \times \left(\int_{\mathbb{R}} \mathrm{sinc}(v)\mathrm{sinc}(v + (m + r_1 - r_2)\pi) dv \right) \\
= \; & \frac{1}{\pi^2} H_0(0, 0; r_1) \left(\int_{\mathbb{R}} \mathrm{sinc}^2(u) du \right) \left(\int_{\mathbb{R}} \mathrm{sinc}(v)\mathrm{sinc}(v + (r_1 - r_2)\pi) dv \right),
\end{aligned}
$$

where the last line of the above is due to orthogonality of the sinc function (see Lemma A.1). If $r_1 \neq r_2$, then by orthogonality of the sinc function we have $C_1(r_1 - r_2; r_1) = 0$. On the other hand if $r_1 = r_2$ we have

$$C_1(0; r) = \frac{1}{2\pi}\int_{-2\pi a/\lambda}^{2\pi a/\lambda} f(\omega)f(\omega+\omega_r)|g(\omega)|^2 d\omega = C_{1,1}(\omega_r).$$

The proof for the remaining terms $A_2(r_1, r_2), A_3(r_1, r_2)$ and $A_4(r_1, r_2)$ is identical, thus we omit the details. □

Theorem B.1 *Suppose Assumptions 2.1, 2.3, 2.5(ii) and 2.6(b,c) hold. Then for* $0 \leq |r_1|, |r_2| \leq C|a|$ *(where* C *is some finite constant) we have*

$$\lambda^d \mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g; \boldsymbol{r}_1), \widetilde{Q}_{a,\lambda}(g; \boldsymbol{r}_2)\right] = \begin{cases} C_1(\boldsymbol{\omega_r}) + O(\ell_{\lambda,a,n}) & \boldsymbol{r}_1 = \boldsymbol{r}_2 (= \boldsymbol{r}) \\ O(\ell_{\lambda,a,n}) & \boldsymbol{r}_1 \neq \boldsymbol{r}_2 \end{cases}$$

$$\lambda^d \mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g; \boldsymbol{r}_1), \overline{\widetilde{Q}_{a,\lambda}(g; \boldsymbol{r}_2)}\right] = \begin{cases} C_2(\boldsymbol{\omega_r}) + O(\ell_{\lambda,a,n}) & \boldsymbol{r}_1 = -\boldsymbol{r}_2 (= \boldsymbol{r}) \\ O(\ell_{\lambda,a,n}) & \boldsymbol{r}_1 \neq -\boldsymbol{r}_2 \end{cases},$$

where $C_1(\boldsymbol{\omega_r})$ *and* $C_2(\boldsymbol{\omega_r})$ *are defined in (B.1) and*

$$\ell_{\lambda,a,n} = \log^2(a)\left[\frac{\log a + \log \lambda}{\lambda}\right] + \frac{\lambda^d}{n}. \tag{B.4}$$

PROOF. By using Lemmas B.1 and B.2 we immediately obtain (in the case $d = 1$)

$$\mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g; r_1), \widetilde{Q}_{a,\lambda}(g; r_2)\right] = \begin{cases} C_1(\omega_r) + O(\ell_{\lambda,a,n}) & r_1 = r_2 (= r) \\ O(\ell_{\lambda,a,n}) & r_1 \neq r_2 \end{cases}$$

and

$$\mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g; r_1), \overline{\widetilde{Q}_{\lambda,a,n}(g; r_2)}\right] = \begin{cases} C_2(\omega_r) + O(\ell_{\lambda,a,n}) & r_1 = -r_2 (= r) \\ O(\ell_{\lambda,a,n}) & r_1 \neq -r_2 \end{cases}.$$

This gives the result for $d = 1$. To prove the result for $d > 1$ we use the same procedure outlined in the proof of Lemma B.2 and the above. □

The above theorem means that the variance for both the bounded and unbounded frequency grid are equivalent (up to the limits of an integral).

Using the above results and the Lipschitz continuity of $g(\cdot)$ and $f(\cdot)$ we can show that

$$C_j(\boldsymbol{\omega_r}) = C_j + O\left(\frac{\|\boldsymbol{r}\|_1}{\lambda}\right),$$

where C_1 and C_2 are defined in Corollary 4.1.

We now derive an expression for the variance for the non-Gaussian case.

Theorem B.2 *Let us suppose that* $\{Z(\boldsymbol{s}); \boldsymbol{s} \in \mathbb{R}^d\}$ *is a fourth order stationary spatial random field that satisfies Assumption 2.1(i). Suppose all the assumptions in Theorem 4.6 hold with the exception of Assumption 2.2 which is replaced with Assumption 2.3 (i.e. we assume the locations are uniformly sampled). Then for* $\|\boldsymbol{r}\|_1, \|\boldsymbol{r}_2\|_1 << \lambda$ *we have*

$$\lambda^d \text{cov}\left[\widetilde{Q}_{a,\lambda}(g; \boldsymbol{r}_1), \widetilde{Q}_{a,\lambda}(g; \boldsymbol{r}_2)\right] = \begin{cases} C_1 + D_1 + O(\ell_{\lambda,a,n} + \frac{(a\lambda)^d}{n^2} + \frac{\|\boldsymbol{r}\|_1}{\lambda}) & \boldsymbol{r}_1 = \boldsymbol{r}_2 (= \boldsymbol{r}) \\ O(\ell_{\lambda,a,n} + \frac{(a\lambda)^d}{n^2}) & \boldsymbol{r}_1 \neq \boldsymbol{r}_2 \end{cases} \tag{B.5}$$

$$\lambda^d \text{cov}\left[\widetilde{Q}_{a,\lambda}(g; \boldsymbol{r}_1), \overline{\widetilde{Q}_{a,\lambda}(g; \boldsymbol{r}_2)}\right] = \begin{cases} C_2 + D_2 + O(\ell_{\lambda,a,n} + \frac{(a\lambda)^d}{n^2} + \frac{\|\boldsymbol{r}\|_1}{\lambda}) & \boldsymbol{r}_1 = -\boldsymbol{r}_2 (= \boldsymbol{r}) \\ O(\ell_{\lambda,a,n} + \frac{(a\lambda)^d}{n^2}) & \boldsymbol{r}_1 \neq -\boldsymbol{r}_2 \end{cases}, \tag{B.6}$$

where C_1 *and* C_2 *are defined in Corollary 4.1 and*

$$\begin{aligned} D_1 &= \frac{1}{(2\pi)^{2d}} \int_{2\pi[-a/\lambda, a/\lambda]^{2d}} g(\boldsymbol{\omega}_1)\overline{g(\boldsymbol{\omega}_2)} f_4(-\boldsymbol{\omega}_1, -\boldsymbol{\omega}_2, \boldsymbol{\omega}_2) d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2 \\ D_2 &= \frac{1}{(2\pi)^{2d}} \int_{2\pi[-a/\lambda, a/\lambda]^{2d}} g(\boldsymbol{\omega}_1) g(\boldsymbol{\omega}_2) f_4(-\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, -\boldsymbol{\omega}_2) d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2. \end{aligned}$$

PROOF We prove the result for the notationally simple case $d = 1$. By using indecomposable partitions, conditional cumulants and (B.3) we have

$$\lambda \text{cov}\left[\widetilde{Q}_{a,\lambda}(g; r_1), \widetilde{Q}_{a,\lambda}(g; r_2)\right] = A_1(r_1, r_2) + A_2(r_1, r_2) + B_1(r_1, r_2) + B_2(r_1, r_2) + O\left(\frac{\lambda}{n}\right), \tag{B.7}$$

where $A_1(r_1, r_2)$ and $A_2(r_1, r_2)$ are defined below equation (B.3) and

$$\begin{aligned} B_1(r_1, r_2) &= \lambda c_4 \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})} \times \\ &\quad \text{E}\left[\kappa_4(s_2 - s_1, s_3 - s_1, s_4 - s_1) e^{is_1\omega_{k_1}} e^{-is_2\omega_{k_1+r_1}} e^{-is_3\omega_{k_2}} e^{is_4\omega_{k_2+r_2}}\right] \\ B_2(r_1, r_2) &= \frac{\lambda}{n^4} \sum_{j_1, \ldots, j_4 \in \mathcal{D}_3} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})} \times \\ &\quad \text{E}\left[\kappa_4(s_{j_2} - s_{j_1}, s_{j_3} - s_{j_1}, s_{j_4} - s_{j_1}) e^{is_{j_1}\omega_{k_1}} e^{-is_{j_2}\omega_{k_1+r_1}} e^{-is_{j_3}\omega_{k_2}} e^{is_{j_4}\omega_{k_2+r_2}}\right] \end{aligned}$$

with $c_4 = n(n-1)(n-2)(n-3)/n^2$ and
$\mathcal{D}_3 = \{j_1, \ldots, j_4; j_1 \neq j_2 \text{ and } j_3 \neq j_4 \text{ but some } j\text{'s are in common}\}$. The limits of $A_1(r_1, r_2)$

and $A_2(r_1, r_2)$ are given in Lemma B.2, therefore, all that remains is to derive bounds for $B_1(r_1, r_2)$ and $B_2(r_1, r_2)$. We will show that $B_1(r_1, r_2)$ is the dominating term, whereas by placing sufficient conditions on the rate of growth of a, we will show that $B_2(r_1, r_2) \to 0$.

In order to analyze $B_1(r_1, r_2)$ we will use the result

$$\int_{\mathbb{R}^3} \text{sinc}(u_1 + u_2 + u_3 + (r_2 - r_1)\pi)\text{sinc}(u_2)\text{sinc}(u_3) du_1 du_2 du_3 = \begin{cases} \pi^3 & r_1 = r_2 \\ 0 & r_1 \neq r_2. \end{cases}, \tag{B.8}$$

which follows from Lemma A.1. In the following steps we will make a series of approximations which will allow us to apply (B.8).

We start by substituting the Fourier representation of the cumulant function

$$\kappa_4(s_1 - s_2, s_1 - s_3, s_1 - s_4) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} f_4(\omega_1, \omega_2, \omega_3) e^{i(s_1 - s_2)\omega_1} e^{i(s_1 - s_3)\omega_2} e^{i(s_1 - s_4)\omega_4} d\omega_1 d\omega_2 d\omega_3,$$

into $B_1(r_1, r_2)$ to give

$$\begin{aligned}
& B_1(r_1, r_2) \\
= & \frac{c_4}{(2\pi)^3 \lambda^3} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) \overline{g(\omega_{k_2})} \int_{\mathbb{R}^3} f_4(\omega_1, \omega_2, \omega_3) \int_{[-\lambda/2, \lambda/2]^4} e^{is_1(\omega_1 + \omega_2 + \omega_3 + \omega_{k_1})} \\
& e^{-is_2(\omega_1 + \omega_{k_1 + r_1})} e^{-is_3(\omega_2 + \omega_{k_2})} e^{is_4(-\omega_3 + \omega_{k_2 + r_2})} ds_1 ds_2 ds_3 ds_4 d\omega_1 d\omega_2 d\omega_3 \\
= & \frac{c_4 \lambda}{(2\pi)^3} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1}) \overline{g(\omega_{k_2})} \int_{\mathbb{R}^3} f_4(\omega_1, \omega_2, \omega_3) \text{sinc}\left(\frac{\lambda(\omega_1 + \omega_2 + \omega_3)}{2} + k_1 \pi\right) \times \\
& \text{sinc}\left(\frac{\lambda\omega_1}{2} + (k_1 + r_1)\pi\right) \text{sinc}\left(\frac{\lambda\omega_2}{2} + k_2\pi\right) \text{sinc}\left(\frac{\lambda\omega_3}{2} - (k_2 + r_2)\pi\right) d\omega_1 d\omega_2 d\omega_3.
\end{aligned}$$

Now we make a change of variables and let $u_1 = \frac{\lambda\omega_1}{2} + (k_1 + r_1)$, $u_2 = \frac{\lambda\omega_2}{2} + k_2\pi$ and $u_3 = \frac{\lambda\omega_3}{2} - (k_2 + r_2)\pi$, this gives

$$\begin{aligned}
B_1(r_1, r_2) = & \frac{c_4}{\pi^3 \lambda^2} \sum_{k_1, k_2 = -a}^{a} \int_{\mathbb{R}^3} g(\omega_{k_1}) \overline{g(\omega_{k_2})} f_4\left(\frac{2u_1}{\lambda} - \omega_{k_1 + r_1}, \frac{2u_2}{\lambda} - \omega_{k_2}, \frac{2u_3}{\lambda} + \omega_{k_2 + r_2}\right) \times \\
& \times \text{sinc}\left(u_1 + u_2 + u_3 + (r_2 - r_1)\pi\right) \text{sinc}(u_1)\text{sinc}(u_2)\text{sinc}(u_3) du_1 du_2 du_3.
\end{aligned}$$

Next we exchange the summand with a double integral and use Lemma I.1(iii) together with Lemma F.1, equation (F.3) to obtain

$$\begin{aligned}
B_1(r_1, r_2) = & \frac{c_4}{(2\pi)^2 \pi^3} \int_{2\pi[-a/\lambda, a/\lambda]^2} \int_{\mathbb{R}^3} g(\omega_1) \overline{g(\omega_2)} f_4\left(\frac{2u_1}{\lambda} - \omega_1 - \omega_{r_1}, \frac{2u_2}{\lambda} - \omega_2, \frac{2u_3}{\lambda} + \omega_2 + \omega_{r_2}\right) \times \\
& \text{sinc}\left(u_1 + u_2 + u_3 + (r_2 - r_1)\pi\right) \text{sinc}(u_1)\text{sinc}(u_2)\text{sinc}(u_3) du_1 du_2 du_3 d\omega_1 d\omega_2 + O\left(\frac{1}{\lambda}\right).
\end{aligned}$$

By using Lemma F.3, we replace $f_4(\frac{2u_1}{\lambda} - \omega_1 - \omega_{r_1}, \frac{2u_2}{\lambda} - \omega_2, \frac{2u_3}{\lambda} + \omega_2 + \omega_{r_2})$ in the integral with $f_4(-\omega_1 - \omega_{r_1}, -\omega_2, \omega_2 + \omega_{r_2})$, this gives

$$\begin{aligned}
&B_1(r_1, r_2) \\
= \ & \frac{c_4}{(2\pi)^2\pi^3} \int_{2\pi[-a/\lambda, a/\lambda]^2} g(\omega_1)\overline{g(\omega_2)} f_4(-\omega_1 - \omega_{r_1}, -\omega_2, \omega_2 + \omega_{r_2}) \times \\
& \int_{\mathbb{R}^3} \mathrm{sinc}(u_1 + u_2 + u_3 + (r_2 - r_1)\pi)\mathrm{sinc}(u_1)\mathrm{sinc}(u_2)\mathrm{sinc}(u_3) du_1 du_2 du_3 d\omega_1 d\omega_2 + O\left(\frac{\log^3 \lambda}{\lambda}\right) \\
= \ & \frac{c_4 I_{r_1 = r_2}}{(2\pi)^2} \int_{2\pi[-a/\lambda, a/\lambda]^2} g(\omega_1)\overline{g(\omega_2)} f_4(-\omega_1 - \omega_{r_1}, -\omega_2, \omega_2 + \omega_{r_2}) d\omega_1 d\omega_2 + O\left(\frac{\log^3 \lambda}{\lambda}\right),
\end{aligned}$$

where the last line follows from the orthogonality relation of the sinc function in equation (B.8).

Finally we make one further approximation. In the case $r_1 = r_2$ we replace $f_4(-\omega_1 - \omega_{r_1}, -\omega_2, \omega_2 + \omega_{r_2})$ with $f_4(-\omega_1, -\omega_2, \omega_2)$ which by using the Lipschitz continuity of f_4 and Lemma F.1, equation (F.3) gives

$$B_1(r_1, r_1) = \frac{c_4}{(2\pi)^2} \int_{2\pi[-a/\lambda, a/\lambda]^2} g(\omega_1)\overline{g(\omega_2)} f_4(-\omega_1, -\omega_2, \omega_2) d\omega_1 d\omega_2 + O\left(\frac{\log^3 \lambda}{\lambda} + \frac{|r_1|}{\lambda}\right).$$

Altogether this gives

$$\begin{aligned}
&B_1(r_1, r_2) \\
= \ & \begin{cases} \frac{1}{(2\pi)^2} \int_{2\pi[-a/\lambda, a/\lambda]^2} g(\omega_1)\overline{g(\omega_2)} f_4(-\omega_1, -\omega_2, \omega_2) d\omega_1 d\omega_2 + O(\frac{\log^3 \lambda}{\lambda} + \frac{|r_1|}{\lambda}) & r_1 = r_2 \\ O(\frac{\log^3 \lambda}{\lambda}) & r_1 \neq r_2. \end{cases}.
\end{aligned}$$

Next we show that $B_2(r_1, r_2)$ is asymptotically negligible. To bound $B_2(r_1, r_2)$ we decompose $\mathcal{D}_3$ into six sets, the set $\mathcal{D}_{3,1} = \{j_1, \ldots, j_4; j_1 = j_3,\ j_2 \text{ and } j_4 \text{ are different})\}$, $\mathcal{D}_{3,2} = \{j_1, \ldots, j_4; j_1 = j_4,\ j_2 \text{ and } j_3 \text{ are different})$, $\mathcal{D}_{3,3} = \{j_1, \ldots, j_4; j_2 = j_3,\ j_1 \text{ and } j_4 \text{ are different})$, $\mathcal{D}_{3,4} = \{j_1, \ldots, j_4; j_2 = j_4,\ j_1 \text{ and } j_3 \text{ are different})$, $\mathcal{D}_{2,1} = \{j_1, \ldots, j_4; j_1 = j_3 \text{ and } j_2 = j_4\}$, $\mathcal{D}_{2,2} = \{j_1, \ldots, j_4; j_1 = j_4 \text{ and } j_2 = j_3\}$. Using this decomposition we have $B_2(r_1, r_2) = \sum_{j=1}^{4} B_{2,(3,j)}(r_1, r_2) + \sum_{j=1}^{2} B_{2,(2,j)}(r_1, r_2)$, where

$$\begin{aligned}
B_{2,(3,1)}(r_1, r_2) &= \frac{|\mathcal{D}_{3,1}|\lambda}{n^4} \sum_{k_1, k_2 = -a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})} \times \\
&\quad \mathrm{E}\Big[\kappa_4(s_{j_2} - s_{j_1}, 0, s_{j_4} - s_{j_1}) e^{is_{j_1}\omega_{k_1}} e^{-is_{j_2}\omega_{k_1 + r_1}} e^{-is_{j_1}\omega_{k_2}} e^{is_{j_4}\omega_{k_2 + r_2}}\Big]
\end{aligned}$$

for $j=2,3,4$, $B_{2,(3,j)}(r_1,r_2)$ are defined similarly,

$$\begin{aligned} B_{2,(2,1)}(r_1,r_2) &= \frac{|\mathcal{D}_{2,1}|\lambda}{n^4}\sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})}\times \\ &\quad \mathrm{E}\left[\kappa_4(s_{j_2}-s_{j_1},0,s_{j_2}-s_{j_1})e^{is_{j_1}\omega_{k_1}}e^{-is_{j_2}\omega_{k_1+r_1}}e^{-is_{j_1}\omega_{k_2}}e^{is_{j_2}\omega_{k_2+r_2}}\right], \end{aligned}$$

$B_{2,(2,2)}(r_1,r_2)$ is defined similarly and $|\cdot|$ denotes the cardinality of a set. By using identical methods to those used to bound $B_1(r_1,r_2)$ we have

$$\begin{aligned} &|B_{2,(3,1)}(r_1,r_2)| \\ \leq\ & \frac{C\lambda}{n(2\pi)^3}\sum_{k_1,k_2=-a}^{a}|g(\omega_{k_1})\overline{g(\omega_{k_2})}|\int_{\mathbb{R}^3}|f_4(\omega_1,\omega_2,\omega_3)|\left|\mathrm{sinc}\left(\frac{\lambda(\omega_1+\omega_3)}{2}+(k_2-k_1)\pi\right)\right|\times \\ & \left|\mathrm{sinc}\left(\frac{\lambda\omega_1}{2}-(k_1+r_1)\pi\right)\right|\times\left|\mathrm{sinc}\left(\frac{\lambda\omega_3}{2}+(k_2+r_2)\pi\right)\right|d\omega_1 d\omega_2 d\omega_3 = O\left(\frac{\lambda}{n}\right). \end{aligned}$$

Similarly we can show $|B_{2,(3,j)}(r_1,r_2)| = O(\frac{\lambda}{n})$ (for $2\leq j\leq 4$) and

$$\begin{aligned} &|B_{2,(2,1)}(r_1,r_2)| \\ \leq\ & \frac{\lambda}{(2\pi)^3 n^2}\sum_{k_1,k_2=-a}^{a}|g(\omega_{k_1})\overline{g(\omega_{k_2})}|\int_{\mathbb{R}^3}|f_4(\omega_1,\omega_2,\omega_3)|\left|\mathrm{sinc}\left(\frac{\lambda(\omega_1+\omega_2)}{2}+(k_2-k_1)\pi\right)\times\right. \\ & \left.\mathrm{sinc}\left(\frac{\lambda(\omega_1+\omega_2)}{2}+(k_2-k_1+r_2-r_1)\pi\right)\right|d\omega_1 d\omega_2 d\omega_3 = O\left(\frac{a\lambda}{n^2}\right). \end{aligned}$$

This immediately gives us (B.5). To prove (B.6) we use identical methods. Thus we obtain the result. □

Note that the term $B_{2,(2,1)}(r_1,r_2)$ in the proof above is important as it does not seem possible to improve on the bound $O(a\lambda/n^2)$.

B.1 Lemmas required to prove Lemma B.2 and Theorem B.1

In this section we give the proofs of the three results used in Lemma B.2 (which in turn proves Theorem B.1).

Lemma B.3 *Suppose Assumptions 2.5(ii) and 2.6(b,c) holds. Let $A_1(\cdot)$ and $B_1(\cdot)$ be defined as in the proof of Lemma B.2. Then for $r_1,r_2\in\mathbb{Z}$ we have*

$$|A_1(r_1,r_2) - B_1(r_1-r_2;r_1)| = O\left(\frac{\log^2(a)}{\lambda}\right)$$

PROOF. To obtain a bound for the difference we use Lemma I.1(ii) to give

$$
\begin{aligned}
& |A_1(r_1,r_2) - B_1(r_1-r_2;r_1)| \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{m=-2a}^{2a} |\mathrm{sinc}(u)\mathrm{sinc}(u-m\pi)\mathrm{sinc}(v)\mathrm{sinc}(v+(m+r_1-r_2)\pi)| \\
& \underbrace{\left|H_{m,\lambda}\left(\frac{2u}{\lambda},\frac{2v}{\lambda};r_1\right) - H_m\left(\frac{2u}{\lambda},\frac{2v}{\lambda};r_1\right)\right|}_{\leq C/\lambda} dudv \\
\leq & \frac{C}{\lambda}\sum_{m=-2a}^{2a}\int_{-\infty}^{\infty}|\mathrm{sinc}(u)\mathrm{sinc}(u-m\pi)|du \underbrace{\int_{-\infty}^{\infty}|\mathrm{sinc}(v)\mathrm{sinc}(v+(m+r_1-r_2)\pi)|dv}_{<\infty} \\
\leq & \frac{C}{\lambda}\sum_{m=-2a}^{2a}\int_{-\infty}^{\infty}|\mathrm{sinc}(u)\mathrm{sinc}(u-m\pi)|du \text{ (from Lemma } F.1(ii)) \\
= & O(\frac{\log^2 a}{\lambda}),
\end{aligned}
$$

thus giving the desired result. □

Lemma B.4 *Suppose Assumptions 2.5(ii) and 2.6(b,c) holds. Let $B_1(\cdot)$ and $C_1(\cdot)$ be defined as in the proof of Lemma B.2 (with $0 \leq |s|,|r| < C|a|$). Then we have*

$$|B_1(s;r) - C_1(s;r)| = O\bigg(\log^2(a)\bigg[\frac{\log a + \log\lambda}{\lambda}\bigg]\bigg).$$

PROOF. Taking differences, it is easily seen that

$$
\begin{aligned}
& B_1(s,r) - C_1(s,r) \\
= & \int_{\mathbb{R}^2}\sum_{m=-2a}^{2a}\mathrm{sinc}(u)\mathrm{sinc}(u-m\pi)\mathrm{sinc}(v)\mathrm{sinc}(v+(m+s)\pi) \\
& \times\frac{1}{(2\pi)}\int_{2\pi\max(-a,-a+m)/\lambda}^{2\pi\min(a,a+m)/\lambda} g(\omega)\overline{g(\omega+\omega_m)}\left[f(\omega-\frac{2u}{\lambda})f(\omega+\frac{2v}{\lambda}+\omega_r) - f(\omega)f(\omega+\omega_r)\right]d\omega dvdu \\
= & I_1 + I_2
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 = & \int_{\mathbb{R}^2}\sum_{m=-2a}^{2a}\mathrm{sinc}(u)\mathrm{sinc}(u-m\pi)\mathrm{sinc}(v)\mathrm{sinc}(v+(m+s)\pi) \\
& \times\frac{1}{(2\pi)}\int_{2\pi\max(-a,-a+m)/\lambda}^{2\pi\min(a,a+m)/\lambda} g(\omega)\overline{g(\omega+\omega_m)}f(\omega+\frac{2v}{\lambda}+\omega_r)\left[f(\omega-\frac{2u}{\lambda}) - f(\omega)\right]d\omega dvdu \\
= & \int_{\mathbb{R}}\sum_{m=-2a}^{2a}\mathrm{sinc}(v)\mathrm{sinc}(v+(m+s)\pi)D_m(v)dv
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}^2} \sum_{m=-2a}^{2a} \mathrm{sinc}(u)\mathrm{sinc}(u-m\pi)\mathrm{sinc}(v)\mathrm{sinc}(v+(m+s)\pi) \\
&\quad \times \frac{1}{(2\pi)} \int_{2\pi\max(-a,-a+m)/\lambda}^{2\pi\min(a,a+m)/\lambda} g(\omega)\overline{g(\omega+\omega_m)} f(\omega) \left[f(\omega + \frac{2v}{\lambda} + \omega_r) - f(\omega+\omega_r) \right] d\omega dv du \\
&= \sum_{m=-2a}^{2a} d_m \int_{\mathbb{R}} \mathrm{sinc}(u)\mathrm{sinc}(u-m\pi) du
\end{aligned}
$$

with

$$
\begin{aligned}
& D_m(v) = \\
& \int_{\mathbb{R}} \mathrm{sinc}(u)\mathrm{sinc}(u-m\pi) \frac{1}{(2\pi)} \int_{2\pi\max(-a,-a+m)/\lambda}^{2\pi\min(a,a+m)/\lambda} g(\omega)\overline{g(\omega+\omega_m)} f(\omega+\frac{2v}{\lambda}+\omega_r) \\
& \times \left[f(\omega - \frac{2u}{\lambda}) - f(\omega) \right] d\omega du
\end{aligned}
$$

and

$$
\begin{aligned}
& d_m = \\
& \int_{\mathbb{R}} \mathrm{sinc}(v)\mathrm{sinc}(v+(m+s)\pi) \frac{1}{(2\pi)} \int_{2\pi\max(-a,-a+m)/\lambda}^{2\pi\min(a,a+m)/\lambda} g(\omega)\overline{g(\omega+\omega_m)} f(\omega) \\
& \times \left[f(\omega + \frac{2v}{\lambda} + \omega_r) - f(\omega+\omega_r) \right] d\omega dv.
\end{aligned}
$$

Since the functions $f(\cdot)$ and $g(\cdot)$ satisfy the conditions stated in Lemma F.2, the lemma can be used to show that

$$
\max_{|m|\leq a} \sup_{v} |D_m(v)| \leq C\left(\frac{\log\lambda + \log a}{\lambda} \right)
$$

and

$$
\max_{|m|\leq a} |d_m| \leq C\left(\frac{\log\lambda + \log a}{\lambda} \right).
$$

Substituting these bounds into I_1 and I_2 give

$$
\begin{aligned}
|I_1| &\leq C\left(\frac{\log\lambda + \log a}{\lambda} \right) \sum_{m=-2a}^{2a} \int_{-\infty}^{\infty} |\mathrm{sinc}(v)\mathrm{sinc}(v+(m+s)\pi)| dv \\
|I_2| &\leq C\left(\frac{\log\lambda + \log a}{\lambda} \right) \sum_{m=-2a}^{2a} \int_{-\infty}^{\infty} |\mathrm{sinc}(u)\mathrm{sinc}(u-m\pi)| du.
\end{aligned}
$$

Therefore, by using Lemma F.1(ii) we have

$$
|I_1| \text{ and } |I_2| = O\left(\log^2(a) \frac{\log a + \log\lambda}{\lambda} \right).
$$

Since $|B_1(s;r) - C_1(s;r)| \leq |I_1| + |I_2|$ this gives the desired result. □

C Non-uniform sampled locations with frequency grid $\boldsymbol{\omega}_{\lambda,\boldsymbol{k}}$

Most of the results derived here are based on the methodology developed in the uniform sampling case.

In order to prove Theorem 4.3(i), we define the quantities $U_1(\cdot)$ and $U_2(\cdot)$

$$\begin{aligned} U_1(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{\omega}_{\boldsymbol{r}_1},\boldsymbol{\omega}_{\boldsymbol{r}_2}) &= U_{1,1}(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{\omega}_{\boldsymbol{r}_1},\boldsymbol{\omega}_{\boldsymbol{r}_2}) + U_{1,2}(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{\omega}_{\boldsymbol{r}_1},\boldsymbol{\omega}_{\boldsymbol{r}_2}) \\ U_2(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{\omega}_{\boldsymbol{r}_1},\boldsymbol{\omega}_{\boldsymbol{r}_2}) &= U_{2,1}(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{\omega}_{\boldsymbol{r}_1},\boldsymbol{\omega}_{\boldsymbol{r}_2}) + U_{2,2}(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{\omega}_{\boldsymbol{r}_1},\boldsymbol{\omega}_{\boldsymbol{r}_2}) \end{aligned} \tag{C.1}$$

where

$$\begin{aligned} & U_{1,1}(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{\omega}_{\boldsymbol{r}_1},\boldsymbol{\omega}_{\boldsymbol{r}_2}) \\ = & \frac{1}{(2\pi)^d}\sum_{\boldsymbol{j}_1+\ldots+\boldsymbol{j}_4=\boldsymbol{r}_1-\boldsymbol{r}_2}\gamma_{\boldsymbol{j}_1}\gamma_{\boldsymbol{j}_2}\gamma_{\boldsymbol{j}_3}\gamma_{\boldsymbol{j}_4}\times \\ & \int_{\mathcal{D}_{(\boldsymbol{j}_1+\boldsymbol{j}_3)}} g(\boldsymbol{\omega})\overline{g(\boldsymbol{\omega}+\boldsymbol{\omega}_{\boldsymbol{j}_1+\boldsymbol{j}_3})}f(\boldsymbol{\omega}+\boldsymbol{\omega}_{\boldsymbol{j}_1})f(\boldsymbol{\omega}+\boldsymbol{\omega}_{\boldsymbol{r}_1-\boldsymbol{j}_2})d\boldsymbol{\omega} \\ & U_{1,2}(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{\omega}_{\boldsymbol{r}_1},\boldsymbol{\omega}_{\boldsymbol{r}_2}) \\ = & \frac{1}{(2\pi)^d}\sum_{\boldsymbol{j}_1+\boldsymbol{j}_2+\boldsymbol{j}_3+\boldsymbol{j}_4=\boldsymbol{r}_1-\boldsymbol{r}_2}\gamma_{\boldsymbol{j}_1}\gamma_{\boldsymbol{j}_2}\gamma_{\boldsymbol{j}_3}\gamma_{\boldsymbol{j}_4}\times \\ & \int_{\mathcal{D}_{(\boldsymbol{r}_1-\boldsymbol{j}_2-\boldsymbol{j}_3)}} g(\boldsymbol{\omega})\overline{g(-\boldsymbol{\omega}-\boldsymbol{\omega}_{\boldsymbol{r}_2-\boldsymbol{j}_3-\boldsymbol{j}_2})}f(\boldsymbol{\omega}+\boldsymbol{\omega}_{\boldsymbol{j}_1})f(\boldsymbol{\omega}+\boldsymbol{\omega}_{\boldsymbol{r}_1-\boldsymbol{j}_2})d\boldsymbol{\omega} \\ & U_{2,1}(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{\omega}_{\boldsymbol{r}_1},\boldsymbol{\omega}_{\boldsymbol{r}_2}) \\ = & \frac{1}{(2\pi)^d}\sum_{\boldsymbol{j}_1+\boldsymbol{j}_2+\boldsymbol{j}_3+\boldsymbol{j}_4=\boldsymbol{r}_1+\boldsymbol{r}_2}\gamma_{\boldsymbol{j}_1}\gamma_{\boldsymbol{j}_2}\gamma_{\boldsymbol{j}_3}\gamma_{\boldsymbol{j}_4}\times \\ & \int_{\mathcal{D}_{(\boldsymbol{j}_1+\boldsymbol{j}_3)}} g(\boldsymbol{\omega})g(\boldsymbol{\omega}_{-\boldsymbol{j}_1-\boldsymbol{j}_3}-\boldsymbol{\omega})f(\boldsymbol{\omega}+\boldsymbol{\omega}_{\boldsymbol{j}_1})f(\boldsymbol{\omega}+\boldsymbol{\omega}_{\boldsymbol{r}_1-\boldsymbol{j}_2})d\boldsymbol{\omega} \\ & U_{2,2}(\boldsymbol{r}_1,\boldsymbol{r}_2;\boldsymbol{\omega}_{\boldsymbol{r}_1},\boldsymbol{\omega}_{\boldsymbol{r}_2}) \\ = & \frac{1}{(2\pi)^d}\sum_{\boldsymbol{j}_1+\boldsymbol{j}_2+\boldsymbol{j}_3+\boldsymbol{j}_4=\boldsymbol{r}_1+\boldsymbol{r}_2}\gamma_{\boldsymbol{j}_1}\gamma_{\boldsymbol{j}_2}\gamma_{\boldsymbol{j}_3}\gamma_{\boldsymbol{j}_4}\times \\ & \int_{\mathcal{D}_{(\boldsymbol{r}_1-\boldsymbol{j}_2-\boldsymbol{j}_3)}} g(\boldsymbol{\omega})g(\boldsymbol{\omega}-\boldsymbol{\omega}_{\boldsymbol{j}_2+\boldsymbol{j}_3-\boldsymbol{r}_1})f(\boldsymbol{\omega}+\boldsymbol{\omega}_{\boldsymbol{j}_1})f(\boldsymbol{\omega}+\boldsymbol{\omega}_{\boldsymbol{r}_1-\boldsymbol{j}_2})d\boldsymbol{\omega}, \end{aligned}$$

and the integral is defined as
$\int_{\mathcal{D}_{\boldsymbol{r}}} = \int_{2\pi\max(-a,-a-r_1)/\lambda}^{2\pi\min(a,a-r_1)/\lambda}\ldots\int_{2\pi\max(-a,-a-r_d)/\lambda}^{2\pi\min(a,a-r_d)/\lambda}$.

PROOF of Theorem 4.3(i) To prove (i) we use Theorem 2.1 and Lemma E.1 which immediately gives the result.

PROOF of Theorem 4.3(ii) We first note that by using Lemma E.1 (generalized to non-uniform sampling), we can show that

$$\lambda \mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g;r_1), \widetilde{Q}_{a,\lambda}(g;r_2)\right] = A_1(r_1,r_2) + A_2(r_1,r_2) + O\left(\frac{\lambda}{n}\right) \tag{C.2}$$

where

$$\begin{aligned}
A_1(r_1,r_2) &= \lambda \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})}\mathrm{cov}\big[Z(s_1)\exp(is_1\omega_{k_1}), Z(s_3)\exp(is_3\omega_{k_2})\big] \times \\
&\quad \mathrm{cov}\big[Z(s_2)\exp(-is_2\omega_{k_1+r_1}), Z(s_4)\exp(-is_4\omega_{k_2+r_2})\big] \\
A_2(r_1,r_2) &= \lambda \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})}\mathrm{cov}\big[Z(s_1)\exp(is_1\omega_{k_1}), Z(s_4)\exp(-is_4\omega_{k_2+r_2})\big] \times \\
&\quad \mathrm{cov}\big[Z(s_2)\exp(-is_2\omega_{k_1+r_1}), Z(s_3)\exp(is_3\omega_{k_2})\big].
\end{aligned}$$

We first analyze $A_1(r_1,r_2)$. Conditioning on the locations $s_1,\ldots,s_4$ gives

$$\begin{aligned}
& A_1(r_1,r_2) \\
= & \sum_{j_1,\ldots,j_4=-\infty}^{\infty} \gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4} \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})}\frac{1}{\lambda^3}\int_{[-\lambda/2,\lambda/2]^4} c(s_1-s_3)c(s_2-s_4) \\
& e^{is_1\omega_{k_1}-is_3\omega_{k_2}}e^{-is_2\omega_{k_1+r_1}+is_4\omega_{k_2+r_2}}e^{i(s_1\omega_{j_1}+s_2\omega_{j_2}+s_3\omega_{j_3}+s_4\omega_{j_4})}ds_1ds_2ds_3ds_4.
\end{aligned}$$

By using the spectral representation theorem and integrating out $s_1,\ldots,s_4$ we can write the above as

$$\begin{aligned}
& A_1(r_1,r_2) \\
= & \frac{\lambda}{(2\pi)^2}\sum_{j_1,\ldots,j_4=-\infty}^{\infty} \gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4} \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)f(y)\mathrm{sinc}\left(\frac{\lambda x}{2}+(k_1+j_1)\pi\right) \\
& \mathrm{sinc}\left(\frac{\lambda y}{2}-(r_1+k_1-j_2)\pi\right)\mathrm{sinc}\left(\frac{\lambda x}{2}+(k_2-j_3)\pi\right)\mathrm{sinc}\left(\frac{\lambda y}{2}-(r_2+k_2+j_4)\pi\right)dxdy \\
= & \frac{1}{\pi^2\lambda}\sum_{j_1,\ldots,j_4=-\infty}^{\infty} \gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4} \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\frac{2u}{\lambda}-\omega_{k_1+j_1})f(\frac{2v}{\lambda}+\omega_{k_1+r_1-j_2}) \\
& \times\mathrm{sinc}(u)\mathrm{sinc}(u+(k_2-k_1-j_1-j_3)\pi)\mathrm{sinc}(v)\mathrm{sinc}(v-(r_2+k_2+j_4-r_1-k_1+j_2)\pi)dudv.
\end{aligned} \tag{C.3}$$

By making a change of variables $m = k_1 - k_2$ we have

$$\begin{aligned}
& A_1(r_1,r_2) \\
= & \frac{1}{\pi^2\lambda}\sum_{j_1,\ldots,j_4=-\infty}^{\infty} \gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4} \sum_{m=-\infty}^{\infty}\sum_{k_1=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{m-k_1})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\frac{2u}{\lambda}-\omega_{k_1+j_1})\times \\
& f(\frac{2v}{\lambda}+\omega_{k_1+r_1-j_2})\mathrm{sinc}(u)\mathrm{sinc}(u+(m+j_1+j_3)\pi)\times \\
& \mathrm{sinc}(v)\mathrm{sinc}(v+(m-r_2-j_4+r_1-j_2)\pi)dudv.
\end{aligned}$$

Thus by taking absolutes we have

$$\begin{aligned}
& |A_1(r_1,r_2)| \\
\leq\ & \frac{1}{\pi^2\lambda}\sum_{j_1,\ldots,j_4=-\infty}^{\infty}|\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}|\sum_{m=-\infty}^{\infty}\sum_{k_1=-a}^{a}\left|g(\omega_{k_1})\overline{g(\omega_{m-k_1})}\right|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\frac{2u}{\lambda}-\omega_{k_1+j_1})\times \\
& f(\frac{2v}{\lambda}+\omega_{k_1+r_1-j_2})\,|\mathrm{sinc}(u)\mathrm{sinc}(u+(m+j_1+j_3)\pi)|\times \\
& |\mathrm{sinc}(v)\mathrm{sinc}(v+(m-r_2-j_4+r_1-j_2)\pi)|\,dudv.
\end{aligned}$$

Finally, by following the same series of bounds used to prove Lemma B.1(iii) we have

$$\begin{aligned}
|A_1(r_1,r_2)| \ \leq\ & \frac{1}{\pi^2\lambda}\sum_{j_1,\ldots,j_4=-\infty}^{\infty}|\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}|\sup_{\omega}|g(\omega)|^2\|f\|_2^2 \\
& \times\sum_{m=-\infty}^{\infty}\int_{\mathbb{R}^2}|\mathrm{sinc}(u-(m+j_1+j_3)\pi\mathrm{sinc}(v+(m-r_2+r_1-j_4-j_2)\pi)\times \\
& \mathrm{sinc}(u)\mathrm{sinc}(v)|dudv<\infty.
\end{aligned}$$

Similarly we can bound $A_2(r_1,r_2)$ and $\lambda\mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g;r_1),\overline{\widetilde{Q}_{a,\lambda}(g;r_2)}\right]$, thus giving the required result. □

PROOF of Theorem 4.3(iii) The proof uses the expansion (C.2). Using this as a basis, we will show that

$$\begin{aligned}
A_1(r_1,r_1) &= U_1(r_1,r_2;\omega_{r_1},\omega_{r_2})+O(\ell_{\lambda,a,n}) \\
A_2(r_1,r_1) &= U_2(r_1,r_2;\omega_{r_1},\omega_{r_2})+O(\ell_{\lambda,a,n}).
\end{aligned}$$

We find an approximation for $A_1(r_1,r_2)$ starting with the expansion given in (C.3). We use the same proof as that used to prove Lemma B.2 to approximate the terms inside the sum $\sum_{j_1,\ldots,j_4}$. More precisely we let $m=k_1-k_2$, replace $\sum_{k_1}$ with an integral and use the same methodology given in the proof of Lemma B.2 (and that $\sum_j|\gamma_j|<\infty$). Altogether this gives

$$\begin{aligned}
& A_1(r_1,r_2) \\
=\ & \frac{1}{2\pi^3}\sum_{j_1,\ldots,j_4=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}\sum_{m=-2a}^{2a}\int_{2\pi\max(-a,-a+m)/\lambda}^{2\pi\min(a,a+m)/\lambda}f(-\omega-\omega_{j_1})f(\omega+\omega_{r_1-j_2}) \\
& g(\omega)\overline{g(\omega-\omega_m)}d\omega\int_{\mathbb{R}^2}\mathrm{sinc}(u)\mathrm{sinc}(u+(m+j_1+j_3)\pi)\times \\
& \mathrm{sinc}(v)\mathrm{sinc}(v-(m-r_2-j_4+r_1-j_2)\pi)dudv+O(\ell_{\lambda,a,n}).
\end{aligned}$$

By orthogonality of the sinc function we see that the above is zero unless $m=-j_1-j_3$ *and*

$m = r_2 - r_1 + j_2 + j_4$ (and using that $f(-\omega - \omega_{j_1}) = f(\omega + \omega_{j_1})$), therefore

$$
\begin{aligned}
& A_1(r_1, r_2) \\
= \; & \frac{1}{2\pi} \sum_{j_1+\ldots+j_4 = r_1 - r_2} \gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4} \\
& \times \int_{2\pi \max(-a,-a-j_1-j_3)/\lambda}^{2\pi \min(a,a-j_1-j_3)/\lambda} g(\omega)\overline{g(\omega + \omega_{j_1+j_3})} f(\omega + \omega_{j_1}) f(\omega + \omega_{r_1 - j_2}) d\omega + O(\ell_{\lambda,a,n}).
\end{aligned}
$$

This gives us $U_{1,1}(r_1, r_2; \omega_{r_1}, \omega_{r_2})$. Next we consider $A_2(r_1, r_2)$

$$
\begin{aligned}
& A_2(r_1, r_2) \\
= \; & \frac{1}{\lambda^3} \sum_{j_1,j_2,j_3,j_4 \in \mathbb{Z}} \gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4} \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})} \times \\
& \int_{[-\lambda/2,\lambda/2]^4} c(s_1 - s_4)c(s_2 - s_3) \exp(is_1\omega_{k_1+j_1}) \exp(is_4\omega_{k_2+r_2+j_4}) \exp(-is_2\omega_{k_1+r_1-j_2}) \times \\
& \exp(-is_3\omega_{k_2-j_3}) ds_1 ds_2 ds_3 ds_4 \\
= \; & \lambda \sum_{j_1,j_2,j_3,j_4 \in \mathbb{Z}} \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})} \int_{\mathbb{R}^2} f(x) f(y) \text{sinc}\left(\frac{\lambda x}{2} + (k_1 + j_1)\pi\right) \times \\
& \text{sinc}\left(\frac{\lambda x}{2} - (k_2 + r_2 + j_4)\pi\right) \text{sinc}\left(\frac{\lambda y}{2} - (k_1 + r_1 - j_2)\pi\right) \text{sinc}\left(\frac{\lambda y}{2} + (k_2 - j_3)\pi\right) dxdy.
\end{aligned}
$$

Making a change of variables $u = \frac{\lambda x}{2} + (k_1 + j_1)\pi$ and $v = \frac{\lambda y}{2} - (k_1 + r_1 - j_2)\pi$ we have

$$
\begin{aligned}
& A_2(r_1.r_2) \\
= \; & \frac{1}{\pi^2 \lambda} \sum_{j_1,j_2,j_3,j_4 \in \mathbb{Z}} \gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4} \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})} \times \\
& \int_{\mathbb{R}^2} f\left(\frac{2u}{\lambda} - \omega_{k_1+j_1}\right) f\left(\frac{2v}{\lambda} + \omega_{k_1+r_1-j_2}\right) \text{sinc}(u) \times \\
& \text{sinc}\left(u - (k_2 + r_2 + j_4 + k_1 + j_1)\pi\right) \text{sinc}(v) \text{sinc}\left(v + (k_2 - j_3 + k_1 + r_1 - j_2)\pi\right) dudv.
\end{aligned}
$$

Again by using the same proof as that given in Lemma B.2 to approximate the terms inside the sum $\sum_{j_1,\ldots,j_4}$ (setting $m = k_1 + k_2$ and relace $\sum_{k_1}$ with an integral), we can approximate $A_2(r_1, r_2)$ with

$$
\begin{aligned}
& A_2(r_1, r_2) \\
= \; & \frac{1}{2\pi^3} \sum_{j_1,j_2,j_3,j_4 \in \mathbb{Z}} \gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4} \sum_{m=-2a}^{2a} \int_{2\pi(\max(-a,-a+m)/\lambda}^{2\pi \min(a,a+m)/\lambda} g(\omega)\overline{g(-\omega + \omega_m)} \times \\
& \int_{\mathbb{R}^2} f(-\omega - \omega_{j_1}) f(\omega + \omega_{r_1 - j_2}) \times \\
& \text{sinc}(u)\text{sinc}\left(u - (m + r_2 + j_4 + j_1)\pi\right) \text{sinc}(v) \text{sinc}\left(v + (m - j_3 + r_1 - j_2)\pi\right) dudvd\omega + O(\ell_{\lambda,a,n}).
\end{aligned}
$$

Using the orthogonality of the sinc function, the inner integral is non-zero when $m - j_3 + r_1 - j_2 = 0$ and $m + r_2 + j_4 + j_1 = 0$. Setting $m = -r_1 + j_2 + j_3$, this implies

$$\begin{aligned}
& A_2(r_1, r_2) \\
= & \frac{1}{2\pi} \sum_{j_1+j_2+j_3+j_4=r_1-r_2} \gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4} \int_{2\pi \max(-a,-a-r_1+j_2+j_3)/\lambda}^{2\pi \min(a,a-r_1+j_2+j_3)/\lambda} g(\omega)\overline{g(-\omega - \omega_{r_1-j_3-j_2})} \\
& \times f(\omega + \omega_{j_1}) f(\omega + \omega_{r_1-j_2}) d\omega + O(\ell_{\lambda,a,n}),
\end{aligned}$$

thus giving us $U_{1,2}(r_1, r_2; \omega_{r_1}, \omega_{r_2})$.

By using Lemma E.1, we can show that

$$\lambda \mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g; r_1), \overline{\widetilde{Q}_{a,\lambda}(g; r_2)}\right] = A_3(r_1, r_2) + A_4(r_1, r_2) + O\left(\frac{\lambda}{n}\right)$$

where

$$\begin{aligned}
A_3(r_1, r_2) &= \lambda \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1}) g(\omega_{k_2}) \mathrm{cov}\big[Z(s_1)\exp(is_1\omega_{k_1}), Z(s_3)\exp(-is_3\omega_{k_2})\big] \times \\
& \quad \mathrm{cov}\big[Z(s_2)\exp(-is_2\omega_{k_1+r_1}), Z(s_4)\exp(is_4\omega_{k_2+r_2})\big] \\
A_4(r_1, r_2) &= \lambda \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1}) g(\omega_{k_2}) \mathrm{cov}\big[Z(s_1)\exp(is_1\omega_{k_1}), Z(s_4)\exp(is_4\omega_{k_2+r_2})\big] \times \\
& \quad \mathrm{cov}\big[Z(s_2)\exp(-is_2\omega_{k_1+r_1}), Z(s_3)\exp(-is_3\omega_{k_2})\big].
\end{aligned}$$

By following the same proof as that used to prove $A_1(r_1, r_2)$ we have

$$\begin{aligned}
& A_3(r_1, r_2) \\
= & \frac{\lambda}{(2\pi)^2} \sum_{j_1,\ldots,j_4=-\infty}^{\infty} \gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4} \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1}) g(\omega_{k_2}) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x) f(y) \mathrm{sinc}\left(\frac{\lambda x}{2} + (k_1 + j_1)\pi\right) \\
& \mathrm{sinc}\left(\frac{\lambda x}{2} - (k_2 + j_3)\pi\right) \mathrm{sinc}\left(\frac{\lambda y}{2} - (r_1 + k_1 - j_2)\pi\right) \mathrm{sinc}\left(\frac{\lambda y}{2} + (r_2 + k_2 - j_4)\pi\right) dxdy \\
= & \frac{1}{\pi^2 \lambda} \sum_{j_1,\ldots,j_4=-\infty}^{\infty} \gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4} \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1}) g(\omega_{k_2}) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\frac{2u}{\lambda} - \omega_{k_1+j_1}) f(\frac{2v}{\lambda} + \omega_{k_1+r_1-j_2}) \\
& \times \mathrm{sinc}(u)\mathrm{sinc}(u - (k_2 + k_1 + j_1 + j_3)\pi)\mathrm{sinc}(v)\mathrm{sinc}(v + (r_2 + k_2 - j_4 + r_1 + k_1 - j_2)\pi) dudv \\
= & \frac{1}{\pi^2 \lambda} \sum_{j_1,\ldots,j_4=-\infty}^{\infty} \gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4} \sum_{m=-2a}^{2a} \sum_{k_1=\max(-a,-a+m)}^{\min(a,a+m)} g(\omega_{k_1}) g(\omega_{m-k_1}) \\
& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\frac{2u}{\lambda} - \omega_{k_1+j_1}) f(\frac{2v}{\lambda} + \omega_{k_1+r_1-j_2}) \\
& \times \mathrm{sinc}(u)\mathrm{sinc}(u - (m + j_1 + j_3)\pi)\mathrm{sinc}(v)\mathrm{sinc}(v + (r_2 + m - j_4 + r_1 - j_2)\pi) dudv.
\end{aligned}$$

Again using the method used to bound $A_1(r_1,r_2)$ gives

$$A_3(r_1,r_2) = \frac{1}{2\pi^3}\sum_{j_1+j_2+j_3+j_4=r_1+r_2}\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}\sum_m\int_{2\pi\max(-a,-a-j_1-j_3)/\lambda}^{2\pi\min(a,a-j_1-j_3)/\lambda} g(\omega)g(\omega_{-j_1-j_3}-\omega)\times$$
$$f(\omega+\omega_{j_1})f(\omega+\omega_{r_1-j_2})d\omega + O(\ell_{\lambda,a,n}) = U_{1,2}(r_1,r_2;\omega_{r_1},\omega_{r_2}) + O(\ell_{\lambda,a,n}).$$

Finally we consider $A_4(r_1,r_2)$. Using the same expansion as the above we have

$$\begin{aligned}
& A_4(r_1,r_2)\\
= & \frac{1}{\lambda^3}\sum_{j_1,j_2,j_3,j_4\in\mathbb{Z}}\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}\sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})g(\omega_{k_2})\int_{[-\lambda/2,\lambda/2]^4} c(s_1-s_4)c(s_2-s_3)\exp(is_1\omega_{k_1+j_1})\\
& \exp(-is_4\omega_{k_2+r_2-j_4})\exp(-is_2\omega_{k_1+r_1-j_2})\exp(is_3\omega_{k_2+j_3})ds_1ds_2ds_3ds_4\\
= & \lambda\sum_{j_1,j_2,j_3,j_4\in\mathbb{Z}}\sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})g(\omega_{k_2})\int_{\mathbb{R}^2} f(x)f(y)\mathrm{sinc}\left(\frac{\lambda x}{2}+(k_1+j_1)\pi\right)\times\\
& \mathrm{sinc}\left(\frac{\lambda x}{2}+(k_2+r_2-j_4)\pi\right)\mathrm{sinc}\left(\frac{\lambda y}{2}-(k_1+r_1-j_2)\pi\right)\mathrm{sinc}\left(\frac{\lambda y}{2}-(k_2+j_3)\pi\right)dxdy\\
= & \frac{1}{\pi^2\lambda}\sum_{j_1,j_2,j_3,j_4\in\mathbb{Z}}\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}\sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})g(\omega_{k_2})\int_{\mathbb{R}^2} f(\frac{2u}{\lambda}-\omega_{k_1+j_1})f(\frac{2v}{\lambda}+\omega_{k_1+r_1-j_2})\times\\
& \mathrm{sinc}(u)\mathrm{sinc}\left(u+(k_2+r_2-j_4-k_1-j_1)\pi\right)\mathrm{sinc}(v)\mathrm{sinc}\left(v-(k_2+j_3-k_1-r_1+j_2)\pi\right)dudv\\
= & \frac{1}{2\pi^3}\sum_{j_1,j_2,j_3,j_4\in\mathbb{Z}}\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}\sum_{m=-2a}^{2a}\int_{2\pi\max(-a,-a+m)/\lambda}^{2\pi\min(a,a+m)} g(\omega)g(\omega-\omega_m)\times\\
& \int_{\mathbb{R}^2} f(-\omega-\omega_{j_1})f(\omega+\omega_{r_1-j_2})\times\\
& \mathrm{sinc}(u)\mathrm{sinc}\left(u+(r_2-j_4-m-j_1)\pi\right)\mathrm{sinc}(v)\mathrm{sinc}\left(v-(j_3-m-r_1+j_2)\pi\right)dudvd\omega + O(\ell_{\lambda,a,n})\\
= & \frac{1}{2\pi}\sum_{j_1+j_2+j_3+j_4=r_1+r_2}\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}\int_{\max(-a,a-r_1+j_2+j_3)/\lambda}^{2\pi\min(a,a-r_1+j_2+j_3)/\lambda} g(\omega)g(\omega-\omega_{j_2+j_3-r_2})\\
& \times f(\omega+\omega_{j_1})f(\omega+\omega_{r_1-j_2})d\omega + O(\ell_{\lambda,a,n}).
\end{aligned}$$

This gives the desired result. □

We now obtain an approximation of $U_1(\cdot)$ and $U_2(\cdot)$.

PROOF of Corollary 4.1 By using Lipschitz continuity of $g(\cdot)$ and $f(\cdot)$ and $|\gamma_{\boldsymbol{j}}| \leq C\prod_{i=1}^{d}|j_i|^{-(1+\delta)}I(j_i\neq 0)$ we obtain the result. □

PROOF of Theorem 4.6 We prove the result for the case $d=1$ and using $A_1(r_1,r_2),\ldots,A_4(r_1,r_2)$ defined in proof of Theorem 4.3. The proof is identical to the proof of Theorem B.2. Following the same notation in proof of Theorem B.2 we have

$$\lambda\mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g;r_1),\widetilde{Q}_{a,\lambda}(g;r_2)\right] = A_1(r_1,r_2)+A_2(r_1,r_2)+B_1(r_1,r_2)+B_2(r_1,r_2)+O\left(\frac{\lambda}{n}\right),$$

with $|B_2(r_1,r_2)| = O((a\lambda)^2/n^2)$ and the main term involving the trispectral density is

$$
\begin{aligned}
& B_1(r_1,r_2) \\
= \ & \lambda c_4 \sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})}\mathrm{E}\Big[\kappa_4(s_1-s_2,s_1-s_3,s_1-s_4)e^{is_1\omega_{k_1}}e^{-is_2\omega_{k_1+r_1}}e^{-is_3\omega_{k_2}}e^{is_4\omega_{k_2+r_2}}\Big] \\
= \ & \frac{c_4}{(2\pi)^3\lambda^3}\sum_{j_1,\ldots,j_4=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}\sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})}\int_{\mathbb{R}^3} f_4(\omega_1,\omega_2,\omega_3)\times \\
& \int_{[-\lambda/2,\lambda/2]^4} e^{is_1(\omega_1+\omega_2+\omega_3+\omega_{k_1+j_1})}e^{-is_2(\omega_1+\omega_{k_1+r_1-j_2})}e^{-is_3(\omega_2+\omega_{k_2-j_3})} \\
& e^{is_4(-\omega_3+\omega_{k_2+r_2+j_4})}ds_1ds_2ds_3ds_4d\omega_1d\omega_2d\omega_3 \\
= \ & \frac{c_4\lambda}{(2\pi)^3}\sum_{j_1,\ldots,j_4=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}\sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})}\int_{\mathbb{R}^3} f_4(\omega_1,\omega_2,\omega_3)\times \\
& \mathrm{sinc}\left(\frac{\lambda(\omega_1+\omega_2+\omega_3)}{2}+(k_1+j_1)\pi\right)\mathrm{sinc}\left(\frac{\lambda\omega_1}{2}+(k_1+r_1-j_2)\pi\right)\times \\
& \mathrm{sinc}\left(\frac{\lambda\omega_2}{2}+(k_2-j_3)\pi\right)\mathrm{sinc}\left(\frac{\lambda\omega_3}{2}-(k_2+r_2+j_4)\pi\right)d\omega_1d\omega_2d\omega_3.
\end{aligned}
$$

Now we make a change of variables and let $u_1 = \frac{\lambda\omega_1}{2}+(k_1+r_1-j_2)$, $u_2 = \frac{\lambda\omega_2}{2}+(k_2-j_3)\pi$ and $u_3 = \frac{\lambda\omega_3}{2}-(k_2+r_2+j_4)\pi$, this gives

$$
\begin{aligned}
& B_1(r_1,r_2) \\
= \ & \frac{c_4}{\pi^3\lambda^2}\sum_{j_1,\ldots,j_4=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}\sum_{k_1,k_2=-a}^{a}\int_{\mathbb{R}^3} g(\omega_{k_1})\overline{g(\omega_{k_2})}\times \\
& f_4\left(\frac{2u_1}{\lambda}-\omega_{k_1+r_1-j_2},\frac{2u_2}{\lambda}-\omega_{k_2-j_3},\frac{2u_3}{\lambda}+\omega_{k_2+r_2+j_4}\right) \\
& \times\mathrm{sinc}\left(u_1+u_2+u_3+(r_2-r_1+j_1+j_2+j_3+j_4)\pi\right)\mathrm{sinc}(u_1)\mathrm{sinc}(u_2)\mathrm{sinc}(u_3)du_1du_2du_3.
\end{aligned}
$$

Next we exchange the summand with a double integral and use Lemma I.1(iii) together with Lemma F.1, equation (F.3) to obtain

$$
\begin{aligned}
& B_1(r_1,r_2) \\
= \ & \frac{c_4}{\pi^3(2\pi)^2}\sum_{j_1,\ldots,j_4=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}\int_{2\pi[-a/\lambda,a/\lambda]^2} g(\omega_1)\overline{g(\omega_2)}\times \\
& \int_{\mathbb{R}^3} f_4\left(\frac{2u_1}{\lambda}-\omega_1-\omega_{r_1-j_2},\frac{2u_2}{\lambda}-\omega_2-\omega_{j_3},\frac{2u_3}{\lambda}+\omega_2+\omega_{r_2+j_4}\right)\times \\
& \mathrm{sinc}\left(u_1+u_2+u_3+(r_2-r_1+j_1+j_2+j_3+j_4)\pi\right)\times \\
& \mathrm{sinc}(u_1)\mathrm{sinc}(u_2)\mathrm{sinc}(u_3)du_1du_2du_3d\omega_1d\omega_2+O\left(\frac{1}{\lambda}\right).
\end{aligned}
$$

By using Lemma F.3, we replace $f_4\left(\frac{2u_1}{\lambda}-\omega_1-\omega_{r_1-j_2},\frac{2u_2}{\lambda}-\omega_2-\omega_{j_3},\frac{2u_3}{\lambda}+\omega_2+\omega_{r_2+j_4}\right)$ with $f_4\left(-\omega_1-\omega_{r_1-j_2},-\omega_2-\omega_{j_3},\omega_2+\omega_{r_2+j_4}\right)$, to give

$$\begin{aligned}
& B_1(r_1,r_2) \\
= \ & \frac{c_4}{\pi^3(2\pi)^2}\sum_{j_1,\ldots,j_4=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}\int_{2\pi[-a/\lambda,a/\lambda]^2}g(\omega_1)\overline{g(\omega_2)}\times \\
& \int_{\mathbb{R}^3}f_4\left(-\omega_1-\omega_{r_1-j_2},-\omega_2-\omega_{j_3},\omega_2+\omega_{r_2+j_4}\right)\times \\
& \text{sinc}\left(u_1+u_2+u_3+(r_2-r_1+j_1+j_2+j_3+j_4)\pi\right) \\
& \times\text{sinc}(u_1)\text{sinc}(u_2)\text{sinc}(u_3)du_1du_2du_3d\omega_1d\omega_2+O\left(\frac{\log^3(\lambda)}{\lambda}\right) \\
= \ & \frac{c_4}{(2\pi)^2}\sum_{j_1+j_2+j_3+j_4=r_1-r_2}\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}\int_{2\pi[-a/\lambda,a/\lambda]^2}g(\omega_1)\overline{g(\omega_2)}\times \\
& f_4\left(-\omega_1-\omega_{r_1-j_2},-\omega_2-\omega_{j_3},\omega_2+\omega_{r_2+j_4}\right)d\omega_1d\omega_2+O\left(\frac{\log^3(\lambda)}{\lambda}\right),
\end{aligned}$$

where the last line follows from (B.8).

To obtain an expression for $\lambda\text{cov}\left[\widetilde{Q}_{a,\lambda}(g;r_1),\overline{\widetilde{Q}_{a,\lambda}(g;r_2)}\right]$ we note that

$$\lambda\text{cov}\left[\widetilde{Q}_{a,\lambda}(g;r_1),\overline{\widetilde{Q}_{a,\lambda}(g;r_2)}\right]=A_3(r_1,r_2)+A_4(r_1,r_2)+B_3(r_1,r_2)+B_4(r_1,r_2)+O\left(\frac{\lambda}{n}\right),$$

just as in the proof of Theorem B.2 we can show that $|B_4(r_1,r_2)|=O((\lambda a)^d/n^2)$ and the leading term involving the trispectral density is

$$\begin{aligned}
& B_3(r_1,r_2) \\
= \ & \lambda c_4\sum_{k_1,k_2=-a}^{a}g(\omega_{k_1})g(\omega_{k_2})\text{E}\left[\kappa_4(s_1-s_2,s_1-s_3,s_1-s_4)e^{is_1\omega_{k_1}}e^{-is_2\omega_{k_1+r_1}}e^{is_3\omega_{k_2}}e^{-is_4\omega_{k_2+r_2}}\right] \\
= \ & \frac{c_4}{(2\pi)^3\lambda^3}\sum_{j_1,\ldots,j_4=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}\sum_{k_1,k_2=-a}^{a}g(\omega_{k_1})g(\omega_{k_2})\int_{\mathbb{R}^3}f_4(\omega_1,\omega_2,\omega_3)\times \\
& \int_{[-\lambda/2,\lambda/2]^4}e^{is_1(\omega_1+\omega_2+\omega_3+\omega_{k_1+j_1})} \\
& e^{-is_2(\omega_1+\omega_{k_1+r_1-j_2})}e^{-is_3(\omega_2-\omega_{k_2+j_3})}e^{is_4(-\omega_3-\omega_{k_2+r_2-j_4})}ds_1ds_2ds_3ds_4d\omega_1d\omega_2d\omega_3
\end{aligned}$$

Integrating out the locations

$$B_3(r_1,r_2) = \frac{c_4\lambda}{(2\pi)^3}\sum_{j_1,\ldots,j_4=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}\sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})\overline{g(\omega_{k_2})}\int_{\mathbb{R}^3} f_4(\omega_1,\omega_2,\omega_3)\times$$
$$\text{sinc}\left(\frac{\lambda(\omega_1+\omega_2+\omega_3)}{2}+(k_1+j_1)\pi\right)\text{sinc}\left(\frac{\lambda\omega_1}{2}+(k_1+r_1-j_2)\pi\right)\times$$
$$\text{sinc}\left(\frac{\lambda\omega_2}{2}-(k_2+j_3)\pi\right)\text{sinc}\left(\frac{\lambda\omega_3}{2}+(k_2+r_2-j_4)\pi\right)d\omega_1 d\omega_2 d\omega_3.$$

We make a change of variables $u_1 = \frac{\lambda\omega_1}{2}+(k_1+r_1-j_2)\pi$, $u_2 = \frac{\lambda\omega_2}{2}-(k_2+j_3)\pi$ and $u_3 = \frac{\lambda\omega_3}{2}+(k_2+r_2-j_4)\pi$. This gives

$$B_3(r_1,r_2)$$
$$= \frac{c_4}{\lambda^2(2\pi)^3}\sum_{j_1,\ldots,j_4=-\infty}^{\infty}\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}\sum_{k_1,k_2=-a}^{a} g(\omega_{k_1})g(\omega_{k_2})\times$$
$$\int_{\mathbb{R}^3} f_4\left(\frac{2u_1}{\lambda}-\omega_{k_1+r_1-j_2},\frac{2u_2}{\lambda}+\omega_{k_2+j_3},\frac{2u_3}{\lambda}-\omega_{k_2-r_2+j_4}\right)$$
$$\times\text{sinc}\left(u_1+u_2+u_3+(j_1+j_2+j_3+j_4-r_1-r_2)\pi\right)\text{sinc}(u_1)\text{sinc}(u_2)\text{sinc}(u_3)du_1du_2du_3$$
$$= \frac{1}{(2\pi)^2}\sum_{j_1+j_2+j_3+j_4=r_1+r_2}\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}\gamma_{j_4}\times$$
$$\int_{2\pi[-a/\lambda,a/\lambda]^2} g(\omega_1)g(\omega_2)f_4(-\omega_1-\omega_{r_1-j_2},\omega_2+\omega_{j_3},-\omega_2-\omega_{-r_2+j_4})d\omega_1 d\omega_2.$$

Finally, by replacing $f_4\left(-\omega_1-\omega_{r_1-j_2},-\omega_2-\omega_{j_3},\omega_2+\omega_{r_2+j_4}\right)$ with $f_4\left(-\omega_1,-\omega_2,\omega_2\right)$ and $f_4\left(-\omega_1-\omega_{r_1-j_2},\omega_2+\omega_{j_3},-\omega_2-\omega_{r_2+j_4}\right)$ with $f_4\left(-\omega_1,\omega_2,-\omega_2\right)$ in $B_1(r_1,r_2)$ and $B_3(r_1,r_2)$ respectively, and using the pointwise Lipschitz continuity of f_4 and that $|\gamma_j| \leq CI(j \neq 0)|j|^{-(1+\delta)}$ we obtain $B_1(r_1,r_2) = D_1+O\left(\frac{\log^3(\lambda)}{\lambda}+\frac{|r_1|+|r_2|}{\lambda}\right)$ and $B_3(r_1,r_2) = D_3+O\left(\frac{\log^3(\lambda)}{\lambda}+\frac{|r_1|+|r_2|}{\lambda}\right)$. Thus giving the required result. □

D Uniformly sampled locations with general frequency grid $\boldsymbol{\omega}_{\Omega,\boldsymbol{k}}$

In this section we calculate the variance of $\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r})$. We prove and expand on the results stated in Section 3. We assume that the spatial process is observed at $\{\boldsymbol{s}_j\}_{j=1}^{n}$ where $\boldsymbol{s}_j$ are iid uniformly distributed random variable defined on $[-\lambda/2,\lambda/2]^d$. We focus on the estimator

$$\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}) = \frac{1}{\Omega^d}\sum_{k_1,\ldots,k_d=-a}^{a} g(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})\overline{J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}+\boldsymbol{r}})}$$
$$-\frac{\lambda^d}{\Omega^d n}\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})\frac{1}{n}\sum_{j=1}^{n} Z(\boldsymbol{s}_j)^2\exp(-i\boldsymbol{s}_j'\boldsymbol{\omega}_{\Omega,\boldsymbol{r}}), \quad \text{(D.1)}$$

and $\boldsymbol{\omega}_{\Omega,\boldsymbol{k}} = 2\pi\boldsymbol{k}/\Omega$. In the previous section we considered the case $\Omega = \lambda$ here we consider the case that $\Omega \neq \lambda$, noting that when $\Omega > \lambda$ the frequency finer grid is finer and $\Omega < \lambda$ corresponds to a coarser frequency grid.

The rate $\mathrm{var}[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r})]$ converges to zero as $\lambda \to \infty$ depends on whether the ratio $\lambda/\Omega < 1$ or $\lambda/\Omega \geq 1$. The reason behind this difference is due to the following result.

Lemma D.1 *Let $T(\cdot)$ denote the triangle kernel where $T(x) = (1-|x|)$ for $|x| \leq 1$ else $T(x) = 0$ and suppose $\alpha > 0$. Then we have*

$$\sum_{k=-a}^{a} \mathrm{sinc}^2(\alpha k\pi) = \frac{1}{\alpha}\int_{-\alpha}^{\alpha} T\left(\frac{x}{\alpha}\right)\frac{\sin(2\pi(a+\frac{1}{2})x)}{\sin(\pi x)}dx, \tag{D.2}$$

$$\sum_{k\in\mathbb{Z}} \mathrm{sinc}^2(\alpha k\pi) = \frac{1}{\alpha}\sum_{|k|\leq\lfloor\alpha\rfloor} T\left(\frac{k}{\alpha}\right) \tag{D.3}$$

and

$$\sum_{k\in\mathbb{Z}} \mathrm{sinc}^2(\alpha k\pi) \to 1 \tag{D.4}$$

as $\alpha \to \infty$.

PROOF. We recall that $\mathrm{sinc}(\alpha x/2) = \frac{1}{\alpha}\int_{-\alpha/2}^{\alpha/2} e^{ix\omega}dx$ (Fourier transform of the rectangle kernel $\alpha^{-1}I_{-\alpha/2,\alpha/2}(x)$). Thus $\mathrm{sinc}^2(\alpha x/2)$ is the convolution of two rectangle kernels (which is the triangle kernel defined on $[-\alpha,\alpha]$) and

$$\mathrm{sinc}^2\left(\frac{\alpha}{2}\omega\right) = \frac{1}{\alpha}\int_{-\alpha}^{\alpha} T\left(\frac{x}{\alpha}\right)\exp(i\omega x)dx,$$

thus setting $\omega = k\pi$ we have

$$\mathrm{sinc}^2(\alpha k\pi) = \frac{1}{\alpha}\int_{-\alpha}^{\alpha} T\left(\frac{x}{\alpha}\right)\exp(i2\pi kx)dx.$$

Substituting this into (D.2) gives

$$\sum_{k=-a}^{a} \mathrm{sinc}^2(\alpha k\pi) = \frac{1}{\alpha}\int_{-\alpha}^{\alpha} T\left(\frac{x}{\alpha}\right)\left[\sum_{k=-a}^{a}\exp(i2\pi kx)\right]dx = \frac{1}{\alpha}\int_{-\alpha}^{\alpha} T\left(\frac{x}{\alpha}\right)\frac{\sin(2\pi(a+\frac{1}{2})x)}{\sin(\pi x)}dx,$$

we note that $D_a(x) = \frac{\sin(2\pi(a+\frac{1}{2})x)}{\sin(\pi x)}$ is the Dirichlet kernel. Thus proving (D.2). Next letting the limit in the sum $a \to \infty$ we use that the Dirichlet kernel limits to the generalized function

$$\frac{\sin(2\pi(a+\frac{1}{2})x)}{\sin(\pi x)} \to \sum_{m\in\mathbb{Z}} \delta_m(x) \qquad \text{as } a \to \infty$$

where $\delta_m(x)$ is the dirac delta function which is zero everywhere but m. Substituting this into the the above integral gives (D.3).

To prove (D.4) we note that using (D.3) we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \text{sinc}^2(\alpha k\pi) &= \frac{1}{\alpha}\sum_{|k|\leq\lfloor\alpha\rfloor} T\left(\frac{k}{\alpha}\right) = \frac{1}{\alpha}\left(1+2\sum_{1\leq|k|\leq\lfloor\alpha\rfloor}\left[1-\frac{|k|}{\alpha}\right]\right) \\
&= \frac{1}{\alpha}\left(1+\frac{2}{\alpha}\sum_{1\leq|k|\leq\lfloor\alpha\rfloor}(\alpha-|k|)\right) \\
&= \frac{1}{\alpha}\left(1+\frac{2}{\alpha}\sum_{1\leq|k|\leq\lfloor\alpha\rfloor}(\alpha-\lfloor\alpha\rfloor)+\frac{2}{\alpha}\sum_{1\leq|k|\leq\lfloor\alpha\rfloor}(\lfloor\alpha\rfloor-|k|)\right) \\
&= \frac{1}{\alpha}\left(1+\frac{2}{\alpha}\lfloor\alpha\rfloor(\alpha-\lfloor\alpha\rfloor)+\frac{1}{\alpha}\lfloor\alpha\rfloor(\lfloor\alpha\rfloor+1)\right) = 1+O\left(\frac{1}{\alpha}\right),
\end{aligned}$$

where $\lfloor x\rfloor$ denotes the smallest integer less than or equal to x. Thus we see that $\sum_{k\in\mathbb{Z}}\text{sinc}^2(\alpha k\pi)$ is uniformly bounded for all $\alpha\geq 1$ and as $\alpha\to\infty$ $\sum_{k\in\mathbb{Z}}\text{sinc}^2(\alpha k\pi)\to 1$, thus proving (D.4). □

We now summarize the pertinent points of the above lemma. If $0<\alpha\leq 1$ we have

$$\alpha\sum_{k\in\mathbb{Z}}\text{sinc}^2(\alpha k\pi) = T(0) = 1 \Rightarrow \sum_{k\in\mathbb{Z}}\text{sinc}^2(\alpha k\pi) = \frac{1}{\alpha}.$$

On the other hand, if $\alpha>1$ then $\sum_{k\in\mathbb{Z}}\text{sinc}^2(\alpha k\pi)$ is uniformly bounded for all α. Setting $\alpha=\frac{\lambda}{\Omega}$ this gives

$$\sum_{k=-\infty}^{\infty}\text{sinc}^2\left(\frac{\lambda}{\Omega}k\pi\right) = \begin{cases} \frac{\Omega}{\lambda} & \frac{\lambda}{\Omega}<1 \\ O(1) & \frac{\lambda}{\Omega}>1 \end{cases}, \tag{D.5}$$

where the $O(1)$ bound is uniform for $\Omega<\lambda$.

We show in the lemmas below this simple result determines the optimal choice of frequency grid. In the following lemma we obtain the first approximation under the assumption of Gaussianity of the spatial process.

Lemma D.2 *Suppose Assumptions 2.1, 2.3 and Assumptions 2.5(ii) and 2.6(b) hold. Then*

(i) If $\lambda<\Omega$ (finer frequency grid)

$$\lambda^d\text{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_1),\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_2)\right] = \frac{\lambda^d}{\Omega^{2d}}\left[A_1(\boldsymbol{r}_1,\boldsymbol{r}_2)+A_2(\boldsymbol{r}_1,\boldsymbol{r}_2)\right]+F_{1,\text{fine}},$$

$$\lambda^d\text{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_1),\overline{\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_2)}\right] = \frac{\lambda^d}{\Omega^{2d}}\left[A_3(\boldsymbol{r}_1,\boldsymbol{r}_2)+A_4(\boldsymbol{r}_1,\boldsymbol{r}_2)\right]+F_{2,\text{fine}},$$

with

$$F_{1,\text{fine}} \text{ and } F_{2,\text{fine}} = O\left(\frac{\lambda^d}{n} + \frac{\log^3(a)}{n}\right)$$

(ii) If $\lambda > \Omega \geq 1$ (coarser frequency grid)

$$\Omega^d \text{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_1), \widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_2)\right] = \frac{1}{\Omega^d}\left[A_1(\boldsymbol{r}_1,\boldsymbol{r}_2) + A_2(\boldsymbol{r}_1,\boldsymbol{r}_2)\right] + F_{1,\text{coarse}},$$

$$\Omega^d \text{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_1), \overline{\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_2)}\right] = \frac{1}{\Omega^d}\left[A_3(\boldsymbol{r}_1,\boldsymbol{r}_2) + A_4(\boldsymbol{r}_1,\boldsymbol{r}_2)\right] + F_{2,\text{coarse}},$$

with

$$F_{1,\text{coarse}} \text{ and } F_{2,\text{coarse}} = O\left(\frac{\lambda^d}{n} + \frac{\log^3(a)}{n} I_{\frac{\lambda}{\Omega}\notin\mathbb{Z}}\right).$$

where

$$A_1(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{1}{\pi^{2d}} \sum_{\boldsymbol{m}=-2a}^{2a} \sum_{\boldsymbol{k}=\max(-a,-a+\boldsymbol{m})}^{\min(a,a+\boldsymbol{m})} \int_{\mathbb{R}^{2d}} f(\frac{2\boldsymbol{u}}{\lambda} - \boldsymbol{\omega}_{\Omega,\boldsymbol{k}}) f(\frac{2\boldsymbol{v}}{\lambda} + \boldsymbol{\omega}_{\Omega,\boldsymbol{k}} + \boldsymbol{\omega}_{\Omega,\boldsymbol{r}_1})$$
$$g(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})\overline{g(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}} - \boldsymbol{\omega}_{\Omega,\boldsymbol{m}})} \text{Sinc}\left(\boldsymbol{u} - \frac{\lambda}{\Omega}\boldsymbol{m}\pi\right) \text{Sinc}\left(\boldsymbol{v} + \frac{\lambda}{\Omega}(\boldsymbol{m} + \boldsymbol{r}_1 - \boldsymbol{r}_2)\pi\right) \text{Sinc}(\boldsymbol{u})\text{Sinc}(\boldsymbol{v}) d\boldsymbol{u} d\boldsymbol{v}$$

$$A_2(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{1}{\pi^{2d}} \sum_{\boldsymbol{m}=-2a}^{2a} \sum_{\boldsymbol{k}=\max(-a,-a+\boldsymbol{m})}^{\min(a,a+\boldsymbol{m})} \int_{\mathbb{R}^{2d}} f(\frac{2\boldsymbol{u}}{\lambda} - \boldsymbol{\omega}_{\Omega,\boldsymbol{k}}) f(\frac{2\boldsymbol{v}}{\lambda} + \boldsymbol{\omega}_{\Omega,\boldsymbol{k}} + \boldsymbol{\omega}_{\Omega,\boldsymbol{r}_1})$$
$$g(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})\overline{g(\boldsymbol{\omega}_{\Omega,\boldsymbol{m}-\boldsymbol{k}})} \text{Sinc}\left(\boldsymbol{u} - \frac{\lambda}{\Omega}(\boldsymbol{m} + \boldsymbol{r}_2)\pi\right) \text{Sinc}\left(\boldsymbol{v} + \frac{\lambda}{\Omega}(\boldsymbol{m} + \boldsymbol{r}_1)\pi\right) \text{Sinc}(\boldsymbol{u})\text{Sinc}(\boldsymbol{v}) d\boldsymbol{u} d\boldsymbol{v}$$

$$A_3(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{1}{\pi^{2d}} \sum_{\boldsymbol{m}=-2a}^{2a} \sum_{\boldsymbol{k}=\max(-a,-a+\boldsymbol{m})}^{\min(a,a+\boldsymbol{m})} \int_{\mathbb{R}^{2d}} f(\frac{2\boldsymbol{u}}{\lambda} - \boldsymbol{\omega}_{\Omega,\boldsymbol{k}}) f(\frac{2\boldsymbol{v}}{\lambda} + \boldsymbol{\omega}_{\Omega,\boldsymbol{k}} + \boldsymbol{\omega}_{\Omega,\boldsymbol{r}_1})$$
$$g(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}}) g(\boldsymbol{\omega}_{\Omega,\boldsymbol{m}-\boldsymbol{k}}) \text{Sinc}\left(\boldsymbol{u} + \frac{\lambda}{\Omega}\boldsymbol{m}\pi\right) \text{Sinc}\left(\boldsymbol{v} + \frac{\lambda}{\Omega}(\boldsymbol{m} + \boldsymbol{r}_2 + \boldsymbol{r}_1)\pi\right) \text{Sinc}(\boldsymbol{u})\text{Sinc}(\boldsymbol{v}) d\boldsymbol{u} d\boldsymbol{v}$$

$$A_4(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{1}{\pi^{2d}} \sum_{\boldsymbol{m}=-2a}^{2a} \sum_{\boldsymbol{k}=\max(-a,-a+\boldsymbol{m})}^{\min(a,a+\boldsymbol{m})} \int_{\mathbb{R}^{2d}} f(\frac{2\boldsymbol{u}}{\lambda} - \boldsymbol{\omega}_{\Omega,\boldsymbol{k}}) f(\frac{2\boldsymbol{v}}{\lambda} + \boldsymbol{\omega}_{\Omega,\boldsymbol{k}} + \boldsymbol{\omega}_{\Omega,\boldsymbol{r}_1})$$
$$g(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}}) g(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}-\boldsymbol{m}}) \text{Sinc}\left(\boldsymbol{u} - \frac{\lambda}{\Omega}(\boldsymbol{m} - \boldsymbol{r}_2)\pi\right) \text{Sinc}\left(\boldsymbol{v} + \frac{\lambda}{\Omega}(\boldsymbol{m} + \boldsymbol{r}_1)\pi\right) \text{Sinc}(\boldsymbol{u})\text{Sinc}(\boldsymbol{v}) d\boldsymbol{u} d\boldsymbol{v}$$

where $\boldsymbol{k} = \max(-a, -a - \boldsymbol{m}) = \{k_1 = \max(-a, -a - m_1), \ldots, k_d = \max(-a, -a - m_d)\}$, $\boldsymbol{k} = \min(-a + \boldsymbol{m}) = \{k_1 = \max(-a, -a + m_1), \ldots, k_d = \min(-a + m_d)\}$.

PROOF. To simplify notation, we prove the result for $d=1$.

Using a similar expansion to that in equation (E.3), for the case $\lambda/\Omega<1$ ("fine" frequency grid) we have

$$\begin{aligned}
&\lambda \mathrm{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;r_1),\widetilde{Q}_{a,\Omega,\lambda}(g;r_2)\right] \\
=\ & \frac{\lambda^3}{\Omega^2 n^4}\sum_{j_1,j_2,j_3,j_4\in\mathcal{D}_4}\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\Big(\\
&\mathrm{cum}\big[Z(s_{j_1})e^{is_{j_1}\omega_{\Omega,k_1}},Z(s_{j_3})e^{-is_{j_3}\omega_{\Omega,k_2}}\big]\mathrm{cum}\big[Z(s_{j_2})e^{-is_{j_2}\omega_{\Omega,k_1+r_1}},Z(s_{j_4})e^{is_{j_4}\omega_{\Omega,k_2+r_2}}\big] \\
&+\mathrm{cum}\big[Z(s_{j_1})e^{is_{j_1}\omega_{\Omega,k_1}},Z(s_{j_4})e^{is_{j_4}\omega_{\Omega,k_2+r_2}}\big]\mathrm{cum}\big[Z(s_{j_2})e^{-is_{j_2}\omega_{\Omega,k_1+r_2}},Z(s_{j_3})e^{-is_{j_3}\omega_{\Omega,k_2}}\big]\Big) \\
&+F_{1,\mathrm{fine}} \qquad \text{(D.6)}
\end{aligned}$$

whereas for the case $\lambda/\Omega\geq 1$ ("coarse" frequency grid)

$$\begin{aligned}
&\Omega \mathrm{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;r_1),\widetilde{Q}_{a,\Omega,\lambda}(g;r_2)\right] \\
=\ & \frac{\lambda^2}{\Omega n^4}\sum_{j_1,j_2,j_3,j_4\in\mathcal{D}_4}\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\Big(\\
&\mathrm{cum}\big[Z(s_{j_1})e^{is_{j_1}\omega_{\Omega,k_1}},Z(s_{j_3})e^{-is_{j_3}\omega_{\Omega,k_2}}\big]\mathrm{cum}\big[Z(s_{j_2})e^{-is_{j_2}\omega_{\Omega,k_1+r_1}},Z(s_{j_4})e^{is_{j_4}\omega_{\Omega,k_2+r_2}}\big] \\
&+\mathrm{cum}\big[Z(s_{j_1})e^{is_{j_1}\omega_{\Omega,k_1}},Z(s_{j_4})e^{is_{j_4}\omega_{\Omega,k_2+r_2}}\big]\mathrm{cum}\big[Z(s_{j_2})e^{-is_{j_2}\omega_{\Omega,k_1+r_2}},Z(s_{j_3})e^{-is_{j_3}\omega_{\Omega,k_2}}\big]\Big) \\
&+F_{1,\mathrm{coarse}} \qquad \text{(D.7)}
\end{aligned}$$

where

$$F_{1,\mathrm{fine}} = \lambda F_1 \quad \text{and} \quad F_{1,\mathrm{coarse}}=\Omega F_1$$

with

$$\begin{aligned}
F_1 =\ & \frac{\lambda^2}{\Omega^2 n^4}\sum_{j_1,j_2,j_3,j_4\in\mathcal{D}_3}\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})} \\
&\times\Big(\mathrm{cum}\big[Z(s_{j_1})e^{is_{j_1}\omega_{\Omega,k_1}},Z(s_{j_3})e^{-is_{j_3}\omega_{\Omega,k_2}}\big]\mathrm{cum}\big[Z(s_{j_2})e^{-is_{j_2}\omega_{\Omega,k_1+r_1}},Z(s_{j_4})e^{is_{j_4}\omega_{\Omega,k_2+r_2}}\big] \\
&+\mathrm{cum}\big[Z(s_{j_1})e^{is_{j_1}\omega_{\Omega,k_1}},Z(s_{j_4})e^{is_{j_4}\omega_{\Omega,k_2+r_2}}\big]\mathrm{cum}\big[Z(s_{j_2})e^{-is_{j_2}\omega_{\Omega,k_1+r_1}},Z(s_{j_3})e^{-is_{j_3}\omega_{\Omega,k_2}}\big] \\
&+\mathrm{cum}\big[Z(s_{j_1})e^{is_{j_1}\omega_{\Omega,k_1}},Z(s_{j_2})e^{-is_{j_2}\omega_{\Omega,k_1+r_1}},Z(s_{j_3})e^{-is_{j_3}\omega_{\Omega,k_2}},Z(s_{j_4})e^{is_{j_4}\omega_{\Omega,k_2+r_2}}\big]\Big),
\end{aligned}$$

$\mathcal{D}_4=\{j_1,\ldots,j_4=\text{all } j\text{s are different}\}$, $\mathcal{D}_3=\{j_1,\ldots,j_4;\text{two } j\text{s are the same but } j_1\neq j_2 \text{ and } j_3\neq j_4\}$ (noting that by definition of $\widetilde{Q}_{a,\Omega,\lambda}(g,0)$ more than two elements in $\{j_1,\ldots,j_4\}$ cannot be the same). By using the same techniques used in the proof of Lemma B.1 we can show that the first sum in (D.6) and (D.7) is equal to the $A_1(r_1,r_2)$ and $A_2(r_1,r_2)$ given at the start of the lemma. We now bound the remainders $F_{1,\mathrm{fine}}=\lambda F_1$ and $F_{1,\mathrm{coarse}}=\Omega F_1$.

To bound F_1 we note that it is comprised of terms which take the form

$$D_1 = \sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})} \times \\ \mathrm{cum}[Z(s_1)e^{is_1\omega_{\Omega,k_1}}, Z(s_2)e^{-is_2\omega_{\Omega,k_1+r_1}}, Z(s_3)e^{-is_3\omega_{\Omega,k_2}}, Z(s_1)e^{is_1\omega_{\Omega,k_2+r_2}}]$$

and

$$D_2 = \sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})} \times \\ \mathrm{cum}\left[Z(s_1)e^{is_1\omega_{\Omega,k_1}}, Z(s_1)e^{-is_1\omega_{\Omega,k_2}}\right] \mathrm{cum}\left[Z(s_2)e^{-is_2\omega_{\Omega,k_1+r_1}}, Z(s_3)e^{is_3\omega_{\Omega,k_2+r_2}}\right].$$

Thus

$$F_1 = O\left(\frac{\lambda^2}{\Omega^2 n}\left[D_1 + D_2\right]\right).$$

Bounds for D_1 and D_2 are given in Lemma E.2. Using these bounds we have

$$\begin{aligned} F_1 &\leq C\frac{\lambda^2}{\Omega^2 n}\begin{cases} \frac{\Omega}{\lambda} + \left(\frac{\Omega}{\lambda}\right)^2 + \left[\frac{\Omega^2}{\lambda^3}\log^3(a)\right] I_{\frac{\lambda}{\Omega}\notin\mathbb{Z}} & \frac{\lambda}{\Omega} \geq 1 \\ \left(\frac{\Omega}{\lambda}\right)^2 + \frac{\Omega}{\lambda^2}\log(a) + \frac{\Omega^2}{\lambda^3}\log^3(a) & \frac{\lambda}{\Omega} < 1 \end{cases} \\ &= C\begin{cases} \frac{\lambda}{n\Omega} + \frac{1}{n} + \left[\frac{1}{n\lambda}\log^3(a)\right] I_{\frac{\lambda}{\Omega}\notin\mathbb{Z}} & \frac{\lambda}{\Omega} \geq 1 \\ \frac{1}{n} + \frac{1}{n\Omega}\log(a) + \frac{1}{n\lambda}\log^3(a) & \frac{\lambda}{\Omega} < 1 \end{cases} \end{aligned}$$

Finally, since $F_{1,\text{fine}} = \lambda F_1$ $(\lambda < \Omega)$ and $F_{1,\text{coarse}} = \Omega F_1$ $(\lambda > \Omega)$ we have

$$F_{1,\text{fine}} = O\left(\frac{\lambda}{n} + \frac{\log^3(a)}{n}\right) \quad \text{and} \quad F_{1,\text{coarse}} = O\left(\frac{\lambda}{n} + \frac{\log^3(a)}{n} I_{\frac{\lambda}{\Omega}\notin\mathbb{Z}}\right).$$

A similar expression holds for $\lambda\mathrm{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;r_1), \overline{\widetilde{Q}_{a,\Omega,\lambda}(g;r_2)}\right]$. Thus we obtain the required result. □

Now we obtain approximations to $A_1(\boldsymbol{r}_1,\boldsymbol{r}_2),\ldots,A_4(\boldsymbol{r}_1,\boldsymbol{r}_2)$ by separating the sinc function from the spectral density. Let

$$C_{11}\left(\frac{a}{\Omega}, \boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right) = \frac{1}{(2\pi)^d}\int_{[-2\pi a/\Omega, 2\pi a/\Omega]^d} f(\boldsymbol{\omega})f(\boldsymbol{\omega}+\boldsymbol{\omega}_{\Omega,\boldsymbol{r}})|g(\boldsymbol{\omega})|^2 d\boldsymbol{\omega}$$

$$C_{12}\left(\frac{a}{\Omega}, \boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right) = \frac{1}{2\pi}\int_{\mathcal{D}_{\Omega,\boldsymbol{r}}} f(\boldsymbol{\omega})f(\boldsymbol{\omega}+\boldsymbol{\omega}_{\Omega,\boldsymbol{r}_1})g(\boldsymbol{\omega})\overline{g(-\boldsymbol{\omega})}d\boldsymbol{\omega}$$

$$C_{21}\left(\frac{a}{\Omega}, \boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right) = \frac{1}{(2\pi)^d}\int_{[-2\pi a/\Omega, 2\pi a/\Omega]^d} f(\boldsymbol{\omega})f(\boldsymbol{\omega}+\boldsymbol{\omega}_{\Omega,\boldsymbol{r}})g(\boldsymbol{\omega})^2 d\boldsymbol{\omega}$$

$$C_{22}\left(\frac{a}{\Omega}, \boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right) = \frac{1}{(2\pi)^d}\int_{\mathcal{D}_{\Omega,\boldsymbol{r}}} f(\boldsymbol{\omega})f(\boldsymbol{\omega}+\boldsymbol{\omega}_{\Omega,\boldsymbol{r}})g(\boldsymbol{\omega})g(-\boldsymbol{\omega})d\boldsymbol{\omega}$$

where $\int_{\mathcal{D}_{\Omega,\boldsymbol{r}}} = \int_{2\pi\max(-a,-a-r_1)/\Omega}^{2\pi\min(a,a-r_1)/\Omega} \cdots \int_{2\pi\max(-a,-a-r_d)/\Omega}^{2\pi\min(a,a-r_d)/\Omega}$ and define

$$\widetilde{A}_1(\boldsymbol{r}_1,\boldsymbol{r}_2) = C_{11}\left(\frac{a}{\Omega},\boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right)\sum_{\boldsymbol{m}=-2a}^{2a}\text{Sinc}\left(\frac{\lambda}{\Omega}\boldsymbol{m}\pi\right)\text{Sinc}\left(\frac{\lambda}{\Omega}(\boldsymbol{m}+\boldsymbol{r}_1-\boldsymbol{r}_2)\pi\right) \tag{D.8}$$

$$\widetilde{A}_2(\boldsymbol{r}_1,\boldsymbol{r}_2) = C_{12}\left(\frac{a}{\Omega},\boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right)\sum_{\boldsymbol{m}=-2a}^{2a}\text{Sinc}\left(\frac{\lambda}{\Omega}(\boldsymbol{m}+\boldsymbol{r}_1)\pi\right)\text{Sinc}\left(\frac{\lambda}{\Omega}(\boldsymbol{m}+\boldsymbol{r}_2)\pi\right) \tag{D.9}$$

$$\widetilde{A}_3(\boldsymbol{r}_1,\boldsymbol{r}_2) = C_{21}\left(\frac{a}{\Omega},\boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right)\sum_{\boldsymbol{m}=-2a}^{2a}\text{Sinc}\left(\frac{\lambda}{\Omega}\boldsymbol{m}\pi\right)\text{Sinc}\left(\frac{\lambda}{\Omega}(\boldsymbol{m}+\boldsymbol{r}_1+\boldsymbol{r}_2)\pi\right) \tag{D.10}$$

and

$$\widetilde{A}_4(\boldsymbol{r}_1,\boldsymbol{r}_2) = C_{22}\left(\frac{a}{\Omega},\boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right)\sum_{\boldsymbol{m}=-2a}^{2a}\text{Sinc}\left(\frac{\lambda}{\Omega}(\boldsymbol{m}+\boldsymbol{r}_1)\pi\right)\text{Sinc}\left(\frac{\lambda}{\Omega}(\boldsymbol{m}-\boldsymbol{r}_2)\pi\right). \tag{D.11}$$

Lemma D.3 *Suppose Assumptions 2.1, 2.3 and Assumptions 2.5(ii) and 2.6(b) hold. Let* $\widetilde{A}_1(\boldsymbol{r}_1,\boldsymbol{r}_2),\ldots,\widetilde{A}_4(\boldsymbol{r}_1,\boldsymbol{r}_2)$ *be defined as in (D.8), (D.9), (D.10) and (D.11). Then for* $\|\boldsymbol{r}_1\|_1, \|\boldsymbol{r}_2\|_1 \leq Ca$, $1 \leq \Omega, \lambda \leq a$ *and* $1 \leq j \leq 4$

(i) if $\lambda/\Omega < 1$ *then*

$$\frac{\lambda^d}{\Omega^{2d}}A_j(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{\lambda^d}{\Omega^d}\widetilde{A}_j(\boldsymbol{r}_1,\boldsymbol{r}_2) + O\left(\frac{\log^2(a)[\log a + \log\lambda]}{\lambda}\right)$$

(ii) if $\lambda/\Omega \geq 1$ *then*

$$\frac{1}{\Omega^d}A_j(\boldsymbol{r}_1,\boldsymbol{r}_2) = \widetilde{A}_j(\boldsymbol{r}_1,\boldsymbol{r}_2) + O\left(\frac{(\log^2 a)(\log\lambda + \log a)}{\Omega}\right)$$

PROOF. We prove the result for $j = 1$ and $d = 1$. We first define a sequence of approximations. We replace the sum $\frac{1}{\Omega}\sum_k$ in $\frac{1}{\Omega}A_1(r_1,r_2)$ with an integral to give

$$\begin{aligned} A_{1,1}(r_1,r_2) = {} & \frac{1}{\pi^2}\sum_{m=-2a}^{2a}\frac{1}{2\pi}\int_{2\pi\max(-a,-a+m)/\Omega}^{2\pi\min(a,a+m)/\Omega}\int_{\mathbb{R}^2} f(\frac{2u}{\lambda}-\omega)f(\frac{2v}{\lambda}+\omega+\omega_{\Omega,r_1}) \times \\ & g(\omega)\overline{g(\omega-\omega_{\Omega,m})}\text{sinc}\left(u-\frac{\lambda}{\Omega}m\pi\right)\text{sinc}\left(v+\frac{\lambda}{\Omega}(m+r_1-r_2)\pi\right) \times \\ & \text{sinc}(u)\text{sinc}(v)dudvd\omega. \end{aligned}$$

Next we replace $f(\frac{2v}{\lambda}+\omega+\omega_{\Omega,r_1})$ in $A_{1,1}(r_1,r_2)$ with $f(\omega+\omega_{\Omega,r_1})$ to give

$$\begin{aligned} A_{1,2}(r_1,r_2) &= \frac{1}{\pi^2}\sum_{m=-2a}^{2a}\frac{1}{2\pi}\int_{2\pi\max(-a,-a+m)/\Omega}^{2\pi\min(a,a+m)/\Omega} f(\omega)f(\omega+\omega_{\Omega,r_1})g(\omega)\overline{g(\omega-\omega_{\Omega,m})}d\omega \\ &\quad\times\int_{\mathbb{R}^2}\text{sinc}\left(u-\frac{\lambda}{\Omega}m\pi\right)\text{sinc}\left(v+\frac{\lambda}{\Omega}(m+r_1-r_2)\pi\right)\text{sinc}(u)\text{sinc}(v)dudv \\ &= \frac{1}{\pi^2}\sum_{m=-2a}^{2a}\text{sinc}\left(\frac{\lambda}{\Omega}m\pi\right)\text{sinc}\left(\frac{\lambda}{\Omega}(m+r_1-r_2)\pi\right) \\ &\quad\times\frac{1}{2\pi}\int_{2\pi\max(-a,-a+m)/\Omega}^{2\pi\min(a,a+m)/\Omega} f(\omega)f(\omega+\omega_{\Omega,r_1})g(\omega)\overline{g(\omega-\omega_{\Omega,m})}d\omega. \end{aligned}$$

In the proof we use the notation $\ell_p(u)$ defined in Lemma F.1.

We first prove part (i), where the fine frequency grid is used with $\lambda/\Omega<1$. By using Lemma I.1 (to replace summand by integral) and Lemma F.1, equation (F.1), we have

$$\begin{aligned} &\frac{\lambda}{\Omega}\left|\frac{1}{\Omega}A_1(r_1,r_2)-A_{1,1}(r_1,r_2)\right| \\ \leq\ & \frac{C\lambda}{\Omega^2}\sum_{m=-2a}^{2a}\int_{\mathbb{R}^2}\left|\text{sinc}\left(u-\frac{\lambda}{\Omega}m\pi\right)\text{sinc}\left(v+\frac{\lambda}{\Omega}(m+r_1-r_2)\pi\right)\text{sinc}(u)\text{sinc}(v)\right|dudv \\ \leq\ & \frac{C\lambda}{\Omega^2}\sum_{m=-2a}^{2a}\ell_1\left(\frac{\lambda}{\Omega}m\pi\right)\ell_1\left(\frac{\lambda}{\Omega}(m+r_1-r_2)\pi\right)=O\left(\frac{\log^2(a)}{\lambda}\right). \end{aligned}$$

Note that the above can also be bounded by $O(\log^3(a)/\Omega)$. We replace $f(\frac{2u}{\lambda}-\omega)f(\frac{2v}{\lambda}+\omega+\omega_{\Omega,r_1})$ with $f(-\omega)f(\omega+\omega_{\Omega,r_1})$. By using Lemma F.2, equation (F.5) (twice) we have

$$\begin{aligned} &\frac{\lambda}{\Omega}\left|A_{1,1}(r_1,r_2)-A_{1,2}(r_1,r_2)\right| \\ \leq\ & \frac{C\lambda}{\Omega}\sum_{m=-2a}^{2a}\left[\ell_1\left(\frac{\lambda m\pi}{\Omega}\right)\frac{\log\lambda+\log\left(1+\frac{\lambda}{\Omega}|m+r_1-r_2|\right)}{\lambda}\right]+ \\ & \frac{C\lambda}{\Omega}\sum_{m=-2a}^{2a}\left[\ell_1\left(\frac{\lambda(m+r_1-r_2)\pi}{\Omega}\right)\frac{\log\lambda+\log\left(1+\frac{\lambda}{\Omega}|m|\right)}{\lambda}\right]=O\left(\frac{\log^2(a)[\log\lambda+\log(a)]}{\lambda}\right). \end{aligned}$$

The final approximation is based on replacing $\overline{g(\omega-\omega_{\Omega,m})}$ with $\overline{g(\omega)}$ and the limits of the integral, $\int_{\max(-a,-a+m)/\Omega}^{\min(a,a+m)/\Omega}$ with $\int_{-a/\Omega}^{a/\Omega}$. To do this we note

$$\begin{aligned} \frac{\lambda}{\Omega}\left(A_{1,2}(r_1,r_2)-\widetilde{A}_1(r_1,r_2)\right) &= \frac{\lambda}{\Omega}\sum_{m=-2a}^{2a}\text{sinc}\left(\frac{\lambda}{\Omega}m\pi\right)\text{sinc}\left(\frac{\lambda}{\Omega}(m+r_1-r_2)\pi\right)\times \\ \frac{1}{2\pi}&\left(\int_{\max(-a,-a+m)/\Omega}^{\min(a,a+m)/\Omega}f(\omega)^2g(\omega)\overline{g(\omega-\omega_{\Omega,m})}d\omega-\int_{-a/\Omega}^{a/\Omega}f(\omega)^2|g(\omega)|^2\right)d\omega. \end{aligned}$$

We assume $m > 0$ (though the same proof holds for $m < 0$) and bound the following difference

$$\begin{aligned}
& \left| \int_{2\pi \max(-a,-a+m)/\Omega}^{2\pi \min(a,a+m)/\Omega} f(\omega) f(\omega + \omega_{\Omega,r}) g(\omega) \overline{g(\omega - \omega_{\Omega,m})} d\omega - \int_{-a/\Omega}^{a/\Omega} f(\omega) f(\omega + \omega_{\Omega,r}) |g(\omega)|^2 \right| \\
\leq & \left| \int_{2\pi \max(-a,-a+m)/\Omega}^{2\pi \min(a,a+m)/\Omega} f(\omega) f(\omega + \omega_{\Omega,r}) g(\omega) \left[\overline{g(\omega - \omega_{\Omega,m})} - \overline{g(\omega)} \right] d\omega \right| + \\
& \left| \int_{-2\pi a/\Omega}^{2\pi(-a+m)/\Omega} f(\omega) f(\omega + \omega_{\Omega,r}) |g(\omega)|^2 d\omega \right|.
\end{aligned}$$

By using the Lipschitz continuity of g and the mean value theorem twice we have

$$\begin{aligned}
& \left| \int_{2\pi \max(-a,-a+m)/\Omega}^{2\pi \min(a,a+m)/\Omega} f(\omega) f(\omega + \omega_{\Omega,r}) g(\omega) \overline{g(\omega - \omega_{\Omega,m})} d\omega - \int_{-a/\Omega}^{a/\Omega} f(\omega) f(\omega + \omega_{\Omega,r}) |g(\omega)|^2 \right| \\
\leq & \ C \frac{|m|}{\Omega} \qquad \text{(D.12)}
\end{aligned}$$

where C is a finite constant which only depends on f and g. This bound gives

$$\begin{aligned}
\frac{\lambda}{\Omega} \left| A_{1,2}(r_1, r_2) - \widetilde{A}_1(r_1, r_2) \right| & \leq \frac{C}{\lambda} \sum_{m=-2a}^{2a} \left| \sin\left(\frac{\lambda}{\Omega} m\pi\right) \sin\left(\frac{\lambda}{\Omega}(m + r_1 - r_2)\pi\right) \right| \ell_0(|m + r_1 - r_2|) \\
& = O\left(\frac{\log(a + |r_1| + |r_2|)}{\lambda}\right).
\end{aligned}$$

Since $|r_1|, |r_2| < a$, altogether the three bounds above give

$$\frac{\lambda}{\Omega^2} A_j(r_1, r_2) = \frac{\lambda}{\Omega} \widetilde{A}_j(r_1, r_2) + O\left(\frac{\log^2(a)[\log a + \log \lambda]}{\lambda}\right)$$

and same sequence of bounds apply to $A_j(r_1, r_2)$ for $2 \leq j \leq 4$, thus proving (i).

Now we prove (ii), where the coarse frequency is used with $\lambda/\Omega > 1$. From Lemma D.1 we observe that the rate of growth of $A_j(r_1, r_2)$ and $\widetilde{A}_j(r_1, r_2)$ are different when $\lambda/\Omega < 1$ and $\lambda/\Omega > 1$, thus requiring different standardisations. By using Lemma I.1 (to replace summand by integral) and Lemma F.1, equation (F.1), we have

$$\begin{aligned}
& \left| \frac{1}{\Omega} A_1(r_1, r_2) - A_{1,1}(r_1, r_2) \right| \\
\leq & \ \frac{C}{\Omega} \sum_{m=-2a}^{2a} \int_{\mathbb{R}^2} \left| \text{sinc}\left(u - \frac{\lambda}{\Omega} m\pi\right) \text{sinc}\left(v + \frac{\lambda}{\Omega}(m + r_1 - r_2)\pi\right) \text{sinc}(u) \text{sinc}(v) \right| du dv \\
\leq & \ \frac{C}{\Omega} \sum_{m=-2a}^{2a} \ell_1\left(\frac{\lambda}{\Omega} m\pi\right) \ell_1\left(\frac{\lambda}{\Omega}(m + r_1 - r_2)\pi\right) = O\left(\frac{\Omega \log^2(a)}{\lambda^2}\right).
\end{aligned}$$

By using Lemma F.2, equation (F.5) (twice) we have

$$
\begin{aligned}
& \left|A_{1,1}(r_1,r_2) - A_{1,2}(r_1,r_2)\right| \\
\leq\ & C\sum_{m=-2a}^{2a}\left[\ell_1\left(\frac{\lambda m\pi}{\Omega}\right)\frac{\log\lambda + \log\left(1+\frac{\lambda}{\Omega}|m+r_1-r_2|\right)}{\lambda}\right] + \\
& C\sum_{m=-2a}^{2a}\left[\ell_1\left(\frac{\lambda(m+r_1-r_2)\pi}{\Omega}\right)\frac{\log\lambda + \log\left(1+\frac{\lambda}{\Omega}|m|\right)}{\lambda}\right] = O\left(\frac{\log^2(a)[\log\lambda + \log(\frac{\lambda}{\Omega}a)]}{\Omega}\right).
\end{aligned}
$$

Replacing $\overline{g(\omega-\omega_{\Omega,m})}$ with $\overline{g(\omega)}$ and limits of the integral, $\int_{\max(-a,-a+m)/\Omega}^{\min(a,a+m)/\Omega}$ with $\int_{-a/\Omega}^{a/\Omega}$ gives the difference

$$
\begin{aligned}
& A_{1,2}(r_1,r_2) - \widetilde{A}_1(r_1,r_2) = \sum_{m=-2a}^{2a}\mathrm{sinc}\left(\frac{\lambda}{\Omega}m\pi\right)\mathrm{sinc}\left(\frac{\lambda}{\Omega}(m+r_1-r_2)\pi\right)\times \\
& \frac{1}{2\pi}\left(\int_{2\pi\max(-a,-a+m)/\Omega}^{2\pi\min(a,a+m)/\Omega} f(\omega)f(\omega+\omega_{\Omega,r})g(\omega)\overline{g(\omega-\omega_{\Omega,m})}d\omega - \int_{-a/\Omega}^{a/\Omega} f(\omega)f(\omega+\omega_{\Omega,r})|g(\omega)|^2\right)d\omega,
\end{aligned}
$$

where $\widetilde{A}_1(r_1,r_2)$ is defined in (D.8). Finally, using (D.12) we have

$$
\begin{aligned}
\left|A_{1,2}(r_1,r_2) - \widetilde{A}_1(r_1,r_2)\right| &\leq \frac{\Omega}{\lambda^2}\sum_{m=-2a}^{2a}\left|\sin\left(\frac{\lambda}{\Omega}m\pi\right)\sin\left(\frac{\lambda}{\Omega}(m+r_1-r_2)\pi\right)\right|\ell_0(m+r_1-r_2) \\
&= O\left(\frac{\Omega\log(\frac{\lambda}{\Omega}a)}{\lambda^2}\right).
\end{aligned}
$$

Thus the three bounds together give

$$
\frac{1}{\Omega}A_1(r_1,r_2) = \widetilde{A}_1(r_1,r_2) + O\left(\frac{(\log^2 a)[\log\lambda + \log(a)]}{\Omega}\right).
$$

The same sequence bounds apply to $A_j(r_1,r_2)$ for $2 \leq j \leq 4$. Thus proving (ii). □

Altogether the above gives the following corollary.

Corollary D.1 *Suppose Assumptions 2.1, 2.3 and Assumptions 2.5(ii) and 2.6(b) hold. Let* $\widetilde{A}_1(\boldsymbol{r}_1,\boldsymbol{r}_2),\ldots,\widetilde{A}_4(\boldsymbol{r}_1,\boldsymbol{r}_2)$ *be defined as in (D.8), (D.9), (D.10) and (D.11). Then for* $\|\boldsymbol{r}_1\|_1, \|\boldsymbol{r}_2\|_1 \leq Ca$

(i) If $\lambda < \Omega$

$$
\begin{aligned}
\lambda^d \mathrm{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_1), \widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_2)\right] &= \frac{\lambda^d}{\Omega^d}\left(\widetilde{A}_1(\boldsymbol{r}_1,\boldsymbol{r}_2) + \widetilde{A}_2(\boldsymbol{r}_1,\boldsymbol{r}_2)\right) \\
&\quad + O\left(\frac{\lambda^d}{n} + \frac{\log^3(a)}{n} + \frac{\log^3(a)}{\lambda}\right),
\end{aligned}
$$

$$\lambda^d \text{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_1),\overline{\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_2)}\right] = \frac{\lambda^d}{\Omega^d}\left(\widetilde{A}_3(\boldsymbol{r}_1,\boldsymbol{r}_2)+\widetilde{A}_4(\boldsymbol{r}_1,\boldsymbol{r}_2)\right) +O\left(\frac{\lambda^d}{n}+\frac{\log^3(a)}{n}+\frac{\log^3(a)}{\lambda}\right),$$

(ii) If $\lambda \geq \Omega$

$$\Omega^d \text{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_1),\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_2)\right] = \left(\widetilde{A}_1(\boldsymbol{r}_1,\boldsymbol{r}_2)+\widetilde{A}_2(\boldsymbol{r}_1,\boldsymbol{r}_2)\right) +O\left(\frac{\lambda^d}{n}+\frac{\log^3(a)}{n}+\frac{\log^3(a)}{\Omega}\right),$$

$$\Omega^d \text{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;r_1),\overline{\widetilde{Q}_{a,\Omega,\lambda}(g;r_2)}\right] = \left(\widetilde{A}_3(\boldsymbol{r}_1,\boldsymbol{r}_2)+\widetilde{A}_4(\boldsymbol{r}_1,\boldsymbol{r}_2)\right) +O\left(\frac{\lambda^d}{n}+\frac{\log^3(a)}{n}+\frac{\log^3(a)}{\Omega}\right),$$

where $\widetilde{A}_1(\boldsymbol{r}_1,\boldsymbol{r}_2)$, $\widetilde{A}_2(\boldsymbol{r}_1,\boldsymbol{r}_2)$, $\widetilde{A}_3(\boldsymbol{r}_1,\boldsymbol{r}_2)$, $\widetilde{A}_3(\boldsymbol{r}_1,\boldsymbol{r}_2)$ are defined in (D.8)-(D.11).

We now summarize the key points in the above result. Using the above result we have

$$\begin{aligned} & \text{var}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r})\right] \\ = \; & C_{11}\left(\frac{a}{\Omega},\boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right)\frac{1}{\Omega^d}\sum_{\boldsymbol{m}=-2a}^{2a}\text{sinc}^2\left(\frac{\lambda}{\Omega}\boldsymbol{m}\pi\right)+C_{12}\left(\frac{a}{\Omega},\boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right)\frac{1}{\Omega^d}\sum_{m=-2a}^{2a}\text{sinc}^2\left(\frac{\lambda}{\Omega}(\boldsymbol{m}+\boldsymbol{r})\pi\right) \\ & +O\left(\widetilde{\ell}_{a,\Omega,\lambda}\left[\frac{1}{\lambda^d}I_{\frac{\lambda}{\Omega}<1}+\frac{1}{\Omega^d}I_{\frac{\lambda}{\Omega}\geq 1}\right]\right). \end{aligned} \tag{D.13}$$

where

$$\widetilde{\ell}_{a,\Omega,\lambda} = \frac{\lambda^d}{n}+\frac{\log^3(a)}{n}I_{\frac{\lambda}{\Omega}\notin\mathbb{Z}}+\frac{\log^2(a)[\log\lambda+\log(a)]}{\Omega}I_{\Omega\leq\lambda}+\frac{\log^2(a)[\log\lambda+\log(a)]}{\lambda}I_{\Omega>\lambda}. \tag{D.14}$$

Note when $\boldsymbol{r}=0$, (D.13) is the same as the expression given in Section 3. This result shows that the rate of convergence of the variance is determined by the term

$$\frac{1}{\Omega}\sum_{m=-2a}^{2a}\text{sinc}^2\left(\frac{\lambda}{\Omega}m\pi\right).$$

Using Lemma D.1 we see that the rate that the above term converges to zero depends on whether $\Omega<\lambda$ or $\Omega\geq\lambda$. In particular setting $a=\infty$

$$\frac{1}{\Omega}\sum_{m=-\infty}^{\infty}\text{sinc}^2\left(\frac{\lambda}{\Omega}m\pi\right) = \begin{cases} \frac{1}{\lambda} & \frac{\lambda}{\Omega}<1 \\ \frac{1}{\Omega} & \frac{\lambda}{\Omega}\in\mathbb{Z} \\ O\left(\frac{1}{\Omega}\right) & \frac{\lambda}{\Omega}>1 \text{ and } \frac{\lambda}{\Omega}\notin\mathbb{Z} \end{cases}$$

and $\sum_{m=-\infty}^{\infty} \text{sinc}^2\left(\frac{\lambda}{\Omega}m\pi\right) \to 1$ as $\frac{\lambda}{\Omega} \to \infty$. Therefore for the fine frequency grid with $\lambda < \Omega$ we have

$$\begin{aligned}
& \lambda^d \text{var}[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r})] \\
= \; & C_{11}\left(\frac{a}{\Omega},\boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right)\frac{\lambda^d}{\Omega^d}\sum_{\boldsymbol{m}=-2a}^{2a}\text{Sinc}^2\left(\frac{\lambda}{\Omega}\boldsymbol{m}\pi\right) + C_{12}\left(\frac{a}{\Omega},\boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right)\frac{\lambda^d}{\Omega^d}\sum_{m=-2a}^{2a}\text{Sinc}^2\left(\frac{\lambda}{\Omega}(\boldsymbol{m}+\boldsymbol{r})\pi\right) \\
& + O\left(\widetilde{\ell}_{a,\Omega,\lambda}\right).
\end{aligned}$$

On the other hand if a coarse frequency grid is used with $\lambda \geq \Omega$ then the rate of convergence is worse with

$$\begin{aligned}
& \Omega^d \text{var}[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r})] \\
= \; & C_{11}\left(\frac{a}{\Omega},\boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right)\sum_{\boldsymbol{m}=-2a}^{2a}\text{Sinc}^2\left(\frac{\lambda}{\Omega}\boldsymbol{m}\pi\right) + C_{12}\left(\frac{a}{\Omega},\boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right)\sum_{m=-2a}^{2a}\text{Sinc}^2\left(\frac{\lambda}{\Omega}(\boldsymbol{m}+\boldsymbol{r})\pi\right) \\
& + O\left(\widetilde{\ell}_{a,\Omega,\lambda}\right).
\end{aligned}$$

The above result implies that

$$\text{var}[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r})] = \begin{cases} O(\lambda^{-d}) & \frac{\lambda}{\Omega} < 1 \\ O(\Omega^{-d}) & \frac{\lambda}{\Omega} \geq 1 \end{cases}$$

The above results assume the spatial process is Gaussian. We now relax the assumption of Gaussianity.

Theorem D.1 *Let us suppose that $\{Z(\boldsymbol{s});\boldsymbol{s}\in\mathbb{R}^d\}$ is a fourth order stationary spatial random field that satisfies Assumption 2.1(i), 2.3, 2.5, 2.7 and 2.6(a,c) or 2.6(b,c). Then for $|\boldsymbol{r}_1|,|\boldsymbol{r}_2| \leq Ca$*

(i) If $\lambda < \Omega$

$$\lambda^d \text{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_1),\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_2)\right] = \frac{\lambda^d}{\Omega^d}\left(\widetilde{A}_1(\boldsymbol{r}_1,\boldsymbol{r}_2)+\widetilde{A}_2(\boldsymbol{r}_1,\boldsymbol{r}_2)\right) + B_1(\boldsymbol{r}_1,\boldsymbol{r}_2) + +O\left(\widetilde{\ell}^{(2)}_{a,\Omega,\lambda}\right),$$

$$\lambda^d \text{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_1),\overline{\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_2)}\right] = \frac{\lambda^d}{\Omega^d}\left(\widetilde{A}_3(\boldsymbol{r}_1,\boldsymbol{r}_2)+\widetilde{A}_4(\boldsymbol{r}_1,\boldsymbol{r}_2)\right) + B_3(\boldsymbol{r}_1,\boldsymbol{r}_2) + O\left(\widetilde{\ell}^{(2)}_{a,\Omega,\lambda}\right),$$

(ii) If $\lambda \geq \Omega$

$$\Omega^d \text{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_1),\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_2)\right] = \left(\widetilde{A}_1(\boldsymbol{r}_1,\boldsymbol{r}_2)+\widetilde{A}_2(\boldsymbol{r}_1,\boldsymbol{r}_2)\right) + \frac{\Omega^d}{\lambda^d}B_1(\boldsymbol{r}_1,\boldsymbol{r}_2) + +O\left(\widetilde{\ell}^{(2)}_{a,\Omega,\lambda}\right),$$

$$\Omega^d \text{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;r_1),\overline{\widetilde{Q}_{a,\Omega,\lambda}(g;r_2)}\right] = \left(\widetilde{A}_3(\boldsymbol{r}_1,\boldsymbol{r}_2)+\widetilde{A}_4(\boldsymbol{r}_1,\boldsymbol{r}_2)\right) + \frac{\Omega^d}{\lambda^d}B_3(\boldsymbol{r}_1,\boldsymbol{r}_2) + O\left(\widetilde{\ell}^{(2)}_{a,\Omega,\lambda}\right),$$

where $\widetilde{A}_1(\boldsymbol{r}_1,\boldsymbol{r}_2)$, $\widetilde{A}_2(\boldsymbol{r}_1,\boldsymbol{r}_2)$, $\widetilde{A}_3(\boldsymbol{r}_1,\boldsymbol{r}_2)$, $\widetilde{A}_3(\boldsymbol{r}_1,\boldsymbol{r}_2)$ *are defined in (D.8)-(D.11)*

$$\begin{aligned} B_1(\boldsymbol{r}_1,\boldsymbol{r}_2) &= \frac{c_4}{\pi^{2d+1}\Omega^{2d}}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2=-a}^{a}\int_{\mathbb{R}^{3d}} g(\boldsymbol{\omega}_{\boldsymbol{k}_1})\overline{g(\boldsymbol{\omega}_{\boldsymbol{k}_2})} \\ & f_4\left(\frac{2\boldsymbol{u}_1}{\lambda}-\boldsymbol{\omega}_{\Omega,\boldsymbol{k}_1+\boldsymbol{r}_1},\frac{2\boldsymbol{u}_2}{\lambda}-\boldsymbol{\omega}_{\Omega,\boldsymbol{k}_2},\frac{2\boldsymbol{u}_3}{\lambda}+\boldsymbol{\omega}_{\Omega,\boldsymbol{k}_2+\boldsymbol{r}_2}\right)\times \\ & \times \mathrm{Sinc}\left(\boldsymbol{u}_1+\boldsymbol{u}_2+\boldsymbol{u}_3+\frac{\lambda}{\Omega}(\boldsymbol{r}_2-\boldsymbol{r}_1)\pi\right)\mathrm{Sinc}(\boldsymbol{u}_1)\mathrm{Sinc}(\boldsymbol{u}_2)\mathrm{Sinc}(\boldsymbol{u}_3)d\boldsymbol{u}_1 d\boldsymbol{u}_2 d\boldsymbol{u}_3, \end{aligned} \tag{D.15}$$

$$\begin{aligned} B_3(\boldsymbol{r}_1,\boldsymbol{r}_2) &= \frac{c_4}{\pi^{2d+1}\Omega^{2d}}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2=-a}^{a}\int_{\mathbb{R}^{3d}} g(\boldsymbol{\omega}_{\boldsymbol{k}_1})g(\boldsymbol{\omega}_{\boldsymbol{k}_2}) \\ & f_4\left(\frac{2\boldsymbol{u}_1}{\lambda}-\boldsymbol{\omega}_{\Omega,\boldsymbol{k}_1+\boldsymbol{r}_1},\frac{2\boldsymbol{u}_2}{\lambda}+\boldsymbol{\omega}_{\Omega,\boldsymbol{k}_2},\frac{2\boldsymbol{u}_3}{\lambda}-\boldsymbol{\omega}_{\Omega,\boldsymbol{k}_2+\boldsymbol{r}_2}\right)\times \\ & \times \mathrm{Sinc}\left(\boldsymbol{u}_1+\boldsymbol{u}_2+\boldsymbol{u}_3-\frac{\lambda}{\Omega}(\boldsymbol{r}_2+\boldsymbol{r}_1)\pi\right)\mathrm{Sinc}(\boldsymbol{u}_1)\mathrm{Sinc}(\boldsymbol{u}_2)\mathrm{Sinc}(\boldsymbol{u}_3)d\boldsymbol{u}_1 d\boldsymbol{u}_2 d\boldsymbol{u}_3 \end{aligned}$$

and

$$\widetilde{\ell}^{(2)}_{a,\Omega,\lambda} = \begin{cases} \frac{\lambda^d}{n}+\frac{\log^3(a)}{n}+\frac{\log^2(a)[\log\lambda+\log a]}{\lambda}+\frac{\lambda^{2d}a^d}{n^2\Omega^d} & \frac{\lambda}{\Omega}<1 \\ \frac{\lambda^d}{n}+\frac{\log^3(a)}{n}I_{\frac{\lambda}{\Omega}\notin\mathbb{Z}}+\frac{\log^2(a)[\log\lambda+\log(a)]}{\Omega}+\frac{\lambda^d a^d}{n^2} & \frac{\lambda}{\Omega}\geq 1 \end{cases}. \tag{D.16}$$

PROOF. The proof is similar to the proof of Theorem B.2. We focus on the case $d=1$. Using Lemma D.3 and expanding out (for both the case $\lambda/\Omega<1$ and $\lambda/\Omega\geq 1$) we have

$$\begin{aligned} & \mathrm{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;r_1),\widetilde{Q}_{a,\Omega,\lambda}(g;r_2)\right] \\ = {} & \frac{1}{\Omega}\widetilde{A}_1(r_1,r_2)+\frac{1}{\Omega}\widetilde{A}_2(r_1,r_2)+\widetilde{B}_1(r_1,r_2)+\widetilde{B}_2(r_1,r_2)+O\left(\widetilde{\ell}_{a,\Omega,\lambda}\left[\frac{1}{\lambda}I_{\frac{\lambda}{\Omega}<1}+\frac{1}{\Omega}I_{\frac{\lambda}{\Omega}\geq 1}\right]\right), \end{aligned}$$

where $\ell_{a,\Omega,\lambda}$, $\widetilde{A}_1(r_1,r_2)$ and $\widetilde{A}_2(r_1,r_2)$ are defined in Lemma D.3 and

$$\begin{aligned} \widetilde{B}_1(r_1,r_2) &= \frac{c_4\lambda^2}{\Omega^2}\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\times \\ & \mathrm{E}\left[\kappa_4(s_2-s_1,s_3-s_1,s_4-s_1)e^{is_1\omega_{\Omega,k_1}}e^{-is_2\omega_{\Omega,k_1+r_1}}e^{-is_3\omega_{\Omega,k_2}}e^{is_4\omega_{\Omega,k_2+r_2}}\right] \\ \widetilde{B}_2(r_1,r_2) &= \frac{\lambda^2}{\Omega^2 n^4}\sum_{j_1,\ldots,j_4\in\mathcal{D}_3}\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\times \\ & \mathrm{E}\left[\kappa_4(s_{j_2}-s_{j_1},s_{j_3}-s_{j_1},s_{j_4}-s_{j_1})e^{is_{j_1}\omega_{\Omega,k_1}}e^{-is_{j_2}\omega_{\Omega,k_1+r_1}}e^{-is_{j_3}\omega_{\Omega,k_2}}e^{is_{j_4}\omega_{\Omega,k_2+r_2}}\right] \end{aligned}$$

with $c_4=n(n-1)(n-2)(n-3)/n^4$ and $\mathcal{D}_3$ and $\mathcal{D}_4$ are defined in the proof of Theorem B.2. The focus in this proof will be on the fourth order cumulant terms $\widetilde{B}_1(r_1,r_2)$ and $\widetilde{B}_2(r_1,r_2)$.

First we consider the "leading term" $\widetilde{B}_1(r_1,r_2)$. Rewriting the fourth order cumulant in terms of the fourth order spectral density we have

$$
\begin{aligned}
& \widetilde{B}_1(r_1,r_2) \\
= & \frac{c_4\lambda^2}{(2\pi)^3\Omega^2\lambda^4}\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\int_{\mathbb{R}^3} f_4(\omega_1,\omega_2,\omega_3)\int_{[-\lambda/2,\lambda/2]^4} e^{is_1(\omega_1+\omega_2+\omega_3+\omega_{\Omega,k_1})} \\
& e^{-is_2(\omega_1+\omega_{\Omega,k_1+r_1})}e^{-is_3(\omega_2+\omega_{\Omega,k_2})}e^{is_4(-\omega_3+\omega_{\Omega,k_2+r_2})}ds_1ds_2ds_3ds_4d\omega_1d\omega_2d\omega_3 \\
= & \frac{c_4\lambda^2}{(2\pi)^3\Omega^2}\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\int_{\mathbb{R}^3} f_4(\omega_1,\omega_2,\omega_3)\mathrm{sinc}\left(\frac{\lambda}{2}(\omega_1+\omega_2+\omega_3+\omega_{\Omega,k_1})\right) \\
& \times\mathrm{sinc}\left(\frac{\lambda}{2}(\omega_1+\omega_{\Omega,k_1+r_1})\right)\mathrm{sinc}\left(\frac{\lambda}{2}(\omega_2+\omega_{\Omega,k_2})\right)\mathrm{sinc}\left(\frac{\lambda}{2}(\omega_3-\omega_{\Omega,k_2+r_2})\right)d\omega_1d\omega_2d\omega_3.
\end{aligned}
$$

Making a change of variables we have

$$
\begin{aligned}
\widetilde{B}_1(r_1,r_2) = & \frac{c_4}{\pi^3\Omega^2\lambda}\sum_{k_1,k_2=-a}^{a}\int_{\mathbb{R}^3} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\times \\
& f_4\left(\frac{2u_1}{\lambda}-\omega_{\Omega,k_1+r_1},\frac{2u_2}{\lambda}-\omega_{\Omega,k_2},\frac{2u_3}{\lambda}+\omega_{\Omega,k_2+r_2}\right)\times \\
& \times\mathrm{sinc}\left(u_1+u_2+u_3+\frac{\lambda}{\Omega}(r_2-r_1)\pi\right)\mathrm{sinc}(u_1)\mathrm{sinc}(u_2)\mathrm{sinc}(u_3)du_1du_2du_3,
\end{aligned}
$$

thus we observe that $\lambda\widetilde{B}_1(r_1,r_2)=B_1(r_1,r_2)$, defined in (D.15).

We now show that $\widetilde{B}_2(r_1,r_2)$ is of lower order. To do so, just as in the proof of Theorem B.2 we have

$$
\widetilde{B}_2(r_1,r_2)=\sum_{j=1}^{4}\widetilde{B}_{2,(3,j)}(r_1,r_2)+\sum_{j=1}^{2}\widetilde{B}_{2,(2,j)}(r_1,r_2) \qquad \text{(D.17)}
$$

where

$$
\begin{aligned}
\widetilde{B}_{2,(3,1)}(r_1,r_2) = & \frac{|\mathcal{D}_{3,1}|\lambda^2}{n^4\Omega^2}\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\times \\
& \mathrm{E}\left[\kappa_4(s_{j_2}-s_{j_1},0,s_{j_4}-s_{j_1})e^{is_{j_1}\omega_{\Omega,k_1}}e^{-is_{j_2}\omega_{\Omega,k_1+r_1}}e^{-is_{j_1}\omega_{\Omega,k_2}}e^{is_{j_4}\omega_{\Omega,k_2+r_2}}\right]
\end{aligned}
$$

for $j=2,3,4$, $\widetilde{B}_{2,(3,j)}(r_1,r_2)$ are defined similarly and

$$
\begin{aligned}
\widetilde{B}_{2,(2,1)}(r_1,r_2) = & \frac{|\mathcal{D}_{2,1}|\lambda^2}{n^4\Omega^2}\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\times \\
& \mathrm{E}\left[\kappa_4(s_{j_2}-s_{j_1},0,s_{j_2}-s_{j_1})e^{is_{j_1}\omega_{\Omega,k_1}}e^{-is_{j_2}\omega_{\Omega,k_1+r_1}}e^{-is_{j_1}\omega_{\Omega,k_2}}e^{is_{j_2}\omega_{\Omega,k_2+r_2}}\right].
\end{aligned}
$$

Integrating out the locations in $\widetilde{B}_{2,(3,1)}(r_1,r_2)$ and making a change of variables we have

$$\begin{aligned}
&\widetilde{B}_{2,(3,1)}(r_1,r_2)\\
=\;& \frac{C\lambda^2|\mathcal{D}_{3,1}|}{n^4\Omega^2(2\pi)^3}\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\int_{\mathbb{R}^3} f_4(\omega_1,\omega_2,\omega_3)\mathrm{sinc}\left(\frac{\lambda}{2}\left(\omega_1+\omega_3+\omega_{\Omega,k_2-k_1}\right)\right)\times\\
&\mathrm{sinc}\left(\frac{\lambda}{2}\left(\omega_1-\omega_{\Omega,k_1+r_1}\right)\right)\times\mathrm{sinc}\left(\frac{\lambda}{2}\left(\omega_3+\omega_{\Omega,k_2+r_2}\right)\right)d\omega_1 d\omega_2 d\omega_3\\
=\;& \frac{C\lambda^2|\mathcal{D}_{3,1}|}{n^4\Omega^2(2\pi)^3}\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\int_{\mathbb{R}^3} f_4\left(\frac{2u_1}{\lambda}+\omega_{\Omega,k_1+r_1},\omega_2,\frac{2u_3}{\lambda}-\omega_{\Omega,k_2+r_2}\right)\\
&\times\mathrm{sinc}\left(u_1+u_3+\frac{\lambda}{\Omega}(r_1-r_2)\right)\mathrm{sinc}(u_1)\mathrm{sinc}(u_3)du_1 d\omega_2 du_3.
\end{aligned}$$

This gives $\widetilde{B}_{2,(3,1)}(r_1,r_2)=O(1/n)$. Next we consider $\widetilde{B}_{2,(2,1)}(r_1,r_2)$. Again by integrating out the locations and changing variables we have

$$\begin{aligned}
&\widetilde{B}_{2,(2,1)}(r_1,r_2)\\
=\;& \frac{\lambda^2}{(2\pi)^3 n^2\Omega^2}\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\int_{\mathbb{R}^3} f_4(\omega_1,\omega_2,\omega_3)\mathrm{sinc}\left(\frac{\lambda}{2}\left(\omega_1+\omega_2+\omega_{\Omega,k_2-k_1}\right)\right)\times\\
&\mathrm{sinc}\left(\frac{\lambda}{2}\left(\omega_1+\omega_2+\omega_{\Omega,k_2-k_1+r_2-r_1}\right)\right)d\omega_1 d\omega_2 d\omega_3\\
=\;& \frac{2\lambda}{(2\pi)^3 n^2\Omega^2}\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\int_{\mathbb{R}^3} f_4\left(\frac{2u_1}{\lambda}-u_2-\omega_{\Omega,k_2-k_1},u_2,u_3\right)\mathrm{sinc}\left(u_1\right)\times\\
&\mathrm{sinc}\left(u_1+\frac{\lambda}{\Omega}(r_2-r_1)\pi)\right)du_1 du_2 du_3.
\end{aligned}$$

Thus $\widetilde{B}_{2,(2,1)}(r_1,r_2)=O(\lambda a/(n^2\Omega))$. Note this bound is the same regardless of the sampling scheme on frequencies used. Thus using (D.17) we have

$$|\widetilde{B}_2(r_1,r_2)|=O\left(\frac{1}{n}+\frac{\lambda a}{n^2\Omega}\right).$$

Thus (D.17) and the above prove (i) and (ii). □

We see from the above lemma that $B_1(r_1,r_2)$ and $B_3(r_1,r_2)$ are the leading higher order cumulant terms. We now obtain some approximations for these rather complex terms. Let

$$\begin{aligned} & D_1\left(\frac{a}{\Omega};\boldsymbol{r}_1,\boldsymbol{r}_2\right) \\ = & \frac{1}{(2\pi)^{2d}}\int_{[-2\pi a/\Omega,2\pi a/\Omega]^{2d}} g(\boldsymbol{\omega}_1)\overline{g(\boldsymbol{\omega}_2)}f_4\left(-\boldsymbol{\omega}_1-\boldsymbol{\omega}_{\Omega,\boldsymbol{r}_1},-\boldsymbol{\omega}_2,\boldsymbol{\omega}_2+\boldsymbol{\omega}_{\Omega,\boldsymbol{r}_2}\right)\mathrm{sinc}\left(\frac{\lambda}{\Omega}(\boldsymbol{r}_2-\boldsymbol{r}_1)\pi\right) \\ & D_2\left(\frac{a}{\Omega};\boldsymbol{r}_1,\boldsymbol{r}_2\right) \\ = & \frac{c_4}{(2\pi)^{2d}}\int_{[-2\pi a/\Omega,2\pi a/\Omega]^{2d}} g(\boldsymbol{\omega}_1)g(\boldsymbol{\omega}_2)f_4\left(-\boldsymbol{\omega}_1-\boldsymbol{\omega}_{\Omega,\boldsymbol{r}_1},\boldsymbol{\omega}_2,-\boldsymbol{\omega}_2-\boldsymbol{\omega}_{\Omega,\boldsymbol{r}_2}\right)\mathrm{sinc}\left(\frac{\lambda}{\Omega}(\boldsymbol{r}_2+\boldsymbol{r}_1)\pi\right). \end{aligned} \tag{D.18}$$

Theorem D.2 *Let us suppose that* $\{Z(\boldsymbol{s});\boldsymbol{s}\in\mathbb{R}^d\}$ *is a fourth order stationary spatial random field that satisfies Assumption 2.1(i), 2.3, 2.5, 2.7 and 2.6(a,c) or 2.6(b,c) are satisfied. Let* $D_1(\cdot)$ *and* $D_2(\cdot)$ *be defined as in (D.18). Then all* $\boldsymbol{r}_1,\boldsymbol{r}_2\in\mathbb{Z}^d$

$$B_1(\boldsymbol{r}_1,\boldsymbol{r}_2) = D_1\left(\frac{a}{\Omega};\boldsymbol{r}_1,\boldsymbol{r}_2\right)+O\left(\frac{1}{\Omega}+\frac{\log^2(\lambda)}{\lambda}+\frac{\log(1+\frac{\lambda}{\Omega}\|\boldsymbol{r}_1-\boldsymbol{r}_2\|_1)}{\lambda}+\frac{1}{n}\right) \tag{D.19}$$

and

$$B_3(\boldsymbol{r}_1,\boldsymbol{r}_2) = D_2\left(\frac{a}{\Omega};\boldsymbol{r}_1,\boldsymbol{r}_2\right)+O\left(\frac{1}{\Omega}+\frac{\log^2(\lambda)}{\lambda}+\frac{\log(1+\frac{\lambda}{\Omega}\|\boldsymbol{r}_1-\boldsymbol{r}_2\|_1)}{\lambda}\right). \tag{D.20}$$

PROOF. We prove (D.19) for $d=1$, the same proof applies to (D.20) and $d>1$. We define the series of approximations. Replacing $\frac{1}{\Omega^2}\sum_{k_1,k_2=-a}^{a}$ with an double integral gives

$$\begin{aligned} B_{1,1}(r_1,r_2) = & \frac{c_4}{\pi^3(2\pi)^3}\int_{[-2\pi a/\Omega,2\pi a/\Omega]^2} g(\omega_1)\overline{g(\omega_2)}\int_{\mathbb{R}^3} \\ & f_4\left(\frac{2u_1}{\lambda}-\omega_1-\omega_{\Omega,r_1},\frac{2u_2}{\lambda}-\omega_2,\frac{2u_3}{\lambda}+\omega_2+\omega_{\Omega,r_2}\right)\times \\ & \times\mathrm{sinc}\left(u_1+u_2+u_3+\frac{\lambda}{\Omega}(r_2-r_1)\pi\right)\mathrm{sinc}(u_1)\mathrm{sinc}(u_2)\mathrm{sinc}(u_3)du_1du_2du_3d\omega_1d\omega_2. \end{aligned}$$

Next replacing $f_4(\frac{2u_1}{\lambda}-\omega_1-\omega_{\Omega,r_1},\frac{2u_2}{\lambda}-\omega_2,\frac{2u_3}{\lambda}+\omega_2+\omega_{\Omega,r_2})$ with $f_4(-\omega_1-\omega_{\Omega,r_1},-\omega_2,\omega_2+\omega_{\Omega,r_2})$ gives

$$\begin{aligned} B_{1,2}(r_1,r_2) = & \frac{c_4}{(2\pi)^2}\int_{[-2\pi a/\Omega,2\pi a/\Omega]^2} g(\omega_1)\overline{g(\omega_2)} \\ & f_4\left(-\omega_1-\omega_{\Omega,r_1},-\omega_2,\omega_2+\omega_{\Omega,r_2}\right)\mathrm{sinc}\left(\frac{\lambda}{\Omega}(r_2-r_1)\pi\right)d\omega_1d\omega_2. \end{aligned}$$

Now we systematically take $B_1(r_1, r_2)$ from the above. By taking differences (see the proof of Theorem B.2) we have

$$|B_1(r_1, r_2) - B_{1,1}(r_1, r_2)| = O\left(\frac{\ell_3(\frac{\lambda}{\Omega}(r_1 - r_2))}{\Omega}\right)$$

and

$$|B_{1,1}(r_1, r_2) - B_{1,2}(r_1, r_2)| = O\left(\frac{\log^2(\lambda)}{\lambda}\right),$$

thus giving the required result. □

The above result implies that in the case the process is non-Gaussian and $\frac{\lambda}{\Omega} < 1$ then

$$\begin{aligned}
& \lambda^d \mathrm{var}[\widetilde{Q}_{a,\Omega,\lambda}(g; \boldsymbol{r})] \\
= \; & C_{11}\left(\frac{a}{\Omega}, \boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right) \frac{\lambda^d}{\Omega^d} \sum_{\boldsymbol{m}=-2a}^{2a} \mathrm{Sinc}^2\left(\frac{\lambda}{\Omega}\boldsymbol{m}\pi\right) + C_{12}\left(\frac{a}{\Omega}, \boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right) \frac{\lambda^d}{\Omega^d} \sum_{m=-2a}^{2a} \mathrm{Sinc}^2\left(\frac{\lambda}{\Omega}(\boldsymbol{m} + \boldsymbol{r})\pi\right) \\
& + D_1\left(\frac{a}{\Omega}; \boldsymbol{r}_1, \boldsymbol{r}_2\right) + O\left(\widetilde{\ell}^{(2)}_{a,\Omega,\lambda} + \frac{\log(1 + \frac{\lambda}{\Omega}|\boldsymbol{r}_1 - \boldsymbol{r}_2|)}{\lambda}\right).
\end{aligned}$$

On the other hand if a coarse frequency grid is used with $\lambda \geq \Omega$ then the rate of convergence is worse with

$$\begin{aligned}
& \Omega^d \mathrm{var}[\widetilde{Q}_{a,\Omega,\lambda}(g; \boldsymbol{r})] \\
= \; & C_{11}\left(\frac{a}{\Omega}, \boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right) \sum_{\boldsymbol{m}=-2a}^{2a} \mathrm{Sinc}^2\left(\frac{\lambda}{\Omega}\boldsymbol{m}\pi\right) + C_{12}\left(\frac{a}{\Omega}, \boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right) \sum_{m=-2a}^{2a} \mathrm{Sinc}^2\left(\frac{\lambda}{\Omega}(\boldsymbol{m} + \boldsymbol{r})\pi\right) \\
& + \frac{\Omega^d}{\lambda^d} D_1\left(\frac{a}{\Omega}; \boldsymbol{r}_1, \boldsymbol{r}_2\right) + O\left(\widetilde{\ell}^{(2)}_{a,\Omega,\lambda} + \frac{\log(1 + \frac{\lambda}{\Omega}\|\boldsymbol{r}_1 - \boldsymbol{r}_2\|_1)}{\lambda}\right).
\end{aligned}$$

E Approximations to the covariance and cumulants of $\widetilde{Q}_{a,\lambda}(g; \boldsymbol{r})$

In this section, our objective is to obtain bounds for $\mathrm{cum}_q\big(\widetilde{Q}_{a,\lambda}(g; \boldsymbol{r}_1), \ldots, \widetilde{Q}_{a,\lambda}(g; \boldsymbol{r}_q)\big)$, these results will be used to prove the asymptotic expression for the variance of $\widetilde{Q}_{a,\lambda}(g; \boldsymbol{r})$ (given in Section B) and asymptotic normality of $\widetilde{Q}_{a,\lambda}(g; \boldsymbol{r})$. (Fox & Taqqu, 1987), (Dahlhaus, 1989), (Giraitis & Surgailis, 1990) (see also (Taqqu & Peccati, 2011)) have developed techniques for dealing with the cumulants of sums of periodograms of Gaussian (discrete time) time series, and one would have expected that these results could be used here. However, in our setting there are a few differences that we now describe (i) despite the spatial random being Gaussian the locations are randomly sampled, thus the composite process $Z(\boldsymbol{s})$ is not Gaussian (we can only exploit the Gaussianity when we condition on the location) (ii) the random field is defined over $\mathbb{R}^d$ (not $\mathbb{Z}^d$) (iii) the number of terms in the sums $\widetilde{Q}_{a,\lambda}(\cdot)$ is not necessarily the sample size. Unfortunately, these differences make it difficult to apply the above mentioned results to our setting. Therefore, in this section we consider cumulant based results for spatial data observed at irregular locations. In order to reduce cumbersome notation we focus on the case that the locations are from a uniform distribution.

As a simple motivation we first consider $\text{var}[\widetilde{Q}_{a,\lambda}(1,0)]$. By using indecomposable partitions we have

$$
\begin{aligned}
& \text{var}[\widetilde{Q}_{a,\lambda}(1,0)] \\
= & \ \frac{1}{n^4}\sum_{\substack{j_1,j_2,j_3,j_4=1\\ j_1\neq j_2, j_3\neq j_4}}^{n}\sum_{k_1,k_2=-a}^{a}\text{cov}\left[Z(s_{j_1})Z(s_{j_2})\exp(i\omega_{k_1}(s_{j_1}-s_{j_2})), Z(s_{j_3})Z(s_{j_4})\exp(i\omega_{k_2}(s_{j_3}-s_{j_4}))\right] \\
= & \ \frac{1}{n^4}\sum_{\substack{j_1,j_2,j_3,j_4=1\\ j_1\neq j_2, j_3\neq j_4}}^{n}\sum_{k_1,k_2=-a}^{a}\Big(\text{cum}\big[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_3})e^{-is_{j_3}\omega_{k_2}}\big]\text{cum}\big[Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_4})e^{is_{j_4}\omega_{k_2}}\big] \\
& +\text{cum}\big[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_4})e^{is_{j_4}\omega_{k_2}}\big]\text{cum}\big[Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_3})e^{-is_{j_3}\omega_{k_2}}\big] \\
& +\text{cum}\big[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_3})e^{-is_{j_3}\omega_{k_2}}, Z(s_{j_4})e^{is_{j_4}\omega_{k_2}}\big]\Big).
\end{aligned} \tag{E.1}
$$

In order to evaluate the covariances in the above we condition on the locations $\{s_j\}$. To evaluate the fourth order cumulant of the above we appeal to a generalisation of the conditional variance method. This expansion was first derived in (Brillinger, 1969), and in the general setting it is stated as

$$
\begin{aligned}
& \text{cum}(Y_1, Y_2, \ldots, Y_q) \\
= & \sum_{\pi}\text{cum}\left[\text{cum}(Y_{\pi_1}|s_1,\ldots,s_q),\ldots,\text{cum}(Y_{\pi_b}|s_1,\ldots,s_q)\right],
\end{aligned} \tag{E.2}
$$

where the sum is over all partitions π of $\{1,\ldots,q\}$ and $\{\pi_1,\ldots,\pi_b\}$ are all the blocks in the partition π. We use (E.2) to evaluate $\text{cum}[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}},\ldots,Z(s_{j_q})e^{is_{j_q}\omega_{k_q}}]$, where $Y_i = Z(s_{j_i})e^{is_{j_i}\omega_{k_i}}$ and we condition on the locations $\{s_j\}$. Using this decomposition we observe that because the spatial process is Gaussian, $\text{cum}[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}},\ldots,Z(s_{j_q})e^{is_{j_q}\omega_{k_q}}]$ can only be composed of cumulants of covariances conditioned on the locations. Moreover, if $s_1,\ldots,s_q$ are independent then by using the same reasoning we see that $\text{cum}[Z(s_1)e^{is_1\omega_{k_1}},\ldots,Z(s_q)e^{is_q\omega_{k_q}}] = 0$. Therefore, $\text{cum}\big[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_3})e^{is_{j_3}\omega_{k_2}}, Z(s_{j_4})e^{-is_{j_4}(\omega_{k_2}}\big]$ will only be non-zero if some elements of $s_{j_1}, s_{j_2}, s_{j_3}, s_{j_4}$ are dependent. Using these rules we have

$$
\begin{aligned}
& \mathrm{var}[\widetilde{Q}_{a,\lambda}(1,0)] \\
= \; & \frac{1}{n^4} \sum_{j_1,j_2,j_3,j_4 \in \mathcal{D}_4} \sum_{k_1,k_2=-a}^{a} \Big(\mathrm{cum}\big[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_3})e^{-is_{j_3}\omega_{k_2}}\big] \times \\
& \mathrm{cum}\big[Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_4})e^{is_{j_4}\omega_{k_2}}\big] \\
& + \mathrm{cum}\big[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_4})e^{is_{j_4}\omega_{k_2}}\big] \mathrm{cum}\big[Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_3})e^{-is_{j_3}\omega_{k_2}}\big] \Big) \\
& + \frac{1}{n^4} \sum_{j_1,j_2,j_3,j_4 \in \mathcal{D}_3} \sum_{k_1,k_2=-a}^{a} \Big(\mathrm{cum}\big[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_3})e^{-is_{j_3}\omega_{k_2}}\big] \times \\
& \mathrm{cum}\big[Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_4})e^{is_{j_4}\omega_{k_2}}\big] \\
& + \mathrm{cum}\big[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_4})e^{is_{j_4}\omega_{k_2}}\big] \mathrm{cum}\big[Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_3})e^{-is_{j_3}\omega_{k_2}}\big] \Big) \\
& + \frac{1}{n^4} \sum_{j_1,j_2,j_3,j_4 \in \mathcal{D}_3} \sum_{k_1,k_2=-a}^{a} \mathrm{cum}\big[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_3})e^{is_{j_3}\omega_{k_2}}, Z(s_{j_4})e^{-is_{j_4}\omega_{k_2}}\big],
\end{aligned}
\tag{E.3}
$$

where $\mathcal{D}_4 = \{j_1, \ldots, j_4 = \text{all } j\text{s are different}\}$, $\mathcal{D}_3 = \{j_1, \ldots, j_4; \text{two } j\text{s are the same but } j_1 \neq j_2 \text{ and } j_3 \neq j_4\}$ (noting that by definition of $\widetilde{Q}_{a,\lambda}(1,0)$ more than two elements in $\{j_1, \ldots, j_4\}$ cannot be the same). We observe that $|\mathcal{D}_4| = O(n^4)$ and $|\mathcal{D}_3| = O(n^3)$, where $|\cdot|$ denotes the cardinality of a set. We will show that the second and third terms are asymptotically negligible with respect to the first term. To show this we require the following lemma.

Lemma E.1 *Suppose Assumptions 2.1, 2.3, 2.5 and 2.6(b,c) hold (note we only use Assumption 2.6(c) to get 'neater expressions in the proofs' it is not needed to obtain the same order). Then we have*

$$
\sup_a \sum_{k_1,k_2=-a}^{a} \mathrm{cum}[Z(s_1)e^{is_1\omega_{k_1}}, Z(s_2)e^{-is_2\omega_{k_1}}, Z(s_3)e^{-is_3\omega_{k_2}}, Z(s_1)e^{is_1\omega_{k_2}}] = O(1) \tag{E.4}
$$

$$
\sup_a \sum_{k_1,k_2=-a}^{a} \mathrm{cum}\left[Z(s_1)e^{is_1\omega_{k_1}}, Z(s_1)e^{is_1\omega_{k_2}}\right] \mathrm{cum}\left[Z(s_2)e^{-is_2\omega_{k_1}}, Z(s_3)e^{-is_3\omega_{k_2}}\right] = O(1). \tag{E.5}
$$

PROOF. To show (E.4) we use conditional cumulants (see (E.2)). By using the conditional cumulant expansion and Gaussianity of $Z(s)$ conditioned on the location we have

$$\begin{aligned}
&\sum_{k_1,k_2=-a}^{a} \mathrm{cum}[Z(s_1)e^{is_1\omega_{k_1}}, Z(s_2)e^{-is_2\omega_{k_1}}, Z(s_3)e^{-is_3\omega_{k_2}}, Z(s_1)e^{is_1\omega_{k_2}}] \\
= &\sum_{k_1,k_2=-a}^{a} \mathrm{cum}[c(s_1-s_2)e^{is_1\omega_{k_1}-is_2\omega_{k_1}}, c(s_1-s_3)e^{-is_3\omega_{k_2}+is_1\omega_{k_2}}] + \\
&\sum_{k_1,k_2=-a}^{a} \mathrm{cum}[c(s_1-s_3)e^{is_1\omega_{k_1}-is_3\omega_{k_2}}, c(s_1-s_2)e^{-is_2\omega_{k_1}+is_1\omega_{k_2}}] + \\
&\underbrace{\sum_{k_1,k_2=-a}^{a} \mathrm{cum}[c(0)e^{is_1(\omega_{k_1}+\omega_{k_2})}, c(s_2-s_3)e^{-is_2\omega_{k_1}-is_3\omega_{k_2}}]}_{=0 \text{ (since } s_1 \text{ is independent of } s_2 \text{ and } s_3)} = I_1 + I_2.
\end{aligned}$$

Writing I_1 in terms of expectations and using the spectral representation of the covariance we have

$$\begin{aligned}
I_1 &= \sum_{k_1,k_2=-a}^{a} \Big(\mathrm{E}\big[c(s_1-s_2)c(s_1-s_3)e^{is_1(\omega_{k_1}+\omega_{k_2})}e^{-is_3\omega_{k_2}}e^{-is_2\omega_{k_1}}\big] - \mathrm{E}\big[c(s_1-s_2)e^{is_1\omega_{k_1}-is_2\omega_{k_1}}\big] \\
&\quad \times \mathrm{E}\big[c(s_1-s_3)e^{is_1\omega_{k_2}-is_3\omega_{k_2}}\big]\Big) \\
&= E_1 - E_2
\end{aligned}$$

where

$$\begin{aligned}
E_1 &= \frac{1}{(2\pi)^2}\sum_{k_1,k_2=-a}^{a} \int\int f(x)f(y)\mathrm{sinc}\Big(\frac{\lambda}{2}(x+y)+(k_1+k_2)\pi\Big) \times \\
&\quad \mathrm{sinc}\Big(\frac{\lambda}{2}x+k_1\pi\Big)\mathrm{sinc}\Big(\frac{\lambda}{2}y+k_2\pi\Big)dxdy \\
E_2 &= \frac{1}{(2\pi)^2}\sum_{k_1,k_2=-a}^{a} \int\int f(x)f(y)\mathrm{sinc}\Big(\frac{\lambda}{2}x+k_2\pi\Big)\mathrm{sinc}\Big(\frac{\lambda}{2}x+k_2\pi\Big) \times \\
&\quad \mathrm{sinc}\Big(\frac{\lambda}{2}y+k_1\pi\Big)\mathrm{sinc}\Big(\frac{\lambda}{2}y+k_1\pi\Big)dxdy.
\end{aligned}$$

To bound E_1 we make a change of variables $u = \frac{\lambda x}{2} + k_1\pi$, $v = \frac{\lambda y}{2} + k_2\pi$, and replace sum with integral (and use Lemma I.1) to give

$$\begin{aligned}
E_1 &= \frac{1}{\pi^2\lambda^2}\int\int \sum_{k_1,k_2=-a}^{a} f(\frac{2u}{\lambda}-\omega_{k_1})f(\frac{2v}{\lambda}-\omega_{k_2})\mathrm{sinc}(u+v)\mathrm{sinc}(u)\mathrm{sinc}(v)dudv \\
&= \frac{1}{(2\pi)^2\pi^2}\int\int \mathrm{sinc}(u+v)\mathrm{sinc}(u)\mathrm{sinc}(v) \times \\
&\quad \Big(\int_{-2\pi a/\lambda}^{2\pi a/\lambda}\int_{-2\pi a/\lambda}^{2\pi a/\lambda} f(\frac{2u}{\lambda}-\omega_1)f(\frac{2v}{\lambda}-\omega_2)d\omega_1 d\omega_2\Big)dudv + O(\frac{1}{\lambda}).
\end{aligned}$$

Let $G(\frac{2u}{\lambda}) = \frac{1}{2\pi}\int_{-2\pi a/\lambda}^{2\pi a/\lambda} f(\frac{2u}{\lambda}-\omega)d\omega$, then substituting this into the above and using equation (F.3) in Lemma F.1 we have

$$E_1 = \frac{1}{\pi^2}\int\int \text{sinc}(u+v)\text{sinc}(u)\text{sinc}(v)G\left(\frac{2u}{\lambda}\right)G\left(\frac{2v}{\lambda}\right)dudv + O(\frac{1}{\lambda}) = O(1).$$

To bound E_2 we use a similar technique and Lemma F.1(iii) to give $E_2 = O(1)$. Altogether, this gives $I_1 = O(1)$. The same proof can be used to show that $I_2 = O(1)$. Altogether this gives (E.4).

To bound (E.5), we observe that if $k_1 \neq -k_2$, then $\text{cum}\big[Z(s_1)e^{is_1\omega_{k_1}}, Z(s_1)e^{is_1\omega_{k_2}}\big] = \text{E}[c(0)e^{is(\omega_{k_1}+\omega_{k_2})}] = 0$ otherwise $\text{cum}\big[Z(s_1)e^{is_1\omega_k}, Z(s_1)e^{-is_1\omega_k}\big] = c(0)$. Using this, (E.5) can be reduced to

$$\begin{aligned}
&\sum_{k_1,k_2=-a}^{a} \text{cum}\left[Z(s_1)e^{is_1\omega_{k_1}}, Z(s_1)e^{is_1\omega_{k_2}}\right]\text{cum}\left[Z(s_2)e^{-is_2\omega_{k_1}}, Z(s_3)e^{-is_3\omega_{k_2}}\right] \\
= \;& c(0)\sum_{k=-a}^{a}\text{cum}\left[Z(s_2)e^{is_2\omega_k}, Z(s_3)e^{-is_3\omega_k}\right] \\
= \;& \frac{c(0)}{2\pi}\int_{-\infty}^{\infty}\sum_{k=-a}^{a} f(x)\text{sinc}\left(\frac{\lambda x}{2}+k\pi\right)\text{sinc}\left(\frac{\lambda x}{2}+k\pi\right)dx \qquad \left(\text{let } \omega = \frac{\lambda x}{2}+k\pi\right) \\
= \;& c(0)\int_{-\infty}^{\infty}\frac{1}{\pi\lambda}\sum_{k=-a}^{a} f\left(\frac{2\omega}{\lambda}-\omega_k\right)\text{sinc}^2(\omega)d\omega \\
= \;& \frac{c(0)}{2\pi^2}\int_{-\infty}^{\infty}\text{sinc}^2(\omega)\bigg(\int_{-2\pi a/\lambda}^{2\pi a/\lambda} f\left(\frac{2\omega}{\lambda}-x\right)dx\bigg)d\omega + O\left(\frac{1}{\lambda}\right) = O(1),
\end{aligned}$$

thus proving (E.5). □

We now consider a generalization of the above result for general frequency grids $\boldsymbol{\omega}_{\Omega,\boldsymbol{k}}$.

Lemma E.2 *Suppose Assumptions 2.1, 2.3 and Assumptions 2.5(ii) and 2.6(b) hold. Suppose that* $\sup_{\omega\in\mathbb{R}}|g(\omega)| < \infty$. *Then for all* $\Omega, \lambda > 1$ *(for simplicity we assume* $1 \leq \Omega, \lambda \leq a$*) we have*

$$\begin{aligned}
&\bigg|\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\times \\
&\text{cum}[Z(s_1)e^{is_1\omega_{\Omega,k_1}}, Z(s_2)e^{-is_2\omega_{\Omega,k_1+r_1}}, Z(s_3)e^{-is_3\omega_{\Omega,k_2}}, Z(s_1)e^{is_1\omega_{\Omega,k_2+r_2}}]\bigg| \\
= \;& O\left(\frac{\Omega^2}{\lambda^2}\right)
\end{aligned} \tag{E.6}$$

and

$$\Big|\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\text{cum}\left[Z(s_1)e^{is_1\omega_{\Omega,k_1}}, Z(s_1)e^{-is_1\omega_{\Omega,k_2}}\right]\times$$
$$\text{cum}\left[Z(s_2)e^{-is_2\omega_{\Omega,k_1+r_1}}, Z(s_3)e^{is_3\omega_{\Omega,k_2+r_2}}\right]\Big|$$
$$= \begin{cases} \frac{\Omega}{\lambda} + \left[\frac{\Omega^2}{\lambda^3}\log^3(a)\right] I_{\frac{\lambda}{\Omega}\notin\mathbb{Z}} & \frac{\lambda}{\Omega}\geq 1 \\ \left(\frac{\Omega}{\lambda}\right)^2 + \frac{\Omega}{\lambda^2}\log(a) + \frac{\Omega^2}{\lambda^3}\log^3(a) & \frac{\lambda}{\Omega} < 1 \end{cases} \tag{E.7}$$

where $I_{\Omega/\lambda\notin\mathbb{Z}} = 1$ *when* $\Omega/\lambda \notin \mathbb{Z}$ *else it is zero.*

PROOF. We start by obtaining the bound in (E.6)

$$D_1 = \sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\text{cum}[Z(s_1)e^{is_1\omega_{\Omega,k_1}}, Z(s_2)e^{-is_2\omega_{\Omega,k_1+r_1}}, Z(s_3)e^{-is_3\omega_{\Omega,k_2}}, Z(s_1)e^{is_1\omega_{\Omega,k_2+r_2}}].$$

Applying the conditional cumulant formula in (E.2) by conditioning on location we have

$$\begin{aligned} D_1 &= \sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\text{cum}[c(s_1-s_2)e^{is_1\omega_{\Omega,k_1}-is_2\omega_{\Omega,k_1+r_1}}, c(s_1-s_3)e^{-is_3\omega_{\Omega,k_2}+is_1\omega_{\Omega,k_2+r_2}}] + \\ &\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\text{cum}[c(s_1-s_3)e^{is_1\omega_{\Omega,k_1}-is_3\omega_{\Omega,k_2}}, c(s_1-s_2)e^{-is_2\omega_{\Omega,k_1+r_1}+is_1\omega_{\Omega,k_2+r_2}}] + \\ &\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\underbrace{\text{cum}[c(0)e^{is_1(\omega_{\Omega,k_1}+\omega_{\Omega,k_2})}, c(s_2-s_3)e^{-is_2\omega_{\Omega,k_1}-is_3\omega_{\Omega,k_2}}]}_{=0 \text{ (since locations are independent)}} = I_1 + I_2. \end{aligned}$$

Expanding I_1 just as was done in the proof of Lemma E.1 gives $I_1 = E_1 - E_2$ where

$$\begin{aligned} E_1 &= \frac{1}{(2\pi)^2}\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\int\int f(x)f(y)\text{sinc}\left(\frac{\lambda}{2}\left[(x+y)+\omega_{\Omega,k_1+k_2+r_2}\right]\right) \\ &\times\text{sinc}\left(\frac{\lambda}{2}\left[x+\omega_{\Omega,k_1+r_2}\right]\right)\text{sinc}\left(\frac{\lambda}{2}\left[y+\omega_{\Omega,k_2}\right]\right)dxdy \end{aligned}$$

and E_2 is defined similarly. Changing variables $u = \frac{\lambda}{2}\left[x+\omega_{\Omega,k_1+r_2}\right]$ and $v = \frac{\lambda}{2}\left[y+\omega_{\Omega,k_2}\right]$ gives

$$\begin{aligned} E_1 &= \frac{1}{\pi^2\lambda^2}\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\int\int f\left(\frac{2u}{\lambda}-\omega_{\Omega,k_1+r_1}\right) \\ &f\left(\frac{2v}{\lambda}-\omega_{\Omega,k_2}\right)\text{sinc}\left(u+v+\frac{\lambda}{\Omega}(r_2-r_2)\pi\right)\text{sinc}(u)\text{sinc}(v)dudv. \end{aligned}$$

Since $\frac{1}{\Omega^2}\sum_{k_1,k_2}|g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}|f(\frac{2u}{\lambda}-\omega_{\Omega,k_1+r_1})f(\frac{2v}{\lambda}-\omega_{\Omega,k_2}) < \infty$ we have $E_1 = O(\frac{\Omega^2}{\lambda^2})$. Similarly we can show that $E_2 = O(\frac{\Omega^2}{\lambda^2})$, thus $I_1 = O(\frac{\Omega^2}{\lambda^2})$. The same proof can be used to show that $I_2 = O(\frac{\Omega^2}{\lambda^2})$. Thus

$$|D_1| = O\left(\frac{\Omega^2}{\lambda^2}\right)$$

and (E.6).

Next we consider D_2, where

$$\begin{aligned} & D_2 \\ = & \sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\text{cum}\left[Z(s_1)e^{is_1\omega_{\Omega,k_1}}, Z(s_1)e^{-is_1\omega_{\Omega,k_2}}\right] \times \\ & \text{cum}\left[Z(s_2)e^{-is_2\omega_{\Omega,k_1+r_1}}, Z(s_3)e^{is_3\omega_{\Omega,k_2+r_2}}\right]. \end{aligned}$$

By conditioning on location and using that $\text{cum}\left[Z(s_1)e^{is_1\omega_{\Omega,k_1}}, Z(s_1)e^{-is_1\omega_{\Omega,k_2}}\right] = c(0)\text{E}\left[e^{is_1\omega_{\Omega,k_1-k_2}}\right]$ we have

$$\begin{aligned} D_2 &= c(0)\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\text{E}\left[e^{is_1\omega_{\Omega,k_1-k_2}}\right]\text{cum}\left[Z(s_2)e^{-is_2\omega_{\Omega,k_1+r_1}}, Z(s_3)e^{is_3\omega_{\Omega,k_2+r_2}}\right] \\ &= \sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\text{sinc}\left(\frac{\lambda}{\Omega}(k_1-k_2)\pi\right) \times \\ &\quad \int f(x)\text{sinc}\left(\frac{\lambda}{2}(x+\omega_{\Omega,k_1+r_1})\right)\text{sinc}\left(\frac{\lambda}{2}(x+\omega_{\Omega,k_2+r_2})\right)dx. \end{aligned}$$

To bound D_2 we need to consider two separate cases, when $\lambda/\Omega \in \mathbb{Z}$ and when $\lambda/\Omega \notin \mathbb{Z}$. If the ratio $\lambda/\Omega \in \mathbb{Z}$, then for $k_1 \neq k_2$, we have $\text{sinc}\left(\frac{\lambda}{\Omega}(k_1-k_2)\pi\right) = 0$. This reduces the above double summand in D_2 to a single summand and

$$\begin{aligned} D_2 &= c(0)\sum_{k=-a}^{a}|g(\omega_{\Omega,k})|^2\int f(x)\text{sinc}\left(\frac{\lambda}{2}(x+\omega_{\Omega,k+r_1})\right)\text{sinc}\left(\frac{\lambda}{2}(x+\omega_{\Omega,k+r_2}\right)dx \\ &= \frac{c(0)}{\lambda}\sum_{k=-a}^{a}|g(\omega_{\Omega,k})|^2\int f\left(\frac{2u}{\lambda}-\omega_{\Omega,k+r_1}\right)\text{sinc}\,(u)\,\text{sinc}\left(u+\frac{\lambda}{\Omega}(r_2-r_2)\pi\right)du = O\left(\frac{\Omega}{\lambda}\right). \end{aligned}$$

On the other hand, if $\lambda/\Omega \notin \mathbb{Z}$, then D_2 remains a double sum. In this case we make a change of variables $u = \frac{\lambda}{2}(x+\omega_{\Omega,k_1+r_1})$

$$\begin{aligned} D_2 &= \frac{c(0)}{\lambda}\sum_{k_1,k_2=-a}^{a} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,k_2})}\text{sinc}\left(\frac{\lambda}{\Omega}(k_1-k_2)\pi\right) \\ &\quad \times\int f\left(\frac{2u}{\lambda}-\omega_{\Omega,k_1+r_1}\right)\text{sinc}\,(u)\,\text{sinc}\left(u+\frac{\lambda}{\Omega}(k_2-k_1+r_2-r_1)\pi\right)du. \end{aligned}$$

We make the change of variable $m = k_2 - k_1$, this gives

$$\begin{aligned}
D_2 &= \frac{c(0)}{\lambda}\sum_{m=-2a}^{2a}\sum_{k_1=\max(-a,-a+m)}^{\min(a,a+m)} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,m-k_1})}\mathrm{sinc}\left(\frac{\lambda}{\Omega}m\pi\right) \\
&\quad \times \int f\left(\frac{2u}{\lambda} - \omega_{\Omega,k_1+r_1}\right)\mathrm{sinc}\left(u + \frac{\lambda}{\Omega}(m + r_2 - r_1)\pi\right)\mathrm{sinc}\,(u)\,du \\
&= \frac{\Omega c(0)}{\lambda}\sum_{m=-2a}^{2a}\mathrm{sinc}\left(\frac{\lambda}{\Omega}m\pi\right)\int_{\mathbb{R}}\mathrm{sinc}\,(u)\,\mathrm{sinc}\left(u + \frac{\lambda}{\Omega}(m + r_2 - r_1)\pi\right) \\
&\quad \times \frac{1}{\Omega}\sum_{k_1=\max(-a,-a+m)}^{\min(a,a+m)} g(\omega_{\Omega,k_1})\overline{g(\omega_{\Omega,m-k_1})}f\left(\frac{2u}{\lambda} - \omega_{\Omega,k_1+r_1}\right)du
\end{aligned}$$

To bound the above we make a series of approximations. To do so, we first use the $\ell_p(u)$ notation introduced in Lemma F.1 and note that

$$\sum_{m=-a}^{a}\ell_p\left(\frac{\lambda}{\Omega}m\pi\right) \leq C\frac{\Omega}{\lambda}\log^{p+1}(a)$$

and

$$\begin{aligned}
&\sum_{m=-a}^{a}\ell_0\left(\frac{\lambda}{\Omega}m\right)\ell_p\left(\frac{\lambda}{\Omega}(m + r_2 - r_1)\pi\right) \\
&= \sum_{m\leq|\lambda e/\Omega|}\ell_0\left(\frac{\lambda}{\Omega}m\right)\ell_p\left(\frac{\lambda}{\Omega}(m + r_2 - r_1)\right) + \sum_{m>|\lambda e/\Omega|}\ell_0\left(\frac{\lambda}{\Omega}m\right)\ell_p\left(\frac{\lambda}{\Omega}(m + r_2 - r_1)\right) \\
&\leq C\begin{cases} \left(\frac{\Omega}{\lambda}\right)^2\log^p\left(\frac{\lambda}{\Omega}a\right) & \frac{\lambda}{\Omega}\geq 1 \\ \left(\frac{\Omega}{\lambda}\right)\log^{p+1}\left(\frac{\lambda}{\Omega}a\right) & \frac{\lambda}{\Omega} < 1 \end{cases}.
\end{aligned} \qquad \text{(E.8)}$$

Returning to D_2, we note that the sum over $\frac{1}{\Omega}\sum_{k_1}$ is finite. Thus we replace the sum with an integral and define

$$\begin{aligned}
D_{2,1} &= \frac{c(0)\Omega}{\lambda 2\pi}\sum_{m=-2a}^{2a}\mathrm{sinc}\left(\frac{\lambda}{\Omega}m\pi\right)\int_{\mathbb{R}}\int_{\max(-a,-a+m)/\Omega}^{\min(a,a+m)/\Omega} g(\omega)\overline{g(\omega - \omega_{\Omega,k_1})}f\left(\frac{2u}{\lambda} - \omega + \omega_{\Omega,r_1}\right) \\
&\quad \mathrm{sinc}\,(u)\,\mathrm{sinc}\left(u + \frac{\lambda}{\Omega}(m + r_2 - r_1)\pi\right)d\omega du.
\end{aligned}$$

To bound the difference we use (E.8)

$$\begin{aligned}
\left|D_2 - D_{2,1}\right| &\leq \frac{C}{\lambda 2\pi}\sum_{m=-2a}^{2a}\left|\mathrm{sinc}\left(\frac{\lambda}{\Omega}m\pi\right)\right|\ell_1\left(\frac{\lambda}{\Omega}(m + r_2 - r_1)\right) \\
&\leq \frac{C}{\lambda 2\pi}\sum_{m=-2a}^{2a}\ell_0\left(\frac{\lambda}{\Omega}m\pi\right)\ell_1\left(\frac{\lambda}{\Omega}(m + r_2 - r_1)\pi\right) \\
&\leq \frac{C}{\lambda}\begin{cases} \left(\frac{\Omega}{\lambda}\right)^2\log\left(\frac{\lambda}{\Omega}a\right) & \frac{\lambda}{\Omega}\geq 1 \\ \left(\frac{\Omega}{\lambda}\right)\log^2\left(\frac{\lambda}{\Omega}a\right) & \frac{\lambda}{\Omega} < 1 \end{cases} \leq C\begin{cases} \frac{\Omega^2}{\lambda^3}\log(a) & \frac{\lambda}{\Omega}\geq 1 \\ \frac{\Omega}{\lambda^2}\log^2(a) & \frac{\lambda}{\Omega} < 1 \end{cases}
\end{aligned} \qquad \text{(E.9)}$$

where the last bound above uses that $1 \leq \Omega, \lambda \leq Ca$.

Next we replace $f\left(\frac{2u}{\lambda} - \omega + \omega_{r_1}\right)$ with $f\left(-\omega + \omega_{r_1}\right)$ in $D_{2,1}$ and define

$$\begin{aligned} D_{2,2} &= \frac{c(0)\Omega}{\lambda 2\pi} \sum_{m=-2a}^{2a} \operatorname{sinc}\left(\frac{\lambda}{\Omega} m\right) \int_{\mathbb{R}} \int_{\max(-a,-a+m)/\Omega}^{\min(a,a+m)/\Omega} g(\omega)\overline{g(\omega - \omega_{\Omega,k_1})} f\left(-\omega + \omega_{\Omega,r_1}\right) \\ & \quad \operatorname{sinc}(u) \operatorname{sinc}\left(u + \frac{\lambda}{\Omega}(m + r_2 - r_1)\pi\right) d\omega du \\ &= \frac{c(0)\Omega}{\lambda 2\pi} \sum_{m=-2a}^{2a} \operatorname{sinc}\left(\frac{\lambda}{\Omega} m\pi\right) \operatorname{sinc}\left(\frac{\lambda}{\Omega}(m + r_2 - r_1)\pi\right) \times \\ & \quad \int_{\max(-a,-a+m)/\Omega}^{\min(a,a+m)/\Omega} g(\omega)\overline{g(\omega - \omega_{\Omega,k_1})} f\left(-\omega + \omega_{\Omega,r_1}\right) d\omega. \end{aligned}$$

By using Lemma F.2, equation (F.5) this gives the bound

$$\begin{aligned} & |D_{2,1} - D_{2,2}| \\ \leq & \frac{c(0)\Omega}{\lambda^2 2\pi} \sum_{m=-2a}^{2a} \left|\operatorname{sinc}\left(\frac{\lambda}{\Omega} m\pi\right)\right| \left(\log\lambda + \log\left(1 + \frac{\lambda}{\Omega}(|m + r_2 - r_1|)\right)\right) \\ \leq & \frac{C\Omega\log\lambda}{\lambda^2} \sum_{m=-2a}^{2a} \ell_0\left(\frac{\lambda}{\Omega} m\pi\right) + \frac{C\Omega}{\lambda^2} \sum_{m=-2a}^{2a} \ell_1\left(\frac{\lambda}{\Omega}(|m| + |r_2 - r_1|)\pi\right). \end{aligned}$$

By using (E.8) we have

$$\begin{aligned} |D_{2,1} - D_{2,2}| &\leq C\frac{\Omega}{\lambda^2} \begin{cases} \log\lambda\left(\frac{\Omega}{\lambda}\right)\log\left(\frac{\lambda}{\Omega}a\right) + \left(\frac{\Omega}{\lambda}\right)\log^2\left(\frac{\lambda}{\Omega}a\right) & \frac{\lambda}{\Omega} \geq 1 \\ \log\lambda\left(\frac{\Omega}{\lambda}\right)\log\left(\frac{\lambda}{\Omega}a\right) + \left(\frac{\Omega}{\lambda}\right)\log^2\left(\frac{\lambda}{\Omega}a\right) & \frac{\lambda}{\Omega} < 1 \end{cases} \\ &\leq C\frac{\Omega^2}{\lambda^3}\log^2\left(\frac{\lambda}{\Omega}a\right). \end{aligned} \tag{E.10}$$

Finally, we bound $D_{2,2}$. By using the Cauchy-Schwarz inequality and (D.5) we have

$$D_{2,2} \leq C\frac{\Omega}{\lambda} \sum_{m=-2a}^{2a} \operatorname{sinc}^2\left(\frac{\lambda}{\Omega} m\right) \leq C \begin{cases} \frac{\Omega}{\lambda} & \frac{\lambda}{\Omega} > 1 \\ \left(\frac{\Omega}{\lambda}\right)^2 & \frac{\lambda}{\Omega} < 1 \end{cases}$$

Altogether, by using (E.9), (E.10) and (E.11) the bound for D_2 is

$$D_2 = \begin{cases} \frac{\Omega}{\lambda} + \frac{\Omega^2}{\lambda^3}\log^3(a) I_{\lambda/\Omega \notin \mathbb{Z}} & \frac{\lambda}{\Omega} > 1 \\ \left(\frac{\Omega}{\lambda}\right)^2 + \frac{\Omega}{\lambda^2}\log(a) + \frac{\Omega^2}{\lambda^3}\log^3(a) & \frac{\lambda}{\Omega} < 1 \end{cases}$$

Thus giving the result. □

We now derive an expression for $\mathrm{var}[\widetilde{Q}_{a,\lambda}(1,0)]$, by using Lemma E.1 we have

$$\begin{aligned}
& \mathrm{var}[\widetilde{Q}_{a,\lambda}(1,0)] \\
= & \frac{1}{n^4} \sum_{j_1,j_2,j_3,j_4 \in \mathcal{D}_4} \sum_{k_1,k_2=-a}^{a} \Big(\mathrm{cum}\big[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_3})e^{-is_{j_3}\omega_{k_2}}\big] \times \qquad \text{(E.11)} \\
& \mathrm{cum}\big[Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_4})e^{is_{j_4}\omega_{k_2}}\big] \\
& + \mathrm{cum}\big[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_4})e^{is_{j_4}\omega_{k_2}}\big] \mathrm{cum}\big[Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_3})e^{-is_{j_3}\omega_{k_2}}\big] \Big) + O\left(\frac{1}{n}\right). \qquad \text{(E.12)}
\end{aligned}$$

In Lemma B.1 we have shown that the covariance terms above are of order $O(\lambda^{-1})$, thus dominating the fourth order cumulant terms which is of order $O(n^{-1})$ (so long as $\lambda << n$).

Lemma E.3 *Suppose that* $\{Z(\boldsymbol{s}); \boldsymbol{s} \in \mathbb{R}^d\}$ *is a mean zero Gaussian random process and* $\{\boldsymbol{s}_j\}$ *are iid random variables. This means conditioned on the location only the second order cumulant is non-zero. Using this and (E.2) the following hold:*

(i) $\mathrm{cum}[Z(\boldsymbol{s}_{j_1})e^{i\boldsymbol{s}_{j_1}\boldsymbol{\omega}_{\boldsymbol{k}_1}}, \ldots, Z(\boldsymbol{s}_{j_q})e^{i\boldsymbol{s}_{j_q}\boldsymbol{\omega}_{\boldsymbol{k}_q}}]$ *can be written as the sum of products of cumulants of the spatial covariance conditioned on location. Therefore, it is easily seen that the odd order cumulant* $\mathrm{cum}_{2q+1}[Z(\boldsymbol{s}_{j_1})\exp(i\boldsymbol{s}_{j_1}\boldsymbol{\omega}_{\boldsymbol{k}_1}), \ldots, Z(\boldsymbol{s}_{j_{2q+1}}\exp(i\boldsymbol{s}_{j_{2q+1}}\boldsymbol{\omega}_{\boldsymbol{k}_{2q+1}}))] = 0$ *for all* q *and regardless of* $\{\boldsymbol{s}_j\}$ *being dependent or not.*

(ii) If more than $(q+1)$ *locations* $\{\boldsymbol{s}_j; j = 1, \ldots, 2q\}$ *are independent, then* $\mathrm{cum}_{2q}[Z(\boldsymbol{s}_{j_1})\exp(i\boldsymbol{s}_{j_1}\boldsymbol{\omega}_{\boldsymbol{k}_1}), \ldots, Z(\boldsymbol{s}_{j_{2q}})\exp(i\boldsymbol{s}_{j_{2q}}\boldsymbol{\omega}_{\boldsymbol{k}_{2q}}))] = 0$.

Lemma E.4 *Suppose Assumptions 2.1, 2.3 2.5 and 2.6(b) are satisfied, and* $d = 1$. *Then we have*

$$\mathrm{cum}_3[\widetilde{Q}_{a,\lambda}(g,r)] = O\left(\frac{\log^2(a)}{\lambda^2}\right) \qquad \text{(E.13)}$$

with $\lambda^d/(\log^2(a)n) \to 0$ *as* $\lambda \to \infty$, $n \to \infty$ *and* $a \to \infty$.

PROOF. We prove the result for $\mathrm{cum}_3[\widetilde{Q}_{a,\lambda}(1,0)]$, noting that the proof is identical for general g and r. We first expand $\mathrm{cum}_3[\widetilde{Q}_{a,\lambda}(1,0)]$ using indecomposable partitions. Using

Lemma E.3(i) we note that the third order cumulant is zero, therefore

$$\begin{aligned}
& \mathrm{cum}_3[\tilde{Q}_{a,\lambda}(1,0)] \\
= & \frac{1}{n^6}\sum_{\underline{j}\in\mathcal{D}}\sum_{k_1,k_2,k_3=-a}^{a}\mathrm{cum}\big[Z(s_{j_1})Z(s_{j_2})\exp(i\omega_{k_1}(s_{j_1}-s_{j_2})), \\
& Z(s_{j_3})Z(s_{j_4})\exp(i\omega_{k_2}(s_{j_3}-s_{j_4})), Z(s_{j_5})Z(s_{j_6})\exp(i\omega_{k_3}(s_{j_5}-s_{j_6}))\big] \\
= & \frac{1}{n^6}\sum_{\underline{j}\in\mathcal{D}}\sum_{\pi_{(2,2,2)},\in\mathcal{P}_{2,2,2}} A^{\underline{j}}_{2,2,2}(\pi_{(2,2,2)}) + \frac{1}{n^6}\sum_{\underline{j}\in\mathcal{D}}\sum_{\pi_{4,2}\in\mathcal{P}_{4,2}} A^{\underline{j}}_{4,2}(\pi_{(4,2)}) + \frac{1}{n^6}\sum_{\underline{j}\in\mathcal{D}} A^{\underline{j}}_{6} \\
= & B_{2,2,2} + B_{4,2} + B_6,
\end{aligned}$$

where $\mathcal{D} = \{j_1,\ldots,j_6 \in \{1,\ldots,n\}$ but $j_1 \neq j_2, j_3 \neq j_4, j_5 \neq j_6\}$, $A^{\underline{j}}_{2,2,2}$ consists of only the product of cumulants of order two and $\mathcal{P}_{2,2,2}$ is the set of all cumulants of order two from the set of indecomposable partitions of $\{(1,2),(3,4),(5,6)\}$, $A^{\underline{j}}_{4,2}$ consists of only the product of 4th and 2nd order cumulants and $\mathcal{P}_{4,2}$ is the set of all 4th order and 2nd order cumulant indecomposable partitions of $\{(1,2),(3,4),(5,6)\}$, finally $A^{\underline{j}}_{6}$ is the 6th order cumulant. Examples of A's are given below

$$\begin{aligned}
& A^{\underline{j}}_{2,2,2}(\pi_{(2,2,2),1}) \\
= & \sum_{k_1,k_2,k_3=-a}^{a}\mathrm{cum}\left[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_3})e^{is_{j_3}\omega_{k_2}}\right]\mathrm{cum}\left[Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_5})e^{is_{j_5}\omega_{k_3}}\right] \\
& \times\mathrm{cum}\left[Z(s_{j_4})e^{-is_{j_4}\omega_{k_2}}, Z(s_{j_6})e^{-is_{j_6}\omega_{k_3}}\right] \\
= & \sum_{k_1,k_2,k_3=-a}^{a}\mathrm{E}[c(s_{j_1}-s_{j_3})e^{i(s_{j_1}\omega_{k_1}+s_{j_3}\omega_{k_2})}]\mathrm{E}[c(s_{j_2}-s_{j_5})e^{i(-s_{j_2}\omega_{k_1}+s_{j_5}\omega_{k_3})}]\times \\
& \mathrm{E}[c(s_{j_4}-s_{j_6})e^{-i(s_{j_4}\omega_{k_2}+s_{j_6}\omega_{k_3})}],
\end{aligned} \tag{E.14}$$

$$\begin{aligned}
& A^{\underline{j}}_{4,2}(\pi_{(4,2),1}) \\
= & \sum_{k_1,k_2,k_3=-a}^{a}\mathrm{cum}[Z(s_{j_4})\exp(-is_{j_4}\omega_{k_2}), Z(s_{j_6})\exp(-is_{j_6}\omega_{k_3})] \\
& \mathrm{cum}[Z(s_{j_1})\exp(is_{j_1}\omega_{k_1}), Z(s_{j_2})\exp(-is_{j_2}\omega_{k_1}), Z(s_{j_3})\exp(is_{j_3}\omega_{k_2}), Z(s_{j_5})\exp(is_{j_5}\omega_{k_3})],
\end{aligned} \tag{E.15}$$

$$\begin{aligned}
& A^{\underline{j}}_{6} \\
= & \sum_{k_1,k_2,k_3=-a}^{a}\mathrm{cum}[Z(s_{j_1})\exp(is_{j_1}\omega_{k_1}), Z(s_{j_2})\exp(-is_{j_2}\omega_{k_1}), Z(s_{j_3})\exp(is_{j_3}\omega_{k_2}), \\
& Z(s_{j_4})\exp(-is_{j_4}\omega_{k_2}), Z(s_{j_5})\exp(is_{j_5}\omega_{k_3}), Z(s_{j_6})\exp(-is_{j_6}\omega_{k_3})],
\end{aligned} \tag{E.16}$$

where $\underline{j} = (j_1,\ldots,j_6)$.

Bound for B_{222}

We will show that B_{222} is the leading term in $\mathrm{cum}_3(\widetilde{Q}_{a,\lambda}(g;0))$. The set $\mathcal{D}$ is split into four sets, $\mathcal{D}_6$ where all the elements of $\underline{j}$ are different, and for $3 \leq i \leq 5$, $\mathcal{D}_i$ where i elements in $\underline{j}$ are different, such that

$$B_{2,2,2} = \frac{1}{n^6}\sum_{i=0}^{3}\sum_{\underline{j}\in\mathcal{D}_{6-i}}\sum_{\pi_{(2,2,2)}\in\mathcal{P}_{(2,2,2)}} A^{\underline{j}}_{2,2,2}(\pi_{(2,2,2)}).$$

We start by bounding the partition given in (E.14), we later explain how the same bounds can be obtained for other indecomposable partitions in $\mathcal{P}_{2,2,2}$. By using the spectral representation of the covariance and that $|\mathcal{D}_6| = O(n^6)$, it is straightforward to show that

$$\begin{aligned}
&\frac{1}{n^6}\sum_{\underline{j}\in\mathcal{D}_6} A^{\underline{j}}_{2,2,2}(\pi_{(2,2,2),1}) \\
=& \sum_{k_1,k_2,k_3}\frac{c_6}{(2\pi)^3\lambda^6}\int_{\mathbb{R}^3}\int_{[-\lambda/2,\lambda/2]^6} f(x)f(y)f(z)\exp(i\omega_{k_1}(s_1-s_2))\times \\
&\exp(i\omega_{k_2}(s_3-s_4))\exp(i\omega_{k_3}(s_5-s_6)) \\
&\exp(ix(s_1-s_3))\exp(iy(s_4-s_6))\exp(iz(s_2-s_5))\prod_{j=1}^{3}ds_{2j-1}ds_{2j}dxdydz \\
=& \frac{c_6}{(2\pi)^3}\sum_{k_1,k_2,k_3}\int_{\mathbb{R}^3} f(x)f(y)f(z)\mathrm{sinc}\left(\frac{\lambda x}{2}+k_1\pi\right)\mathrm{sinc}\left(\frac{\lambda z}{2}-k_1\pi\right)\times \\
&\mathrm{sinc}\left(\frac{\lambda x}{2}-k_2\pi\right)\mathrm{sinc}\left(\frac{\lambda y}{2}-k_2\pi\right)\mathrm{sinc}\left(\frac{\lambda z}{2}-k_3\pi\right)\mathrm{sinc}\left(\frac{\lambda y}{2}+k_3\pi\right)dxdydz
\end{aligned} \tag{E.17}$$

where $c_6 = n(n-1)\ldots(n-5)/n^6$. By changing variables $x = \frac{\lambda x}{2}+k_1\pi$, $y = \frac{\lambda y}{2}-k_2\pi$ and $z = \frac{\lambda z}{2}-k_3\pi$ we have

$$\begin{aligned}
&\frac{1}{n^6}\sum_{\underline{j}\in\mathcal{D}_6} A^{\underline{j}}_{2,2,2}(\pi_{(2,2,2),1}) \\
=& \frac{c_6}{(2\pi)^3}\sum_{k_1,k_2,k_3}\frac{1}{\lambda^3}\int_{\mathbb{R}^3} f(\frac{2x}{\lambda}-\omega_{k_1})f(\frac{2y}{\lambda}+\omega_{k_2})f(\frac{2z}{\lambda}+\omega_{k_3})\mathrm{sinc}(x)\mathrm{sinc}(z)\mathrm{sinc}(y) \\
&\mathrm{sinc}(x-(k_2+k_1)\pi)\mathrm{sinc}(y+(k_3+k_2)\pi)\mathrm{sinc}(z-(k_1-k_3)\pi)dxdydz.
\end{aligned} \tag{E.18}$$

In order to understand how this case can generalise to other partitions in $\mathcal{P}_{2,2,2}$, we represent the ks inside the sinc function using the linear equations

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}, \tag{E.19}$$

where we observe that the above is a rank two matrix. Based on this we make the following change of variables $k_1 = k_1$, $m_1 = k_2 + k_1$ and $m_2 = k_1 - k_3$, and rewrite the sum as

$$\begin{aligned}
&\frac{1}{n^6}\sum_{\underline{j}\in\mathcal{D}_6} A^{\underline{j}}_{2,2,2}(\pi_{(2,2,2),1}) \\
=\ & \frac{c_6}{(2\pi)^3}\sum_{k_1,m_1,m_2}\frac{1}{\lambda^3}\int_{\mathbb{R}^3} f\left(\frac{2x}{\lambda}-\omega_{k_1}\right) f\left(\frac{2y}{\lambda}+\omega_{m_1-k_1}\right) f\left(\frac{2z}{\lambda}+\omega_{k_1-m_2}\right)\mathrm{sinc}(x)\mathrm{sinc}(z)\mathrm{sinc}(y) \\
& \mathrm{sinc}(x-m_1\pi)\mathrm{sinc}(y-(m_2-m_1)\pi)\mathrm{sinc}(z-m_2\pi)dxdydz \\
=\ & \frac{c_6}{\lambda^2(2\pi)^3}\sum_{m_1,m_2}\int_{\mathbb{R}^3}\underbrace{\mathrm{sinc}(x)\mathrm{sinc}(x-m_1\pi)\mathrm{sinc}(y)\mathrm{sinc}(y+(m_1-m_2)\pi)\mathrm{sinc}(z)\mathrm{sinc}(z-m_2\pi)}_{\text{contains two linearly independent } m \text{ terms}} \\
& \times\frac{1}{\lambda}\sum_{k_1} f\left(\frac{2x}{\lambda}-\omega_{k_1}\right) f\left(\frac{2y}{\lambda}+\omega_{m_1-k_1}\right) f\left(\frac{2z}{\lambda}-\omega_{k_1-m_2}\right) dxdydz.
\end{aligned} \tag{E.20}$$

Finally, we apply Lemma F.1(iv) to give

$$\frac{1}{n^6}\sum_{\underline{j}\in\mathcal{D}_6} A^{\underline{j}}_{2,2,2}(\pi_{(2,2,2),1}) = O\left(\frac{\log^2 a}{\lambda^2}\right). \tag{E.21}$$

The above only gives the bound for one partition of $\mathcal{P}_{2,2,2}$, but we now show that the same bound applies to all the other partitions. Looking back at (E.18) and comparing with (E.20), the reason that only one of the three λs in the denominator of (E.18) gets 'swallowed' is because the matrix in (E.19) has rank two. Therefore, there are two linearly independent ms in the sinc function of (E.18), thus by applying Lemma F.1(iv) the sum $\sum_{m_1,m_2}$ only grows with rate $O(\log^2 a)$. Moreover, it can be shown that all indecomposable partitions of $\mathcal{P}_{2,2,2}$ correspond to rank two matrices (for a proof see equation (A.13) in (Deo & Chen, 2000)). Thus all indecomposable partitions in $\mathcal{P}_{2,2,2}$ will have the same order, which altogether gives

$$\frac{1}{n^6}\sum_{\underline{j}\in\mathcal{D}_6}\sum_{\pi_{(2,2,2)}\in\mathcal{P}_1} A^{\underline{j}}_{2,2,2}(\pi_{(2,2,2)}) = O\left(\frac{\log^2 a}{\lambda^2}\right).$$

Now we consider the case that $\underline{j}\in\mathcal{D}_5$. In this case, there are two 'typical' cases $\underline{j} = (j_1, j_2, j_3, j_4, j_1, j_6)$, which gives

$$\begin{aligned}
& A^{\underline{j}}_{2,2,2}(\pi_{(2,2,2),1}) = \\
& \mathrm{cum}\left[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_3})e^{is_{j_3}\omega_{k_2}}\right]\mathrm{cum}\left[Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_1})e^{is_{j_1}\omega_{k_3}}\right]\times \\
& \mathrm{cum}\left[Z(s_{j_4})e^{-is_{j_4}\omega_{k_2}}, Z(s_{j_6})e^{-is_{j_6}\omega_{k_3}}\right]
\end{aligned}$$

and $\underline{j} = (j_1, j_2, j_1, j_4, j_5, j_6)$, which gives

$$A^{\underline{j}}_{2,2,2}(\pi_{(2,2,2),1}) =$$
$$\mathrm{cum}\left[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, Z(s_{j_1})e^{is_{j_1}\omega_{k_2}}\right] \mathrm{cum}\left[Z(s_{j_2})e^{-is_{j_2}\omega_{k_1}}, Z(s_{j_5})e^{is_{j_5}\omega_{k_3}}\right] \times$$
$$\mathrm{cum}\left[Z(s_{j_4})e^{-is_{j_4}\omega_{k_2}}, Z(s_{j_6})e^{-is_{j_6}\omega_{k_3}}\right].$$

Using the same method used to bound (E.21), when $\underline{j} = (j_1, j_2, j_3, j_4, j_1, j_6)$ $A^{\underline{j}}_{2,2,2}(\pi_{(2,2,2),1}) = O(\frac{\log^2 a}{\lambda^2})$. However, when $\underline{j} = (j_1, j_2, j_1, j_4, j_5, j_6)$ we use the same proof used to prove (E.5) to give $A^{\underline{j}}_{2,2,2}(\pi_{(2,2,2),1}) = O(\frac{1}{\lambda})$. As we get similar expansions for all $\underline{j} \in \mathcal{D}_5$ and $|\mathcal{D}_5| = O(n^5)$ we have

$$\frac{1}{n^6}\sum_{\underline{j}\in\mathcal{D}_5}\sum_{\pi_{(2,2,2)}\in\mathcal{P}_{2,2,2}} A^{\underline{j}}_{2,2,2}(\pi_{(2,2,2)}) = O\left(\frac{1}{\lambda n} + \frac{\log^2(a)}{n\lambda^2}\right).$$

Similarly we can show that

$$\frac{1}{n^6}\sum_{\underline{j}\in\mathcal{D}_4}\sum_{\pi_{(2,2,2)}\in\mathcal{P}_{2,2,2}} A^{\underline{j}}_{2,2,2}(\pi_{(2,2,2)}) = O\left(\frac{1}{\lambda n^2} + \frac{\log^2(a)}{n^2\lambda^2}\right)$$

and

$$\frac{1}{n^6}\sum_{\underline{j}\in\mathcal{D}_3}\sum_{\pi_{(2,2,2)},\in\mathcal{P}_{2,2,2}} A^{\underline{j}}_{2,2,2}(\pi_{(2,2,2)}) = O\left(\frac{1}{n^3} + \frac{\log^2(a)}{n^3\lambda^2}\right).$$

Therefore, if $n >> \lambda/\log^2(a)$ we have $B_{2,2,2} = O(\frac{\log^2(a)}{\lambda^2})$.

Bound for $B_{4,2}$

To bound $B_{4,2}$ we consider the 'typical' partition given in (E.15). Since $A^{\underline{j}}_{4,2}(\pi_{(4,2),1})$ involves fourth order cumulants by Lemma E.3(ii) it will be zero in the case that the $\underline{j}$ are all different. Therefore, only a maximum of five terms in $\underline{j}$ can be different, which gives

$$\frac{1}{n^6}\sum_{\underline{j}\in\mathcal{D}} A^{\underline{j}}_{4,2}(\pi_{(4,2),1}) = \frac{1}{n^6}\sum_{i=1}^{3}\sum_{\underline{j}\in\mathcal{D}_{6-i}} A^{\underline{j}}_{4,2}(\pi_{(4,2),1}).$$

We will show that for $\underline{j} \in \mathcal{D}_5$, $A^{\underline{j}}_{4,2}(\pi_{(4,2),1})$ will not be as small as $O(\log^2(a)/\lambda^2)$, however, this will be compensated by $|\mathcal{D}_5| = O(n^5)$ (noting that $|\mathcal{D}_6| = O(n^6)$). Let $\underline{j} = (j_1, j_2, j_3, j_4, j_1, j_6)$, then expanding the fourth order cumulant in $A^{\underline{j}}_{4,2}(\pi_{(4,2),1})$ and using conditional cumulants (see (E.2)) we have

$$
\begin{aligned}
& A^{j}_{4,2}(\pi_{(4,2),1}) \\
= & \sum_{k_1,k_2,k_3=-a}^{a} \mathrm{cum}[Z(s_{j_4})\exp(-is_{j_4}\omega_{k_2}), Z(s_{j_6})\exp(-is_{j_6}\omega_{k_3})] \\
& \mathrm{cum}[Z(s_{j_1})\exp(is_{j_1}\omega_{k_1}), Z(s_{j_2})\exp(-is_{j_2}\omega_{k_1}), Z(s_{j_3})\exp(is_{j_3}\omega_{k_2}), Z(s_{j_5})\exp(is_{j_5}\omega_{k_3})] \\
= & \sum_{k_1,k_2,k_3=-a}^{a} \Big\{ \mathrm{cum}\big[c(s_1-s_2)e^{i\omega_{k_1}(s_1-s_2)}, c(s_3-s_1)e^{is_3\omega_{k_2}+is_1\omega_{k_3}}\big] \\
& +\mathrm{cum}\big[c(s_1-s_3)e^{i(s_1\omega_{k_1}+s_3\omega_{k_2})}, c(s_1-s_2)e^{is_1\omega_{k_3}-is_2\omega_{k_1}}\big] \\
& +\mathrm{cum}\big[c(0)e^{is_1\omega_{k_1}+is_1\omega_{k_3}}, c(s_2-s_3)e^{-is_2\omega_{k_1}+is_3\omega_{k_2}}\big]\Big\} \mathrm{E}\big[c(s_4-s_6)e^{-is_4\omega_{k_2}-is_6\omega_{k_3}}\big] \\
= & A^{j}_{4,2}(\pi_{(4,2),1},\Omega_1) + A^{j}_{4,2}(\pi_{(4,2),1},\Omega_2) + A^{j}_{4,2}(\pi_{(4,2),1},\Omega_3),
\end{aligned} \tag{E.22}
$$

where we use the notation Ω to denote the partition of the fourth order cumulant into its conditional cumulants. To bound each term we expand the covariances as expectations, this gives

$$
\begin{aligned}
& A^{j}_{4,2}(\pi_{(4,2),1},\Omega_1) \\
= & \sum_{k_1,k_2,k_3=-a}^{a} \Big\{ \mathrm{E}\big[c(s_1-s_2)c(s_3-s_1)e^{i\omega_{k_1}(s_1-s_2)}e^{is_3\omega_{k_2}+is_1\omega_{k_3}}\big]\mathrm{E}\big[c(s_4-s_6)e^{-is_4\omega_{k_2}-is_6\omega_{k_3}}\big] - \\
& \mathrm{E}\big[c(s_1-s_2)e^{i\omega_{k_1}(s_1-s_2)}\big]\mathrm{E}\big[c(s_3-s_1)e^{is_3\omega_{k_2}+is_1\omega_{k_3}}\big]\mathrm{E}\big[c(s_4-s_6)e^{-is_4\omega_{k_2}-is_6\omega_{k_3}}\big]\Big\} \\
= & A^{j}_{4,2}(\pi_{(4,2),1},\Omega_1,\Pi_1) + A^{j}_{4,2}(\pi_{(4,2),1},\Omega_1,\Pi_2),
\end{aligned}
$$

where we use the notation Π to denote the expansion of the cumulants of the spatial covariances expanded into expectations. To bound $A^{j}_{4,2}(\pi_{(4,2),1},\Omega_1,\Pi_1)$ we use the spectral representation theorem to give

$$
\begin{aligned}
& A^{j}_{4,2}(\pi_{(4,2),1},\Omega_1,\Pi_1) = \\
& \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\sum_{k_1,k_2,k_3=-a}^{a} f(x)f(y)f(z)\mathrm{sinc}\left(\frac{\lambda(x-y)}{2}+(k_1+k_3)\pi\right)\mathrm{sinc}\left(\frac{\lambda x}{2}+k_1\pi\right)\times \\
& \mathrm{sinc}\left(\frac{\lambda y}{2}+k_2\pi\right)\mathrm{sinc}\left(\frac{\lambda z}{2}-k_2\pi\right)\mathrm{sinc}\left(\frac{\lambda z}{2}+k_3\pi\right)dxdydz.
\end{aligned}
$$

By changing variables

$$
\begin{aligned}
A^{j}_{4,2}(\pi_{(4,2),1},\Omega_1,\Pi_1) = & \frac{1}{\pi^3\lambda^3}\sum_{k_1,k_2,k_3=-a}^{a}\int_{\mathbb{R}^3} f\left(\frac{2u}{\lambda}-\omega_{k_1}\right) f\left(\frac{2v}{\lambda}-\omega_{k_2}\right) f\left(\frac{2w}{\lambda}-\omega_{k_3}\right)\times \\
& \mathrm{sinc}(u-v+(k_2+k_3)\pi)\mathrm{sinc}(u)\mathrm{sinc}(v)\mathrm{sinc}(w)\mathrm{sinc}(w-(k_2+k_3)\pi)dudvdw.
\end{aligned}
$$

Just as in the bound for $B_{2,2,2}$, we represent the ks inside the sinc function as a set of linear equations

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}, \tag{E.23}$$

observing the matrix has rank one. We make a change of variables $m = k_2 + k_3$, $k_1 = k_1$ and $k_2 = k_2$ to give

$$\begin{aligned} & A^{j}_{4,2}(\pi_{(4,2),1}, \Omega_1, \Pi_1) \\ = \;& \frac{1}{\pi^3 \lambda} \int_{\mathbb{R}^3} \frac{1}{\lambda^2} \sum_{k_1,k_2=-a}^{a} f\left(\frac{2u}{\lambda} - \omega_{k_1}\right) f\left(\frac{2v}{\lambda} - \omega_{k_2}\right) f\left(\frac{2w}{\lambda} - \omega_{m_1-k_2}\right) \times \\ & \sum_{m} \mathrm{sinc}(u - v + m\pi)\mathrm{sinc}(u)\mathrm{sinc}(v)\mathrm{sinc}(w)\mathrm{sinc}(w - m\pi) du dv dw \\ = \;& \frac{2^3}{\lambda} \int_{\mathbb{R}^3} \sum_{m} G_{\lambda,m}\left(\frac{2u}{\lambda}, \frac{2v}{\lambda}, \frac{2w}{\lambda}\right) \mathrm{sinc}(u - v + m\pi)\mathrm{sinc}(u)\mathrm{sinc}(v)\mathrm{sinc}(w)\mathrm{sinc}(w - m\pi) du dv dw, \end{aligned}$$

where $G_{\lambda,m}(\frac{2u}{\lambda}, \frac{2v}{\lambda}, \frac{2w}{\lambda}) = \frac{1}{\lambda^2} \sum_{k_1,k_2=-a}^{a} f(\frac{2u}{\lambda} - \omega_{k_1}) f(\frac{2v}{\lambda} - \omega_{k_2}) f(\frac{2w}{\lambda} - \omega_{m-k_2})$. Taking absolutes gives

$$|A^{j}_{4,2}(\pi_{(4,2),1}, \Omega_1, \Pi_1)| \leq \frac{C}{\lambda} \int_{\mathbb{R}^3} \sum_{m} |\mathrm{sinc}(u - v + m\pi)\mathrm{sinc}(w - m\pi)\mathrm{sinc}(u)\mathrm{sinc}(v)\mathrm{sinc}(w)| du dv dw$$

Since the above contains m in the sinc function we use Lemma F.1(i), equations (F.1) and (F.2), to show

$$\begin{aligned} |A^{j}_{4,2}(\pi_{(4,2),1}, \Omega_1, \Pi_1)| \;& \leq \; \frac{C}{\lambda} \int_{\mathbb{R}^3} \sum_{m} |\mathrm{sinc}(u - v + m\pi)\mathrm{sinc}(w - m\pi)\mathrm{sinc}(u)\mathrm{sinc}(v)\mathrm{sinc}(w)| du dv dw \\ & \leq \; \frac{C}{\lambda} \sum_{m} \ell_1(m\pi)\ell_2(m\pi) = O(\lambda^{-1}), \end{aligned}$$

where the functions $\ell_1(\cdot)$ and $\ell_2(\cdot)$ are defined in Lemma F.1(i). Thus $|A^{j}_{4,2}(\pi_{(4,2),1}, \Omega_1, \Pi_1)| = O(\frac{1}{\lambda})$. We use the same method used to bound (E.21) to show that $A^{j}_{4,2}(\pi_{(4,2),1}, \Omega_1, \Pi_2) = O(\frac{\log^2(a)}{\lambda^2})$ and $|A^{j}_{4,2}(\pi_{(4,2),1}, \Omega_2)| = O(\frac{1}{\lambda})$. Furthermore, it is straightforward to see that by the independence of s_1, s_2 and s_3 that $A^{j}_{4,2}(\pi_{(4,2),1}, \Omega_3) = 0$ (recalling that $A^{j}_{4,2}(\pi_{(4,2),1}, \Omega_3)$ is defined in equation (E.22)). Thus altogether we have for $\underline{j} = (j_1, j_2, j_3, j_4, j_1, j_6)$ and the partition $\pi_{(4,2),1}$, that $|A^{j}_{4,2}(\pi_{(4,2),1})| = O(\frac{1}{\lambda})$. However, it is important to note for all other $\underline{j} \in \mathcal{D}_5$ and partitions in $\mathcal{P}_{4,2}$ the same method will lead to a similar decomposition given in (E.22) and the rank one matrix given in (E.23). The rank one matrix means one linearly independent m in the sinc functions, thus $|A^{\underline{j}}_{4,2}(\pi_{4,2})| = O(\frac{1}{\lambda})$ for all $\underline{j} \in \mathcal{D}_5$ and $\pi_{4,2} \in \mathcal{P}_{4,2}$. Since $|\mathcal{D}_5| = O(n^5)$ we have

$$\frac{1}{n^6}\sum_{\underline{j}\in\mathcal{D}_5}\sum_{\pi_{4,2}\in\mathcal{P}_{4,2}} A^{\underline{j}}_{4,2}(\pi_{4,2}) = O\left(\frac{1}{\lambda n} + \frac{\log^2 a}{\lambda^2 n}\right) = O\left(\frac{\log^2 a}{\lambda^2}\right)$$

if $n >> \lambda/\log^2(a)$ (i.e. $\frac{\lambda}{n\log^2 a} \to 0$). For $\underline{j} \in \mathcal{D}_4$ and $\underline{j} \in \mathcal{D}_3$ we use the same argument, noting that the number of linearly independent m's in the sinc functions goes down but to compensate, $|\mathcal{D}_4| = O(n^4)$ and $|\mathcal{D}_3| = O(n^3)$. Therefore, if $n >> \log^2(a)/\lambda$, then

$$B_{4,2} = \frac{1}{n^6}\sum_{i=1}^{3}\sum_{\underline{j}\in\mathcal{D}_{6-i}}\sum_{\pi_{4,2}\in\mathcal{P}_{4,2}} A^{\underline{j}}_{4,2}(\pi_{4,2}) = O\left(\frac{\log^2 a}{\lambda^2}\right).$$

Bound for B_6

Finally, we bound B_6. By using Lemma E.3(ii) we observe that $A^{\underline{j}}_6(k_1,k_2,k_3) = 0$ if more than four elements of $\underline{j}$ are different. Thus

$$B_6 = \frac{1}{n^6}\sum_{i=2}^{3}\sum_{\underline{j}\in\mathcal{D}_{6-i}} A^{\underline{j}}_6.$$

We start by considering the case that $\underline{j} = (j_1, j_2, j_1, j_4, j_1, j_6)$ (three elements in $\underline{j}$ are the same), then by using conditional cumulants we have

$$\begin{aligned} & A^{\underline{j}}_6 \\ = & \sum_{k_1,k_2,k_3=-a}^{a} \mathrm{cum}[Z(s_1)e^{is_1\omega_{k_1}}, Z(s_2)e^{-is_2\omega_{k_1}}, Z(s_1)e^{is_1\omega_{k_2}}, \\ & Z(s_4)e^{-is_4\omega_{k_2}}, Z(s_1)e^{is_1\omega_{k_3}}, Z(s_6)e^{-is_6\omega_{k_3}}] \\ = & \sum_{k_1,k_2,k_3=-a}^{a}\sum_{\Omega\in\mathcal{R}} A^{\underline{j}}_6(\Omega), \end{aligned}$$

where $\mathcal{R}$ is the set of all pairwise partitions of $\{1,2,1,4,1,6\}$, for example

$$A^{\underline{j}}_6(\Omega_1) = \mathrm{cum}\big[c(s_1-s_2)e^{i\omega_{k_1}(s_1-s_2)}, c(s_1-s_4)e^{i\omega_{k_2}(s_1-s_4)}, c(s_1-s_6)e^{i\omega_{k_3}(s_1-s_6)}\big].$$

We will first bound the above and then explain how this generalises to the other $\Omega \in \mathcal{R}$ and $\underline{j} \in \mathcal{D}_4$. Expanding the above third order cumulant in terms of expectations gives

$$
\begin{aligned}
& A_6^{\underline{j}}(\Omega_1) \\
= & \sum_{k_1,k_2,k_3=-a}^{a} \Big\{ \mathrm{E}\big[c(s_1-s_2)e^{i\omega_{k_1}(s_1-s_2)}c(s_1-s_4)e^{i\omega_{k_2}(s_1-s_4)}c(s_1-s_6)e^{i\omega_{k_3}(s_1-s_6)}\big] - \\
& \mathrm{E}\big[c(s_1-s_2)e^{i\omega_{k_1}(s_1-s_2)}\big]\mathrm{E}\big[c(s_1-s_4)e^{i\omega_{k_2}(s_1-s_4)}c(s_1-s_6)e^{i\omega_{k_3}(s_1-s_6)}\big] - \\
& \mathrm{E}\big[c(s_1-s_2)e^{i\omega_{k_1}(s_1-s_2)}c(s_1-s_4)e^{i\omega_{k_2}(s_1-s_4)}\big)\mathrm{E}\big[c(s_1-s_6)e^{i\omega_{k_3}(s_1-s_6)}\big] - \\
& \mathrm{E}\big[c(s_1-s_2)e^{i\omega_{k_1}(s_1-s_2)}c(s_1-s_6)e^{i\omega_{k_3}(s_1-s_6)}\big]\mathrm{E}\big[c(s_1-s_4)e^{i\omega_{k_2}(s_1-s_4)}\big] + \\
& 2\mathrm{E}\big[c(s_1-s_2)e^{i\omega_{k_1}(s_1-s_2)}\big]\mathrm{E}\big[c(s_1-s_4)e^{i\omega_{k_2}(s_1-s_4)}\big]\mathrm{E}\big[c(s_1-s_6)e^{i\omega_{k_3}(s_1-s_6)}\big]\Big\} \\
= & \sum_{\ell=1}^{5} A_6^{\underline{j}}(\Omega_1,\Pi_\ell).
\end{aligned}
$$

We observe that for $2 \leq \ell \leq 5$, $A_6^{\underline{j}}(\Omega_1,\Pi_\ell)$ resembles $A_{4,2}^{\underline{j}}(\pi_{(4,2),1},\Omega_1,\Pi_1)$ defined in (E.23), thus the same proof used to bound the terms in (E.23) can be use to show that for $2 \leq \ell \leq 5$ $A_6^{\underline{j}}(\Omega_1,\Pi_\ell) = O(\frac{1}{\lambda})$. However, the first term $A_6^{\underline{j}}(\Omega_1,\Pi_1)$ involves just one expectation, and is not included in the previous cases. By using the spectral representation theorem we have

$$
\begin{aligned}
& A_6^{\underline{j}}(\Omega_1,\Pi_1) \\
= & \sum_{k_1,k_2,k_3=-a}^{a} \mathrm{E}\left[c(s_1-s_2)e^{i\omega_{k_1}(s_1-s_2)}c(s_1-s_4)e^{i\omega_{k_2}(s_1-s_4)}c(s_1-s_6)e^{i\omega_{k_3}(s_1-s_6)}\right] \\
= & \frac{1}{(2\pi)^3}\sum_{k_1,k_2,k_3=-a}^{a}\int\int\int f(x)f(y)f(z)\mathrm{sinc}\left(\frac{\lambda(x+y+z)}{2}+(k_1+k_2+k_3)\pi\right)\mathrm{sinc}\left(\frac{\lambda x}{2}+k_1\pi\right) \\
& \mathrm{sinc}\left(\frac{\lambda y}{2}+k_2\pi\right)\mathrm{sinc}\left(\frac{\lambda z}{2}+k_3\pi\right)dxdydz \\
= & \frac{2^3}{(2\pi)^3\lambda^3}\sum_{k_1,k_2,k_3=-a}^{a}\int\int\int f(\frac{2u}{\lambda}-\omega_{k_1})f(\frac{2v}{\lambda}-\omega_{k_2})f(\frac{2w}{\lambda}-\omega_{k_3})\times \\
& \mathrm{sinc}(u+v+w)\mathrm{sinc}(u)\mathrm{sinc}(v)\mathrm{sinc}(w)dudvdw.
\end{aligned}
$$

It is obvious that the ks within the sinc function correspond to a rank zero matrix, and thus $A_6(\Omega_1,\Pi_1) = O(1)$. Therefore, $A_6^{\underline{j}}(\Omega_1) = O(1)$. A similar bound holds for all $\underline{j} \in \mathcal{D}_4$, this we have

$$
\frac{1}{n^6}\sum_{\underline{j}\in\mathcal{D}_4} A_6^{\underline{j}}(\Omega_1) = O\left(\frac{1}{n^2}\right),
$$

since $|\mathcal{D}_4| = O(n^4)$. Indeed, the same argument applies to the other partitions Ω and $\underline{j} \in \mathcal{D}_3$, thus altogether we have

$$\mathcal{B}_6 = \frac{1}{n^6} \sum_{\Omega \in \mathcal{R}} \sum_{i=2}^{3} \sum_{\underline{j} \in \mathcal{D}_{6-i}} A_6^{\underline{j}}(\Omega) = O\left(\frac{1}{n^2}\right).$$

Altogether, using the bounds derived for $\mathcal{B}_{2,2,2}$, $\mathcal{B}_{4,2}$ and $\mathcal{B}_6$ we have

$$\mathrm{cum}_3(\widetilde{Q}_{a,\lambda}(1,0)) = O\bigg(\frac{\log^2(a)}{\lambda^2} + \frac{1}{n\lambda} + \frac{\log^2(a)}{\lambda^2 n} + \frac{1}{n^2}\bigg) = O\left(\frac{\log^2(a)}{\lambda^2}\right),$$

where the last bound is due to the conditions on a, n and λ. This gives the result. $\square$

We now generalize the above results to higher order cumulants.

Lemma E.5 *Suppose Assumptions 2.1, 2.3, 2.5 and 2.6(b) are satisfied, and $d = 1$. Then for $q \geq 3$ we have*

$$\mathrm{cum}_q[\widetilde{Q}_{a,\lambda}(g,r)] \;=\; O\bigg(\frac{\log^{2(q-2)}(a)}{\lambda^{q-1}}\bigg), \tag{E.24}$$

$$\mathrm{cum}_q[\widetilde{Q}_{a,\lambda}(g,r_1),\ldots,\widetilde{Q}_{a,\lambda}(g,r_q)] \;=\; O\bigg(\frac{\log^{2(q-2)}(a)}{\lambda^{q-1}}\bigg) \tag{E.25}$$

and in the case $d > 1$, we have

$$\mathrm{cum}_q[\widetilde{Q}_{a,\lambda}(g,\boldsymbol{r}_1),\ldots,\widetilde{Q}_{a,\lambda}(g,\boldsymbol{r}_q)] = O\bigg(\frac{\log^{2d(q-2)}(a)}{\lambda^{d(q-1)}}\bigg) \tag{E.26}$$

with $\lambda^d/(\log^2(a)n) \to 0$ as $\lambda \to \infty$, $n \to \infty$ and $a \to \infty$.

PROOF. The proof essentially follows the same method used to bound the second and third cumulants. We first prove (E.24). To simplify the notation we prove the result for $g = 1$ and $r = 0$, noting that the proof in the general case is identical. Expanding out $\mathrm{cum}_q[\widetilde{Q}_{a,\lambda}(1,0)]$ and using indecomposable partitions gives

$$\begin{aligned} & \mathrm{cum}_q[\widetilde{Q}_{a,\lambda}(1,0)] \\ = \; & \frac{1}{n^{2q}} \sum_{j_1,\ldots,j_{2q} \in \mathcal{D}} \sum_{k_1,\ldots,k_q=-a}^{a} \mathrm{cum}\left[Z(s_{j_1})Z(s_{j_2})e^{i\omega_{k_1}(s_{j_1}-s_{j_2})},\ldots,Z(s_{j_{2q-1}})Z(s_{j_{2q}})e^{i\omega_{k_q}(s_{j_{2q-1}}-s_{j_{2q}})}\right] \\ = \; & \frac{1}{n^{2q}} \sum_{\underline{j} \in \mathcal{D}} \sum_{\underline{b} \in \mathcal{B}_q} \sum_{\pi_{\underline{b}} \in \mathcal{P}_{2\underline{b}}} A_{2\underline{b}}^{\underline{j}}(\pi_{2\underline{b}}) \end{aligned}$$

where $\mathcal{B}_q$ corresponds to the set of integer partitions of q (more precisely, each partition is a sequence of positive integers which sums to q). The notation used here is simply a generalization of the notation used in Lemma E.4. Let $\underline{b} = (b_1,\ldots,b_m)$ (noting

$\sum_j b_j = q$) denote one of these partitions, then $\mathcal{P}_{2\underline{b}}$ is the set of all indecomposable partitions of $\{(1,2),(3,4),\ldots,(2q-1,2q)\}$ where the size of each partition is $2b_1, 2b_2, \ldots, 2b_m$. For example, if $q=3$, then one example of an element of $\mathcal{B}_3$ is $\underline{b}=(1,1,1)$ and $\mathcal{P}_{2\underline{b}} = \mathcal{P}_{(2,2,2)}$ corresponds to all pairwise indecomposable partitions of $\{(1,2),(3,4),(5,6)\}$. Finally, $A^{\underline{j}}_{2\underline{b}}(\pi_{2\underline{b}})$ corresponds to the product of one indecomposable partition of the cumulant $\mathrm{cum}\big[Z(s_{j_1})Z(s_{j_2})e^{i\omega_{k_1}(s_{j_1}-s_{j_2})},\ldots,Z(s_{j_{2q-1}})Z(s_{j_{2q}})e^{i\omega_{k_q}(s_{j_{2q-1}}-s_{j_{2q}})}\big]$, where the cumulants are of order $2b_1, 2b_2, \ldots, 2b_m$ (examples, in the case $q=3$ are given in equation (E.14)-(E.16)). Let

$$B_{2\underline{b}} = \frac{1}{n^{2q}} \sum_{\underline{j}\in\mathcal{D}} \sum_{\pi_{2\underline{b}}\in\mathcal{P}_{2\underline{b}}} A^{\underline{j}}_{2\underline{b}}(\pi_{2\underline{b}}),$$

therefore $\mathrm{cum}_q[\widetilde{Q}_{a,\lambda}(1,0)] = \sum_{\underline{b}\in\mathcal{B}_q} B_{2\underline{b}}$.

Just as in the proof of Lemma E.4, we will show that under the condition $n >> \lambda/\log^2(a)$, the pairwise decomposition $B_{2,\ldots,2}$ is the denominating term. We start with a 'typical' decomposition $\pi_{(2,\ldots,2),1} \in \mathcal{P}_{2,\ldots,2}$,

$$\begin{aligned} A^{\underline{j}}_{2,\ldots,2}(\pi_{(2,\ldots,2),1}) &= \sum_{k_1,\ldots,k_q=-a}^{a} \mathrm{cum}[Z(s_1)e^{is_1\omega_{k_1}}, Z(s_{2q})e^{-is_{2q}\omega_{k_q}}] \times \\ &\quad \prod_{c=1}^{q-1} \mathrm{cum}[Z(s_{2c})e^{-is_{2c}\omega_{k_c}}, Z(s_{2c+1})e^{is_{2c+1}\omega_{k_{c+1}}}]. \end{aligned}$$

and

$$\frac{1}{n^{2q}} \sum_{\underline{j}\in\mathcal{D}} A^{\underline{j}}_{2,\ldots,2}(\pi_{(2,\ldots,2),1}) = \frac{1}{n^{2q}} \sum_{i=0}^{q-1} \sum_{\underline{j}\in\mathcal{D}_{2q-i}} A^{\underline{j}}_{2,\ldots,2}(\pi_{(2,\ldots,2),1}),$$

where $\mathcal{D}_{2q}$ denotes the set where all elements of $\underline{j}$ are different and $\mathcal{D}_{2q-i}$ denotes the set that $(2q-i)$ elements in $\underline{j}$ are different. We first consider the case that $\underline{j}=(1,2,\ldots,2q)\in\mathcal{D}_{2q}$. Using identical arguments to those used for $\mathrm{var}[\widetilde{Q}_{a,\lambda}(1,0)]$ and $\mathrm{cum}_3[\widetilde{Q}_{a,\lambda}(1,0)]$ we can show that

$$\begin{aligned} A^{\underline{j}}_{2,\ldots,2}(\pi_{(2,\ldots,2),1}) &= \sum_{k_1,\ldots,k_q=-a}^{a} \int_{\mathbb{R}^q} f(x_q)\mathrm{sinc}\left(\frac{2x_q}{\lambda}+k_1\pi\right)\mathrm{sinc}\left(\frac{2x_q}{\lambda}+k_q\pi\right) \times \\ &\quad \prod_{c=1}^{q-1} f(x_c)\mathrm{sinc}\left(\frac{2x_c}{\lambda}-k_c\pi\right)\mathrm{sinc}\left(\frac{2x_c}{\lambda}-k_{c+1}\right)\prod_{c=1}^{q} dx_c. \end{aligned} \quad \text{(E.27)}$$

By a change of variables we get

$$\begin{aligned} A^{\underline{j}}_{2,\ldots,2}(\pi_{(2,\ldots,2),1}) &= \frac{1}{\lambda^q} \sum_{k_1,\ldots,k_q=-a}^{a} \int_{\mathbb{R}^q} \prod_{c=1}^{q-1} f\left(\frac{\lambda u_c}{2}+\omega_c\right)\mathrm{sinc}(u_c)\mathrm{sinc}(u_c-(k_{c+1}-k_c)\pi) \\ &\quad \times f\left(\frac{\lambda u_q}{2}-\omega_1\right)\mathrm{sinc}(u_q)\mathrm{sinc}(u_q+(k_q-k_1)\pi)\prod_{c=1}^{q} dx_c. \end{aligned}$$

As in the proof of the third order cumulant we can rewrite the ks in the above as a matrix equation

$$\begin{pmatrix} -1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 \\ -1 & 0 & \ldots & \ldots & 0 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_q \end{pmatrix},$$

noting that that above is a $(q-1)$-rank matrix. Therefore applying the same arguments that were used in the proof of $\mathrm{cum}_3[\widetilde{Q}_{a,\lambda}(1,0)]$ and also Lemma F.1(iii) we can show that $A^{\underline{j}}_{2,\ldots,2}(\pi_{(2,\ldots,2),1}) = O(\frac{\log^{2(q-2)}(a)}{\lambda^{q-1}})$. Thus for $\underline{j} \in \mathcal{D}_{2q}$ we have $\frac{1}{n^{2q}} \sum_{\underline{j}\in\mathcal{D}_{2q}} A^{\underline{j}}_{2,\ldots,2}(\pi_{(2,\ldots,2),1}) = O(\frac{\log^{2(q-2)}(a)}{\lambda^{q-1}})$.

In the case that $\underline{j} \in \mathcal{D}_{2q-1}$ ($(2q-1)$-terms in $\underline{j}$ are different) by using the same arguments as those used to bound $A_{2,2,2}$ (in the proof of $\mathrm{cum}[\widetilde{Q}_{a,\lambda}(1,0)]$) we have $A^{\underline{j}}_{2,\ldots,2}(\pi_{(2,\ldots,2),1}) = O(\frac{\log^{2(q-3)}(a)}{\lambda^{q-2}})$, similarly if $(2q-2)$-terms in $\underline{j}$ are different, then $A^{\underline{j}}_{2,\ldots,2}(\pi_{(2,\ldots,2),1}) = O(\frac{\log^{2(q-4)}(a)}{\lambda^{q-3}})$ and so forth. Therefore, since $|\mathcal{D}_{2q-i}| = O(n^{2q-i})$ we have

$$\frac{1}{n^{2q}} \sum_{\underline{j}\in\mathcal{D}} |A^{\underline{j}}_{2,\ldots,2}(\pi_{(2,\ldots,2),1})| \leq C \sum_{i=0}^{q} \frac{\log^{2(q-2-i)}(a)}{\lambda^{q-1-i} n^i}.$$

Now by using that $n >> \lambda/\log^2(a)$ we have

$$\frac{1}{n^{2q}} \sum_{\underline{j}\in\mathcal{D}} |A^{\underline{j}}_{2,\ldots,2}(\pi_{(2,\ldots,2),1})| = O\left(\frac{\log^{2(q-2)}(a)}{\lambda^{q-1}}\right).$$

The same argument holds for every other second order cumulant indecomposable partition, because the corresponding matrix will always have rank $(q-1)$ in the case that $\underline{j} \in \mathcal{D}_{2q}$ or for $\underline{j} \in \mathcal{D}_{2q-i}$ and the dependent s_j's lie in different cumulants (see (Deo & Chen, 2000)), thus $B_{2,\ldots,2} = O(\frac{\log^{2(q-1)}(a)}{\lambda^{q-1}})$.

Now, we bound the other extreme B_{2q}. Using the conditional cumulant expansion (E.2) and noting that $\mathrm{cum}_{2q}[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, \ldots, Z(s_{j_{2q}})e^{-is_{j_{2q}}\omega_{k_q}}]$ is non-zero, only when at most $(q+1)$ elements of $\underline{j}$ are different we have

$$\begin{aligned} B_{2q} &= \frac{1}{n^{2q}} \sum_{\underline{j}\in\mathcal{D}} \sum_{k_1,\ldots,k_q=-a}^{a} \mathrm{cum}_{2q}[Z(s_{j_1})e^{is_{j_1}\omega_{k_1}}, \ldots, Z(s_{j_{2q}})e^{-is_{j_{2q}}\omega_{k_q}}] \\ &= \frac{1}{n^{2q}} \sum_{\underline{j}\in\mathcal{D}_{q+1}} \sum_{\Omega\in\mathcal{R}_{2q}} A^{\underline{j}}_{2q}(\Omega). \end{aligned}$$

where $\mathcal{R}_{2q}$ is the set of all pairwise partitions of $\{1,2,1,3,\ldots,1,q\}$. We consider a 'typical' partition $\Omega_1 \in \mathcal{R}_{2q}$

$$A^{\underline{j}}_{2q}(\Omega_1) = \sum_{k_1,\ldots,k_q=-a}^{a} \mathrm{cum}\big[c(s_1-s_2)e^{i(s_1-s_2)\omega_{k_1}}, \ldots, c(s_1-s_{q+1})e^{i(s_1-s_{q+1})\omega_{k_q}}\big]. \quad \text{(E.28)}$$

By expanding the above the cumulant as the sum of the product of expectations we have

$$A_{2q}^{\underline{j}}(\Omega_1) = \frac{1}{n^{2q}} \sum_{\underline{j}\in\mathcal{D}_{q+1}} \sum_{\Omega\in\mathcal{R}_{2q}} \sum_{\Pi\in\mathcal{S}_q} A_{2q}^{\underline{j}}(\Omega_1, \Pi),$$

where $\mathcal{S}_q$ is the set of all partitions of $\{1,\ldots,q\}$. As we have seen in both the $\mathrm{var}[\widetilde{Q}_{a,\lambda}(1,0)]$ and $\mathrm{cum}_3[\widetilde{Q}_{a,\lambda}(1,0)]$ calculations, the leading term in the cumulant expansion is the expectation over all the covariance terms. The same result holds true for higher order cumulants, the expectation over all the covariances in that cumulant is the leading term because the it gives the linear equation of the ks in the sinc function with the lowest order rank (we recall the lower the rank the less 'free' λs). Based on this we will only derive bounds for the expectation over all the covariances. Let $\Pi_1 \in \mathcal{S}_q$, where

$$A_{2q}^{\underline{j}}(\Omega_1, \Pi_1) = \sum_{k_1,\ldots,k_q=-a}^{a} \mathrm{E}\big[\prod_{c=1}^{q} c(s_1 - s_{c+1}) e^{i(s_1 - s_{c+1})\omega_{k_c}}\big].$$

Representing the above expectation as an integral and using the spectral representation theorem and a change of variables gives

$$\begin{aligned}
A_{2q}^{\underline{j}}(\Omega_1, \Pi_1) &= \sum_{k_1,\ldots,k_q=-a}^{a} \mathrm{E}\big[\prod_{c=1}^{q} c(s_1 - s_{c+1}) e^{i(s_1 - s_{c+1})\omega_{k_c}}\big] \\
&= \frac{1}{(2\pi)^q} \sum_{k_1,k_2,k_3=-a}^{a} \int_{\mathbb{R}^q} \mathrm{sinc}\bigg(\frac{\lambda(\sum_{c=1}^{q} x_c)}{2} + \pi \sum_{c=1}^{q} k_c\bigg) \prod_{c=1}^{q} f(x_c)\mathrm{sinc}(x_c + k_c\pi) \prod_{c=1}^{q} dx_c \\
&= \frac{2^q}{(2\pi)^q \lambda^q} \sum_{k_1,\ldots,k_q=-a}^{a} \int_{\mathbb{R}^q} \mathrm{sinc}(\sum_{c=1}^{q} u_c) \prod_{c=1}^{q} f\left(\frac{2u_c}{\lambda} - \omega_{k_c}\right) \mathrm{sinc}(u_c) du_c = O(1),
\end{aligned}$$

where the last line follows from Lemma F.1, equation (F.3). Therefore, $A_{2q}^{\underline{j}}(\Omega_1) = O(1)$. By using the same method on every partition $\Omega \in \mathcal{R}_{q+1}$ and $\underline{j} \in \mathcal{D}_{q+1}$ and $|\mathcal{D}_{q+1}| = O(n^{q+1})$, we have

$$B_{2q} = \frac{1}{n^{2q}} \sum_{\underline{j}\in\mathcal{D}_{q+1}} \sum_{\Omega\in\mathcal{R}_{2q}} \sum_{\Pi\in\mathcal{S}_{2q}} A^{\underline{j}}(\Omega_1, \Pi_1) = O(\frac{1}{n^{q-1}}).$$

Finally, we briefly discuss the terms $B_{2\underline{b}}$ which lie between the two extremes $B_{2,\ldots,2}$ and B_{2q}. Since $B_{2\underline{b}}$ is the product of $2b_1,\ldots,2b_m$ cumulants, by Lemma E.3(ii) at most $\sum_{j=1}^{m}(b_j + 1) = q + m$ elements of $\underline{j}$ can be different. Thus

$$B_{2\underline{b}} = \frac{1}{n^{2q}} \sum_{i=q}^{q+m} \sum_{\underline{j}\in\mathcal{D}_i} \sum_{\pi_{2\underline{b}}\in\mathcal{P}_{2\underline{b}}} A_{2\underline{b}}^{\underline{j}}(\pi_{2\underline{b}}).$$

By expanding the cumulants in terms of the cumulants of covariances conditioned on the location (which is due to Gaussianity of the random field, see for example, (E.28)) we have

$$B_{2\underline{b}} = \frac{1}{n^{2q}} \sum_{i=q}^{q+m} \sum_{\underline{j} \in \mathcal{D}_i} \sum_{\pi_{2\underline{b}} \in \mathcal{P}_{2\underline{b}}} \sum_{\Omega \in \mathcal{R}_{2\underline{b}}} A^{\underline{j}}_{2\underline{b}}(\pi_{2\underline{b}}, \Omega),$$

where $\mathcal{R}_{2\underline{b}}$ is the set of all paired partitions of $\{(1,\ldots,2b_1),\ldots,(2b_{m-1}+1,\ldots,2b_m)\}$. The leading terms are the highest order expectations. This term leads to a matrix equation for the k's within the sinc functions, where the rank of the corresponding matrix is at least $(m-1)$ (we do not give a formal proof of this). Therefore, $B_{2\underline{b}} = O(\frac{\log^{2(m-1)}(a)}{n^{q-m}\lambda^{m-1}}) = O(\frac{\log^{2(q-2)}(a)}{\lambda^{q-1}})$ (since $n >> \lambda/\log^2(a)$). This concludes the proof of (E.24).

The proof of (E.25) is identical and we omit the details.

To prove the result for $d > 1$, (E.26) we use the same method, the main difference is that the spectral density function in (E.27) is a multivariate function of dimension d, there are $2dp$ sinc functions and the integral is over $\mathbb{R}^{dp}$, however the analysis is identical. □

Theorem E.1 *[CLT on real and imaginary parts] Suppose Assumptions 2.1, 2.3, 2.6(b,c) and 2.5(i) or 2.5(ii) hold. Let C_1 and C_2, be defined as in Corollary 4.1. We define the m-dimension complex random vectors $\widetilde{\boldsymbol{Q}}_m = (\widetilde{Q}_{a,\lambda}(g,\boldsymbol{r}_1),\ldots,\widetilde{Q}_{a,\lambda}(g,\boldsymbol{r}_m))$, where $\boldsymbol{r}_1,\ldots,\boldsymbol{r}_m$ are such that $\boldsymbol{r}_i \neq -\boldsymbol{r}_j$ and $\boldsymbol{r}_i \neq 0$. Under these conditions we have*

$$\frac{2\lambda^{d/2}}{C_1}\left(\frac{C_1}{C_1 + \Re C_2}\Re\widetilde{Q}_{a,\lambda}(g,0), \Re\widetilde{\boldsymbol{Q}}_m, \Im\widetilde{\boldsymbol{Q}}_m\right) \xrightarrow{\mathcal{P}} \mathcal{N}\big(0, I_{2m+1}\big) \quad \text{(E.29)}$$

with $\frac{\log^2(a)}{\lambda^{1/2}} \to 0$, $\lambda^d/n \to 0$ as $\lambda \to \infty$, $n \to \infty$ and $a \to \infty$.

PROOF By using the well known identities

$$\begin{aligned}
\mathrm{cov}(\Re A, \Re B) &= \frac{1}{2}\big(\Re\mathrm{cov}(A,B) + \Re\mathrm{cov}(A,\bar{B})\big) \\
\mathrm{cov}(\Im A, \Im B) &= \frac{1}{2}\big(\Re\mathrm{cov}(A,B) - \Re\mathrm{cov}(A,\bar{B})\big), \\
\mathrm{cov}(\Re A, \Im B) &= \frac{-1}{2}\big(\Im\mathrm{cov}(A,B) - \Im\mathrm{cov}(A,\bar{B})\big), \quad \text{(E.30)}
\end{aligned}$$

and equation (**??**), we immediately obtain

$$\lambda^d \mathrm{var}[\Re\widetilde{Q}_{a,\lambda}(g;0)] = \frac{1}{2}\left(C_1(0) + \Re C_2(0)\right) + O(\ell_{\lambda,a,n}),$$

$$\lambda^d \mathrm{cov}\left[\Re\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1), \Re\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)\right] = \begin{cases} \frac{\Re}{2}C_1(\boldsymbol{\omega}_{\boldsymbol{r}}) + O(\ell_{\lambda,a,n}) & \boldsymbol{r}_1 = \boldsymbol{r}_2 (= \boldsymbol{r}) \\ \frac{\Re}{2}C_2(\boldsymbol{\omega}_{\boldsymbol{r}}) + O(\ell_{\lambda,a,n}) & \boldsymbol{r}_1 = -\boldsymbol{r}_2 (= \boldsymbol{r}) \\ O(\ell_{\lambda,a,n}) & \text{otherwise} \end{cases}$$

$$\lambda^d \text{cov}\left[\Im\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1), \Im\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)\right] = \begin{cases} \frac{\Re}{2}C_1(\boldsymbol{\omega_r}) + O(\ell_{\lambda,a,n}) & \boldsymbol{r}_1 = \boldsymbol{r}_2 (= \boldsymbol{r}) \\ \frac{-\Re}{2}C_2(\boldsymbol{\omega_r}) + O(\ell_{\lambda,a,n})) & \boldsymbol{r}_1 = -\boldsymbol{r}_2 (= \boldsymbol{r}) \\ O(\ell_{\lambda,a,n}) & \text{otherwise} \end{cases}$$

and

$$\lambda^d \text{cov}\left[\Re\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1), \Im\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)\right] = \begin{cases} O(\ell_{\lambda,a,n}) & \boldsymbol{r}_1 \neq -\boldsymbol{r}_2 \\ \frac{\Im}{2}C_2(\boldsymbol{\omega_r}) + O(\ell_{\lambda,a,n}) & \boldsymbol{r}_1 = -\boldsymbol{r}_2 (= \boldsymbol{r}) \end{cases}$$

Similar expressions for the covariances of $\Re Q_{a,\lambda}(g;\boldsymbol{r})$ and $\Im Q_{a,\lambda}(g;\boldsymbol{r})$ can also be derived.

Finally, asymptotic normality of $\Re\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$ and $\Im\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})$ follows from Lemma E.5. Thus giving (E.29). □

PROOF of Theorem 4.4 The proof is identical to the proof Lemma E.5, we omit the details. □

PROOF of Theorem 4.5 The proof is similar to the proof of Theorem E.1, we omit the details.

F Technical Lemmas

We first prove Lemma A.1, then state four lemmas, which form an important component in the proofs of this paper. Through out this section we use C to denote a finite generic constant. It is worth mentioning that many of these results build on the work of T. Kawata (see (Kawata, 1959)).

PROOF of Lemma A.1 We first prove (A.2). By using partial fractions and the definition of the sinc function we have

$$\begin{aligned} \int_{-\infty}^{\infty} \text{sinc}(u)\text{sinc}(u+x)du &= \frac{1}{x}\int_{-\infty}^{\infty} \sin(u)\sin(u+x)\left(\frac{1}{u} - \frac{1}{u+x}\right)du \\ &= \frac{1}{x}\int_{-\infty}^{\infty} \frac{\sin(u)\sin(u+x)}{u}du - \frac{1}{x}\int_{-\infty}^{\infty} \frac{\sin(u)\sin(u+x)}{u+x}du. \end{aligned}$$

For the second integral we make a change of variables $u' = u + x$, this gives

$$\begin{aligned} \int_{-\infty}^{\infty} \text{sinc}(u)\text{sinc}(u+x)du &= \frac{1}{x}\int_{-\infty}^{\infty} \frac{\sin(u)\sin(u+x)}{u}du - \frac{1}{x}\int_{-\infty}^{\infty} \frac{\sin(u')\sin(u'-x)}{u'}du' \\ &= \frac{1}{x}\int_{-\infty}^{\infty} \frac{\sin(u)}{u}\Big(\sin(u+x) - \sin(u-x)\Big)du \\ &= \frac{2\sin(x)}{x}\int_{-\infty}^{\infty} \frac{\cos(u)\sin(u)}{u}du = \frac{\pi\sin(x)}{x}. \end{aligned}$$

To prove (A.3), it is clear that for $x = s\pi$ (with $s \in \mathbb{Z}/\{0\}$) $\frac{\pi\sin(s\pi)}{s\pi} = 0$, which gives the result. □

The following result is used to obtain bounds for the variance and higher order cumulants.

Lemma F.1 *Define the function* $\ell_p(x) = C/e$ *for* $|x| \leq e$ *and* $\ell_p(x) = C\log^p |x|/|x|$ *for* $|x| \geq e$.

(i) We have

$$\int_{-\infty}^{\infty} \frac{|\sin(x)\sin(x+y)|}{|x(x+y)|}dx \leq \begin{cases} C\frac{\log|y|}{|y|} & |y| \geq e \\ C & |y| < e \end{cases} = \ell_1(y), \tag{F.1}$$

$$\int_{-\infty}^{\infty} |\text{sinc}(x)|\ell_p(x+y)dx \leq \ell_{p+1}(y) \tag{F.2}$$

and

$$\int_{\mathbb{R}^p} \left| \text{sinc}\left(\sum_{j=1}^{p} x_j\right) \prod_{j=1}^{p} \text{sinc}(x_j) \right| dx_1 \ldots dx_p \leq C, \tag{F.3}$$

(ii)

$$\sum_{m=-a}^{a} \int_{-\infty}^{\infty} \left| \frac{\sin^2(x)}{x(x+m\pi)} \right| dx \leq C\log^2 a$$

(iii)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{m=-a}^{a} \frac{\sin^2(x)}{|x(x+m\pi)|}\frac{\sin^2(y)}{|y(y+m\pi)|}dxdy \leq C, \tag{F.4}$$

(iv)

$$\sum_{m_1,\ldots,m_{q-1}=-a}^{a} \int_{\mathbb{R}^q} \left| \prod_{j=1}^{q-1} \text{sinc}(x_j)\text{sinc}(x_j+m_j\pi) \times \right.$$
$$\left. \text{sinc}(x_q)\text{sinc}(x_q + \pi\sum_{j=1}^{q-1} m_j) \right| \prod_{j=1}^{q} dx_j \leq C\log^{2(q-2)}(a),$$

where C *is a finite generic constant which is independent of* a.

PROOF. We first prove (i), equation (F.1). It is clear that for $|y| \leq e$ that $\int_{-\infty}^{\infty} \frac{|\sin(x)\sin(x+y)|}{|x(x+y)|}dx \leq C$. Therefore we now consider the case $|y| > e$, without loss of generality we prove the result for $y > e$. Partitioning the integral we have

$$\int_{-\infty}^{\infty} \frac{|\sin(x)\sin(x+y)|}{|x(x+y)|}dx = I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$
\begin{aligned}
I_1 &= \int_0^y \frac{|\sin(x)\sin(x+y)|}{|x(x+y)|}dx \qquad I_2 = \int_{-y}^0 \frac{|\sin(x)\sin(x+y)|}{|x(x+y)|}dx \\
I_3 &= \int_{-2y}^{-y} \frac{|\sin(x)\sin(x+y)|}{|x(x+y)|}dx \qquad I_4 = \int_{-\infty}^{-2y} \frac{|\sin(x)\sin(x+y)|}{|x(x+y)|}dx \\
I_5 &= \int_y^\infty \frac{|\sin(x)\sin(x+y)|}{|x(x+y)|}dx.
\end{aligned}
$$

To bound I_1 we note that for $y > 1$ and $x > 0$ that $|\sin(x+y)/(x+y)| \leq 1/y$, thus

$$
I_1 = \frac{1}{y}\int_0^y \frac{|\sin(x)|}{|x|}dx \leq C\frac{\log y}{y}.
$$

To bound I_2, we further partition the integral

$$
\begin{aligned}
I_2 &= \int_{-y}^{-y/2} \frac{|\sin(x)\sin(x+y)|}{|x(x+y)|}dx + \int_{-y/2}^0 \frac{|\sin(x)\sin(x+y)|}{|x(x+y)|}dx \\
&\leq \frac{2}{y}\int_{-y}^{-y/2} \frac{|\sin(x+y)|}{|(x+y)|}dx + \frac{2}{y}\int_{-y/2}^0 \frac{|\sin(x)|}{|x|}dx \leq C\frac{\log y}{y}.
\end{aligned}
$$

To bound I_3, we use the bound

$$
I_3 \leq \frac{1}{y}\int_{-2y}^{-y} \frac{|\sin(x+y)|}{|(x+y)|}dx \leq C\frac{\log y}{y}.
$$

To bound I_4 we use that for $y > 0$, $\int_y^\infty x^{-2}dx \leq C|y|^{-1}$, thus

$$
I_4 \leq \int_{-\infty}^{-y} \frac{1}{x^2}dx \leq C|y|^{-1}
$$

and using a similar argument we have $I_5 \leq C|y|^{-1}$. Altogether, this gives (F.1).

We now prove (F.2). It is clear that for $|y| \leq e$ that $\int_{-\infty}^\infty |\text{sinc}(x)|\ell_p(x+y)|dx \leq C$. Therefore we now consider the case $|y| > e$, without loss of generality we prove the result for $y > e$. As in (F.1) we partition the integral

$$
\int_{-\infty}^\infty |\text{sinc}(x)\ell_p(x+y)|dx = II_1 + \ldots + II_5,
$$

where $II_1, \ldots, II_5$ are defined in the same way as $I_1, \ldots, I_5$ just with $|\text{sinc}(x)\ell_p(x+y)|$ replacing $\frac{|\sin(x)\sin(x+y)|}{|x(x+y)|}$. To bound II_1 we note that

$$
II_1 = \int_0^y |\text{sinc}(x)\ell_p(x+y)|dx \leq \frac{\log^p(y)}{y}\int_0^y |\text{sinc}(x)|dx \leq \frac{\log^{p+1}(y)}{y},
$$

we use similar method to show $II_2 \leq C\frac{\log^{p+1}(y)}{y}$ and $II_3 \leq C\frac{\log^{p+1}(y)}{y}$. Finally to bound II_4 and II_5 we note that by using a change of variables $x = yz$, we have

$$\begin{aligned} II_5 &= \int_y^\infty \frac{|\sin(x)|\log^p(x+y)}{x(x+y)}dx \leq \int_y^\infty \frac{\log^p(x+y)}{x(x+y)}dx \\ &= \frac{1}{y}\int_1^\infty \frac{[\log(y)+\log(z+1)]^p}{z(z+1)}dz \leq C\frac{\log^p(y)}{y}. \end{aligned}$$

Similarly we can show that $II_4 \leq C\frac{\log^p(y)}{y}$. Altogether, this gives the result.

To prove (F.3) we recursively apply (F.2) to give

$$\begin{aligned} &\int_{\mathbb{R}^p} |\text{sinc}(x_1+\ldots+x_p)|\prod_{j=1}^{p}|\text{sinc}(x_j)|dx_1\ldots dx_p \\ \leq &\int_{\mathbb{R}^{p-1}} |\ell_1(x_1+\ldots+x_{p-1})\prod_{j=1}^{p-1}\text{sinc}(x_j)|dx_1\ldots dx_{p-1} \\ \leq &\int_{\mathbb{R}} |\ell_{p-1}(x_1)\text{sinc}(x_1)|dx_1 = O(1), \end{aligned}$$

thus we have the required the result.

To bound (ii), without loss of generality we derive a bound over $\sum_{m=1}^{a}$, the bounds for $\sum_{m=-a}^{-1}$ are identical. Using (F.1) we have

$$\begin{aligned} &\sum_{m=1}^{a}\int_{-\infty}^{\infty}\frac{\sin^2(x)}{|x(x+m\pi)|}dx = \sum_{m=1}^{a}\int_{-\infty}^{\infty}\frac{|\sin(x)\sin(x+m\pi)|}{|x(x+m\pi)|}dx \\ \leq &\sum_{m=1}^{a}\ell_1(m\pi) = C\sum_{m=1}^{a}\frac{\log(m\pi)}{m\pi} \\ \leq &C\log(a\pi)\sum_{m=1}^{a}\frac{1}{m\pi} = C\log(a\pi)\log(a) \leq C\log^2 a. \end{aligned}$$

Thus we have shown (ii).

To prove (iii) we use (F.1) to give

$$\begin{aligned} &\sum_{m=-a}^{a}\left(\int_{-\infty}^{\infty}\frac{\sin^2(x)}{|x(x+m\pi)|}dx\right)\left(\int_{-\infty}^{\infty}\frac{\sin^2(y)}{|y(y+m\pi)|}dy\right) \\ \leq &C\sum_{m=-a}^{a}(\frac{\log m}{m})^2 \leq C\sum_{m=-\infty}^{\infty}(\frac{\log m}{m})^2 \leq C. \end{aligned}$$

To prove (iv) we apply (F.1) to each of the integrals this gives

$$\begin{aligned} &\sum_{m_1,\ldots,m_{q-1}=-a}^{a}\int_{\mathbb{R}^q}\left|\prod_{j=1}^{q-1}\text{sinc}(x_j)\text{sinc}(x_j+m_j\pi)\text{sinc}(x_q)\text{sinc}(x_q+\pi\sum_{j=1}^{q-1}m_j)\right|\prod_{j=1}^{q}dx_j \\ \leq &\sum_{m_1,\ldots,m_{q-1}=-a}^{a}\ell_1(m_1\pi)\ldots\ell_1(m_{q-1}\pi)\ell_1(m_{q-1}\pi)\ell_1(\pi\sum_{j=1}^{q-1}m_j) \leq C\log^{2(q-2)}a, \end{aligned}$$

thus we obtain the desired result. □.

The proofs of Theorem 4.1, Theorems B.1 (ii), Theorem 2.1 involve integrals of $\text{sinc}(u)\text{sinc}(u+m\pi)$, where $|m| \leq a + |r_1| + |r_2|$ and that $a \to \infty$ as $\lambda \to \infty$. In the following lemma we obtain bounds for these integrals. We note that all these results involve an inner integral difference of the form

$$\int_{-b}^{b} g(\omega)\left[h\left(\omega + \frac{2u}{\lambda}\right) - h(\omega)\right] d\omega du \Big|.$$

By using the mean value theorem heuristically it is clear that $O(\lambda^{-1})$ should come out of the integral however the $2u$ in the numerator of $h\left(\omega + \frac{2u}{\lambda}\right)$ makes the analysis quite involved.

Lemma F.2 *Suppose h is a function which is absolutely integrable and $|h'(\omega)| \leq \beta(\omega)$ (where β is a montonically decreasing function that is absolutely integrable), $m \in \mathbb{R}$ and $g(\omega)$ is a bounded function. b can take any value, and the bounds given below are independent of b. Then we have*

$$\begin{aligned} &\Big|\int_{-\infty}^{\infty} \text{sinc}(u)\text{sinc}(u+m\pi) \int_{-b}^{b} g(\omega)\left[h\left(\omega + \frac{2u}{\lambda}\right) - h(\omega)\right] d\omega du\Big| \\ \leq\ & C\frac{\log(\lambda) + \log(1+|m|)}{\lambda}, \end{aligned} \tag{F.5}$$

where C is a finite constant independent of m and b. If $g(\omega)$ is a bounded function with a bounded first derivative, then we have

$$\begin{aligned} &\Big|\int_{-\infty}^{\infty} \text{sinc}(u)\text{sinc}(u+m\pi) \int_{-b}^{b} h(\omega)\Big(g(\omega + \frac{2u}{\lambda}) - g(\omega)\Big) d\omega du\Big| \\ \leq\ & C\frac{\log(\lambda) + \log(1+|m|)}{\lambda}. \end{aligned} \tag{F.6}$$

In the case of double integrals, we assume $h(\cdot,\cdot)$ is such that $\int_{\mathbb{R}^2} |h(\omega_1,\omega_2)| d\omega_1 d\omega_2 < \infty$, and $|\frac{\partial h(\omega_1,\omega_2)}{\partial \omega_1}| \leq \beta(\omega_1)\beta(\omega_2)$, $|\frac{\partial h(\omega_1,\omega_2)}{\partial \omega_2}| \leq \beta(\omega_1)\beta(\omega_2)$ and $|\frac{\partial^2 h(\omega_1,\omega_2)}{\partial \omega_1 \partial \omega_2}| \leq \beta(\omega_1)\beta(\omega_2)$ (where β is a montonically decreasing function that is absolutely integrable) and $g(\cdot,\cdot)$ is a bounded function. Then if $m_1 \neq 0$ and $m_2 \neq 0$ and $m_1, m_2 \in \mathbb{Z}$ we have

$$\begin{aligned} &\Big|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{sinc}(u_1)\text{sinc}(u_1+m_1\pi)\text{sinc}(u_2)\text{sinc}(u_2+m_2\pi) \int_{-b}^{b}\int_{-b}^{b} g(\omega_1,\omega_2) \times \\ &\left[h\left(\omega_1 + \frac{2u_1}{\lambda}, \omega_2 + \frac{2u_2}{\lambda}\right) - h(\omega_1,\omega_2)\right] d\omega_1 d\omega_2 du_1 du_2\Big| \leq C\frac{\prod_{i=1}^{2}[\log(\lambda) + \log|m_i|]}{\lambda^2}. \end{aligned} \tag{F.7}$$

and

$$\Big| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{sinc}(u_1)\mathrm{sinc}(u_1+m_1\pi)\mathrm{sinc}(u_2)\mathrm{sinc}(u_2+m_2\pi)\frac{1}{\lambda^2}\sum_{-b\lambda}^{b\lambda}\sum_{-b\lambda}^{b\lambda} g(\omega_{k_1},\omega_{k_2}) \times$$

$$\left[h\left(\omega_{k_1}+\frac{2u_1}{\lambda},\omega_{k_2}+\frac{2u_2}{\lambda}\right) - h(\omega_{k_1},\omega_{k_2})\right]du_1du_2\Big| \leq C\frac{\prod_{i=1}^{2}[\log(\lambda)+\log|m_i|]}{\lambda^2}. \quad \text{(F.8)}$$

where $a \to \infty$ *as* $\lambda \to \infty$.

PROOF. To simplify the notation in the proof, we'll prove (F.5) for $m > 0$ (the proof for $m \leq 0$ is identical).

The proof is based on considering the cases that $|u| \leq \lambda$ and $|u| > \lambda$ separately. For $|u| \leq \lambda$ we apply the mean value theorem to the difference $h(\omega + \frac{2u}{\lambda}) - h(\omega)$ and for $|u| > \lambda$ we exploit that the integral $\int_{u>|\lambda|} |\mathrm{sinc}(u)\mathrm{sinc}(u+m\pi)|du$ decays as $\lambda \to \infty$. We now make these argument precise. We start by partitioning the integral

$$\int_{-\infty}^{\infty} \mathrm{sinc}(u)\mathrm{sinc}(u+m\pi)\int_{-b}^{b} g(\omega)\Big(h(\omega+\frac{2u}{\lambda}) - h(\omega)\Big)d\omega du = I_1 + I_2, \quad \text{(F.9)}$$

where

$$\begin{aligned}
I_1 &= \int_{|u|>\lambda} \mathrm{sinc}(u)\mathrm{sinc}(u+m\pi)\int_{-b}^{b} g(\omega)\Big(h(\omega+\frac{2u}{\lambda}) - h(\omega)\Big)d\omega du \\
I_2 &= \int_{-\lambda}^{\lambda} \mathrm{sinc}(u)\mathrm{sinc}(u+m\pi)\int_{-b}^{b} g(\omega)\Big(h(\omega+\frac{2u}{\lambda}) - h(\omega)\Big)d\omega du.
\end{aligned}$$

We further partition the integral $I_1 = I_{11} + I_{12} + I_{13}$, where

$$\begin{aligned}
I_{11} &= \int_{\lambda}^{\infty} \mathrm{sinc}(u)\mathrm{sinc}(u+m\pi)\int_{-b}^{b} g(\omega)\Big(h(\omega+\frac{2u}{\lambda}) - h(\omega)\Big)d\omega du \\
I_{12} &= \int_{-\lambda-m\pi}^{-\lambda} \mathrm{sinc}(u)\mathrm{sinc}(u+m\pi)\int_{-b}^{b} g(\omega)\Big(h(\omega+\frac{2u}{\lambda}) - h(\omega)\Big)d\omega du \\
I_{13} &= \int_{-\infty}^{-\lambda-m\pi} \mathrm{sinc}(u)\mathrm{sinc}(u+m\pi)\int_{-b}^{b} g(\omega)\Big(h(\omega+\frac{2u}{\lambda}) - h(\omega)\Big)d\omega du
\end{aligned}$$

and partition $I_2 = I_{21} + I_{22} + I_{23}$, where

$$\begin{aligned}
I_{21} &= \int_{-\lambda}^{\lambda} \mathrm{sinc}(u)\mathrm{sinc}(u+m\pi)\int_{-\min(4|u|/\lambda,b)}^{\min(4|u|/\lambda,b)} g(\omega)\Big(h(\omega+\frac{2u}{\lambda}) - h(\omega)\Big)d\omega du \\
I_{22} &= \int_{-\lambda}^{\lambda} \mathrm{sinc}(u)\mathrm{sinc}(u+m\pi)\int_{-b}^{-\min(4|u|/\lambda,b)} g(\omega)\Big(h(\omega+\frac{2u}{\lambda}) - h(\omega)\Big)d\omega du \\
I_{23} &= \int_{-\lambda}^{\lambda} \mathrm{sinc}(u)\mathrm{sinc}(u+m\pi)\int_{\min(4|u|/\lambda,b)}^{b} g(\omega)\Big(h(\omega+\frac{2u}{\lambda}) - h(\omega)\Big)d\omega du.
\end{aligned}$$

We start by bounding I_1. Taking absolutes of I_{11}, and using that $h(\omega)$ is absolutely integrable we have

$$|I_{11}| \ \leq \ 2\Gamma \int_{\lambda}^{\infty} \frac{|\sin(u)\sin(u+\pi)|}{u(u+m\pi)} du,$$

where $\Gamma = \sup_u |g(u)| \int_0^\infty |h(u)| du$. Since $m > 0$, it is straightforward to show that $\int_{\lambda}^{\infty} \frac{\sin^2(u)}{u(u+m\pi)} du \leq C\lambda^{-1}$, where C is some finite constant. This implies $|I_{11}| \leq 2C\Gamma\lambda^{-1}$. Similarly it can be shown that $|I_{13}| \leq 2C\Gamma\lambda^{-1}$. To bound I_{12} we note that

$$\begin{aligned} |I_{12}| \ &\leq \ \frac{2\Gamma}{\lambda} \int_{-\lambda-m\pi}^{-\lambda} \frac{|\sin(u+m\pi)|}{|u+m\pi|} du = \frac{2\Gamma}{\lambda} \int_{-\lambda}^{-\lambda+m\pi} \frac{|\sin(y)|}{|y|} dy \\ &\leq \ \frac{2\Gamma}{\lambda} \times \begin{cases} \log \lambda & -\lambda + m\pi \in [-e, e] \\ \log\left(\frac{\lambda}{\lambda - m\pi}\right) & -\lambda + m\pi < -e \\ \log \lambda + \log(m\pi - \lambda) & -\lambda + m\pi > e \end{cases} . \end{aligned}$$

Thus, we have $|I_{12}| \leq C\lambda^{-1}\big[\log\lambda + \log(1+m)\big]$ (where C is a finite constant). Altogether, the bounds for I_{11}, I_{12}, I_{13} give

$$|I_1| \leq \frac{C(\log \lambda + \log(1+|m|))}{\lambda}.$$

To bound I_2 we apply the mean value theorem to $h(\omega + \frac{2u}{\lambda}) - h(\omega) = \frac{2u}{\lambda} h'(\omega + \zeta(\omega,u)\frac{2u}{\lambda})$, where $0 \leq \zeta(\omega,u)| \leq 1$. Substituting this into I_{23} gives

$$|I_{23}| \ \leq \ \frac{2}{\lambda} \int_{-\lambda}^{\lambda} \frac{|\sin(u)\sin(u+m\pi)|}{|u+m\pi|} \int_{\min(4|u|/\lambda, b)}^{b} \left| h'\left(\omega + \zeta(\omega,u)\frac{2u}{\lambda}\right)\right| d\omega du.$$

Since the limits of the inner integral are greater than $4u/\lambda$, and the derivative is bounded by $\beta(\omega)$, this means $|h'(\omega + \zeta(\omega,u)\frac{2u}{\lambda})| \leq \max[\beta(\omega), \beta(\omega + \frac{2u}{\lambda})] = \beta(\omega)$. Altogether, this gives

$$|I_{23}| \ \leq \ \frac{2}{\lambda}\left(\int_{\min(4|u|/\lambda, b)}^{b} \beta(\omega) d\omega\right) \int_{-\lambda}^{\lambda} \frac{|\sin(u)\sin(u+m\pi)|}{|u+m\pi|} du \leq \frac{2\Gamma\log(\lambda+m\pi)}{\lambda}.$$

Using the same method we obtain $I_{22} \leq \frac{2\Gamma\log(\lambda+m\pi)}{\lambda}$. Finally, to bound I_{21}, we cannot bound $h'(\omega + \zeta(\omega,u)\frac{2u}{\lambda})$ by a monotonic function since ω and $\omega + \frac{2u}{\lambda}$ can have different signs. Therefore we simply bound $h'(\omega + \zeta(\omega,u)\frac{2u}{\lambda})$ with a constant, this gives

$$|I_{21}| \leq \frac{8C}{\lambda^2} \int_{-\lambda}^{\lambda} \frac{|u\sin(u)\sin(u+m\pi)|}{|u+m\pi|} du \leq \frac{16C}{\lambda}.$$

Altogether, the bounds for I_{21}, I_{22}, I_{23} give

$$|I_2| \leq C\Gamma \frac{\log\lambda + \log(1+m) + \log(\lambda + m\pi)}{\lambda}.$$

Finally, we recall that if $\lambda > 2$ and $m\pi > 2$, then $(\lambda + m\pi) < \lambda m\pi$, thus $\log(\lambda + m\pi) \leq C \log \lambda + \log(1+m)$. Therefore, we obtain (F.5). The proof of (F.6) is similar, but avoids some of the awkward details that are required to prove (F.5).

Now we prove (F.7). We note that both $m_1 \neq 0$ and $m_2 \neq 0$ *and* $m_1, m_2 \in \mathbb{Z}$ (if one of these values were zero or non-integer valued only the slower bound given in (F.5) holds). Without loss of generality we will prove the result for $m_1 > 0$ and $m_2 > 0$. We first note since $m_1 \neq 0$ and $m_2 \neq 0$, by orthogonality of the sinc function at integer shifts (see Lemma A.1) we have

$$\begin{aligned}
& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{sinc}(u_1)\text{sinc}(u_1+m_1\pi)\text{sinc}(u_2)\text{sinc}(u_2+m_2\pi) \int_{-b}^{b}\int_{-b}^{b} g(\omega_1,\omega_2) \times \\
& \left[h\left(\omega_1+\frac{2u_1}{\lambda}, \omega_2+\frac{2u_2}{\lambda}\right) - h(\omega_1,\omega_2)\right] d\omega_1 d\omega_2 du_1 du_2 \\
= & \int_{-\infty}^{\infty} \text{sinc}(u_2)\text{sinc}(u_2+m_2\pi) \int_{-\infty}^{\infty} \text{sinc}(u_1)\text{sinc}(u_1+m_1\pi) \int_{-b}^{b}\int_{-b}^{b} g(\omega_1,\omega_2) \\
& \times h\left(\omega_1+\frac{2u_1}{\lambda}, \omega_2+\frac{2u_2}{\lambda}\right) d\omega_1 du_1 d\omega_2 du_2,
\end{aligned} \tag{F.10}$$

since in the last line $h(\omega_1,\omega_2)$ comes outside the integral over u_1 and u_2. To further the proof, we note that for $m_2 \neq 0$ we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} \text{sinc}(u_2)\text{sinc}(u_2+m_2\pi) \int_{|u_1|>\lambda} \text{sinc}(u_1)\text{sinc}(u_1+m_1\pi) \int_{-b}^{b}\int_{-b}^{b} g(\omega_1,\omega_2) \\
& \times h\left(\omega_1+\frac{2u_1}{\lambda}, \omega_2\right) d\omega_1 du_1 d\omega_2 du_2 = 0.
\end{aligned} \tag{F.11}$$

We use this zero-equality at relevant parts in the proof.

Now we subtract (F.11) from (F.10). We use the same decomposition and notation used in (F.9) to decompose the integral over u_1 into $\int_{|u_1|>\lambda} + \int_{|u_1|\leq\lambda}$, to give

$$\begin{aligned}
= & \int_{-\infty}^{\infty} \text{sinc}(u_2)\text{sinc}(u_2+m_2\pi)\Bigg(\int_{-\infty}^{\infty} \text{sinc}(u_1)\text{sinc}(u_1+m_1\pi) \int_{-b}^{b}\int_{-b}^{b} g(\omega_1,\omega_2) \times \\
& \left[h\left(\omega_1+\frac{2u_1}{\lambda}, \omega_2+\frac{2u_2}{\lambda}\right) - h\left(\omega_1+\frac{2u_1}{\lambda}, \omega_2\right)\right] d\omega_1 du_1\Bigg) d\omega_2 du_2 \\
= & \int_{-\infty}^{\infty} \text{sinc}(u_2)\text{sinc}(u_2+m_2\pi) \int_{-b}^{b}\Bigg(\int_{-\infty}^{\infty} \text{sinc}(u_1)\text{sinc}(u_1+m_1\pi) \int_{-b}^{b} g(\omega_1,\omega_2) \times \\
& \left[h\left(\omega_1+\frac{2u_1}{\lambda}, \omega_2+\frac{2u_2}{\lambda}\right) - h\left(\omega_1+\frac{2u_1}{\lambda}, \omega_2\right)\right] d\omega_1 du_1\Bigg) d\omega_2 du_2 \\
= & \int_{-\infty}^{\infty} \text{sinc}(u_2)\text{sinc}(u_2+m_2\pi) \int_{-b}^{b} \left[I_1(u_2,\omega_2) + I_2(u_2,\omega_2)\right] du_2 d\omega_2 = J_1 + J_2,
\end{aligned}$$

where

$$
\begin{aligned}
I_1(u_2,\omega_2) &= \int_{|u_1|>\lambda} \mathrm{sinc}(u_1)\mathrm{sinc}(u_1+m_1\pi)\times \\
&\quad \int_{-b}^{b} g(\omega_1,\omega_2)\left[h\left(\omega_1+\frac{2u_1}{\lambda},\omega_2+\frac{2u_2}{\lambda}\right)-h\left(\omega_1+\frac{2u_1}{\lambda},\omega_2\right)\right]d\omega_1 du_1 \\
I_2(u_2,\omega_2) &= \int_{|u_1|\leq\lambda} \mathrm{sinc}(u_1)\mathrm{sinc}(u_1+m_1\pi)\int_{-b}^{b} g(\omega_1,\omega_2) \\
&\quad \times\frac{2u_2}{\lambda}\frac{\partial h}{\partial\omega_2}\left(\omega_1+\frac{2u_1}{\lambda},\omega_2+\zeta_1(\omega_2,u_2)\frac{2u_2}{\lambda}\right)d\omega_1 du_1
\end{aligned}
$$

and $|\zeta_1(\omega_1,u_1)|\leq 1$. Note that the expression for $I_2(u_2,\omega_2)$ applies the mean value theorem to the difference $h\left(\omega_1+\frac{2u_1}{\lambda},\omega_2+\frac{2u_2}{\lambda}\right)-h\left(\omega_1+\frac{2u_1}{\lambda},\omega_2\right)=\frac{2u_2}{\lambda}\frac{\partial h}{\partial\omega_2}\left(\omega_1+\frac{2u_1}{\lambda},\omega_2+\zeta_1(\omega_2,u_2)\frac{2u_2}{\lambda}\right)$; there is a slight abuse of notation since the derivative is applied to h before evaluating it at $(\omega_1+\frac{2u_1}{\lambda},\omega_2+\zeta_1(\omega_2,u_2)\frac{2u_2}{\lambda})$. Next to bound J_1 we decompose the outer integral over u_2 into $\int_{|u_2|>\lambda}$ and $\int_{|u_2|\leq\lambda}$ to give $J_1=J_{11}+J_{12}$, where

$$
\begin{aligned}
& J_{11} \\
=\; & \int_{|u_1|>\lambda}\int_{|u_2|>\lambda} \mathrm{sinc}(u_1)\mathrm{sinc}(u_1+m_1\pi)\mathrm{sinc}(u_2)\mathrm{sinc}(u_2+m_2\pi)\times \\
& \int_{-b}^{b}\int_{-b}^{b} g(\omega_1,\omega_2)\left[h\left(\omega_1+\frac{2u_1}{\lambda},\omega_2+\frac{2u_2}{\lambda}\right)-h\left(\omega_1+\frac{2u_1}{\lambda},\omega_2\right)\right]d\omega_1 du_1 du_2 d\omega_2
\end{aligned}
$$

$$
\begin{aligned}
& J_{12} \\
=\; & \int_{|u_1|>\lambda}\int_{-b}^{b} \mathrm{sinc}(u_1)\mathrm{sinc}(u_1+m_1\pi)\int_{|u_2|\leq\lambda}\mathrm{sinc}(u_2)\mathrm{sinc}(u_2+m_2\pi)\times \\
& \int_{-b}^{b} g(\omega_1,\omega_2)\frac{2u_2}{\lambda}\frac{\partial h}{\partial\omega_2}\left(\omega_1+\frac{2u_1}{\lambda},\omega_2+\zeta_2(\omega_2,u_2)\frac{2u_2}{\lambda}\right)d\omega_1 du_1 du_2 d\omega_2,
\end{aligned}
$$

and $|\zeta_2(\omega_2,u_2)|<1$ (applying the mean value theorem to $h\left(\omega_1+\frac{2u_1}{\lambda},\omega_2+\frac{2u_2}{\lambda}\right)-h\left(\omega_1+\frac{2u_1}{\lambda},\omega_2\right)$). By using the same methods used to bound I_1 and I_2 in (F.9) we can show that

$$
|J_{11}| \text{ and } |J_{12}| = O\left(\frac{(\log(\lambda)+\log|m_1|)(\log(\lambda)+\log|m_2|)}{\lambda^2}\right).
$$

To bound J_2 we again use that $m_2\neq 0$ and orthogonality of the sinc function to subtract the term $\frac{\partial}{\partial\omega_2}h\left(\omega_1,\omega_2+\zeta_1(\omega_2,u_2)\frac{2u_2}{\lambda}\right)$ (whose total contribution is zero, see (F.11)) from J_2 to give

$$
\begin{aligned}
J_2 &= \int_{-\infty}^{\infty}\int_{-b}^{b}\mathrm{sinc}(u_2)\mathrm{sinc}(u_2+m_2\pi)\int_{|u_1|\leq\lambda}\mathrm{sinc}(u_1)\mathrm{sinc}(u_1+m_1\pi)\int_{-b}^{b} g(\omega_1,\omega_2)\times \\
&\quad \frac{2u_2}{\lambda}\left[\frac{\partial h}{\partial\omega_2}\left(\omega_1+\frac{2u_1}{\lambda},\omega_2+\zeta_1(\omega_2,u_2)\frac{2u_2}{\lambda}\right)-\frac{\partial h}{\partial\omega_2}\left(\omega_1,\omega_2+\zeta_1(\omega_2,u_2)\frac{2u_2}{\lambda}\right)\right]d\omega_1 d\omega_2 du_1 du_2.
\end{aligned}
$$

By decomposing the outer integral of J_2 over u_2 into $\int_{|u_2|>\lambda}$ and $\int_{|u_2|\leq\lambda}$ we have $J_2 = J_{21}+J_{22}$, where

$$\begin{aligned} J_{21} &= \int_{|u_2|>\lambda}\int_{-b}^{b} \mathrm{sinc}(u_2)\mathrm{sinc}(u_2+m_2\pi)\int_{|u_1|\leq\lambda} \mathrm{sinc}(u_1)\mathrm{sinc}(u_1+m_1\pi)\int_{-b}^{b} g(\omega_1,\omega_2)\times \\ & \frac{2u_2}{\lambda}\left[\frac{\partial h}{\partial\omega_2}\left(\omega_1+\frac{2u_1}{\lambda},\omega_2+\zeta_1(\omega_2,u_2)\frac{2u_2}{\lambda}\right)-\frac{\partial h}{\partial\omega_2}\left(\omega_1,\omega_2+\zeta_1(\omega_2,u_2)\frac{2u_2}{\lambda}\right)\right]d\omega_1 d\omega_2 du_1 du_2 \end{aligned}$$

and

$$\begin{aligned} J_{22} &= \int_{|u_2|\leq\lambda}\int_{-b}^{b} \mathrm{sinc}(u_2)\mathrm{sinc}(u_2+m_2\pi)\int_{|u_1|\leq\lambda} \mathrm{sinc}(u_1)\mathrm{sinc}(u_1+m_1\pi)\times \\ & \int_{-b}^{b} g(\omega_1,\omega_2)\frac{4u_1u_2}{\lambda^2}\frac{\partial^2 h}{\partial\omega_1\partial\omega_2}\left(\omega_1+\zeta_3(\omega_1,u_1)\frac{2u_1}{\lambda},\omega_2+\zeta_1(\omega_2,u_2)\frac{2u_2}{\lambda}\right)d\omega_1 d\omega_2 du_1 du_2 \end{aligned}$$

with $|\zeta_3(\omega_2,u_2)|<1$. Note that the expression for J_{22} is obtained by applying the mean value theorem to $\frac{\partial h}{\partial\omega_2}\left(\omega_1+\frac{2u_1}{\lambda},\omega_2+\zeta_1(\omega_2,u_2)\frac{2u_2}{\lambda}\right)-\frac{\partial h}{\partial\omega_2}\left(\omega_1,\omega_2+\zeta_1(\omega_2,u_2)\frac{2u_2}{\lambda}\right)$.

Again by using the methods used to bound I_1 and I_2 in (F.9) we can show that $|J_{21}| = O(\prod_{i=1}^{2}[\log(\lambda)+\log|m_i|]/\lambda^2)$ and $|J_{22}| = O(\prod_{i=1}^{2}[\log(\lambda)+\log|m_i|]/\lambda^2)$. Altogether this proves (F.7).

The proof of (F.8) follows exactly the same method used to prove (F.7) the only difference is that the summand rather than the integral makes the notation more cumbersome. For this reason we omit the details. □

The following result is used in to obtain expression for the fourth order cumulant term (in the case that the spatial random field is not Gaussian). It is used in the proofs of Theorems 4.6 and B.2.

Lemma F.3 *Suppose h is a function which is absolutely integrable and $|h'(\omega)|\leq\beta(\omega)$ (where β is a monotonically decreasing function that is absolutely integrable), $m\in\mathbb{Z}$ and $g(\omega)$ is a bounded function. Then we have*

$$\begin{aligned} &\int_{\mathbb{R}^3} \mathrm{sinc}(u_1+u_2+u_3+m\pi)\mathrm{sinc}(u_1)\mathrm{sinc}(u_2)\mathrm{sinc}(u_3)\times \\ &\int_{-a/\lambda}^{a/\lambda} g(\omega)\left[h\left(\frac{2u_1}{\lambda}-\omega\right)-h(-\omega)\right]d\omega du_1 du_2 du_3 = O\left(\frac{[\log(\lambda)+\log|m|]^3}{\lambda}\right). \end{aligned} \tag{F.12}$$

$$\begin{aligned} &\int_{\mathbb{R}^2} \mathrm{sinc}(u_1+u_2+m\pi)\mathrm{sinc}(u_1)\mathrm{sinc}(u_2)\int_{-a/\lambda}^{a/\lambda} g(\omega)\left[h\left(\frac{2u_1}{\lambda}-\omega\right)-h(-\omega)\right]d\omega du_1 du_2 \\ &= O\left(\frac{[\log(\lambda)+\log|m|]^3}{\lambda}\right). \end{aligned} \tag{F.13}$$

PROOF. The proof of (F.12) is very similar to the proof of Lemma F.2. We start by partitioning the integral over u_1 into $\int_{|u_1|\leq\lambda}$ and $\int_{|u_1|>\lambda}$

$$\int_{\mathbb{R}^3} \text{sinc}(u_1+u_2+u_3+m\pi)\text{sinc}(u_1)\text{sinc}(u_2)\text{sinc}(u_3) \times$$
$$\int_{-a/\lambda}^{a/\lambda} g(\omega)\left(h(\frac{2u_1}{\lambda}-\omega)-h(-\omega)\right)d\omega du_1 du_2 du_3 = I_1 + I_2,$$

where

$$I_1 =$$
$$\int_{|u_1|>\lambda}\int_{\mathbb{R}^3} \text{sinc}(u_1+u_2+u_3+m\pi)\text{sinc}(u_1)\text{sinc}(u_2)\text{sinc}(u_3)$$
$$\times \int_{-a/\lambda}^{a/\lambda} g(\omega)\left(h(\frac{2u_1}{\lambda}-\omega)-h(-\omega)\right)d\omega du_1 du_2 du_3$$
$$I_2 =$$
$$\int_{|u_1|\leq\lambda}\int_{\mathbb{R}^3} \text{sinc}(u_1+u_2+u_3+m\pi)\text{sinc}(u_1)\text{sinc}(u_2)\text{sinc}(u_3)$$
$$\times \int_{-a/\lambda}^{a/\lambda} g(\omega)\left(h(\frac{2u_1}{\lambda}-\omega)-h(-\omega)\right)d\omega du_1 du_2 du_3.$$

Taking absolutes of I_1 and using Lemma F.1, equations (F.1) and (F.2) we have

$$|I_1| \leq 4\Gamma\int_{|u_1|>\lambda}|\text{sinc}(u_1+m\pi)|\ell_2(u_1)du_1 \leq C\int_{|u|>\lambda}\frac{\log^2(u)}{|u|}\times\frac{|\text{sinc}(u+m\pi)|}{|u+m\pi|}du,$$

where $\Gamma = \sup_\omega|g(\omega)|\int_0^\infty|h(\omega)|d\omega$ and C is a finite constant (which has absorbed Γ). Decomposing the above integral we have

$$|I_1| \leq C\int_{|u|>\lambda}\frac{\log^2(u)}{|u|}\times\frac{|\text{sinc}(u+m\pi)|}{|u+m\pi|}du = I_{11}+I_{12},$$

where

$$I_{11} = \int_{\lambda<|u|\leq\lambda(1+|m|)}\frac{\log^2(u)}{|u|}\times\frac{|\text{sinc}(u_1+m\pi)|}{|u_1+m\pi|}du$$
$$I_{12} = \int_{|u|>\lambda(1+|m|)}\frac{\log^2(u)}{|u|}\times\frac{|\text{sinc}(u_1+m\pi)|}{|u_1+m\pi|}du.$$

We first bound I_{11}

$$I_{11} \leq \frac{\log^2[\lambda(1+|m|\pi)]}{\lambda}\int_{\lambda<|u|\leq\lambda(1+|m|)}\frac{|\text{sinc}(u_1+m\pi)|}{|u_1+m\pi|}du$$
$$\leq \frac{2C\log^2[\lambda(1+|m|\pi)]}{\lambda}\times\log(\lambda+m\pi) = C\frac{(\log|m|+\log\lambda)^3}{\lambda}$$

To bound I_{12} we make a change of variables $u = \lambda z$, the above becomes

$$I_{12} \leq \frac{C}{\lambda} \int_{|z|>1+|m|\lambda} \frac{[\log \lambda + \log z]^2}{z(z+\frac{m}{\lambda})} dz = O\left(\frac{\log^2(\lambda)}{\lambda}\right).$$

Altogether the bounds for I_{11} and I_{12} give $|I_1| \leq C(\log |m| + \log \lambda)^3/\lambda$.

To bound I_2, just as in Lemma F.2, equation (F.5), we decompose it into three parts $I_2 = I_{21} + I_{22} + I_{23}$, where using Lemma F.1, equations (F.1) and (F.2) we have the bounds

$$|I_{21}| \leq \int_{|u|\leq\lambda} |\text{sinc}(u+m\pi)|\ell_2(u) \int_{-\min(a,4|u|)/\lambda}^{\min(a,4u)/\lambda} |g(\omega)| \left| h\left(\frac{2u}{\lambda} - \omega\right) - h(-\omega)\right| d\omega du$$

$$|I_{22}| \leq \int_{|u|\leq\lambda} |\text{sinc}(u+m\pi)|\ell_2(u) \int_{-a/\lambda}^{-\min(a,4|u|)/\lambda} |g(\omega)| \left| h\left(\frac{2u}{\lambda} - \omega\right) - h(-\omega)\right| d\omega du$$

$$|I_{23}| \leq \int_{|u|\leq\lambda} |\text{sinc}(u+m\pi)|\ell_2(u) \int_{\min(a,4|u|)/\lambda}^{a/\lambda} |g(\omega)| \left| h\left(\frac{2u}{\lambda} - \omega\right) - h(-\omega)\right| d\omega du.$$

Using the same method used to bound $|I_{21}|, |I_{22}|, |I_{23}|$ in Lemma F.2, we have $|I_{21}|, |I_{22}|, |I_{23}| \leq C[\log(\lambda) + \log(|m|)]^3/\lambda$. Having bounded all partitions of the integral, we have the result.

The proof of (F.13) is identical and we omit the details. □

G Fixed domain asymptotics

In this section our aim is to investigate the sampling properties of $Q_{a,\lambda,\Omega}(g;0)$ in the case that the domain, λ, over which is the spatial process is defined in kept fixed but the number of locations that are sampled grows ($n \to \infty$).

In the following theorem we evaluate an expression for the covariance between the Fourier transforms when the domain is fixed. We use the following result to prove Lemma 3.1 and the first part of Theorem 4.7.

Theorem G.1 (Fixed domain) *Suppose that $\{Z(\boldsymbol{s}); \boldsymbol{s} \in \mathbb{R}^d\}$ is a second order stationary process defined in $[-\lambda/2, \lambda/2]^d$. Suppose that λ and Ω are both fixed. Then*

(i) Under Assumption 2.2 (general random design of locations) we have

$$\begin{aligned}
&\text{cov}\left[J_n\left(\frac{2\pi \boldsymbol{k}_1}{\Omega}\right), J_n\left(\frac{2\pi \boldsymbol{k}_2}{\Omega}\right)\right] \\
= &\left(\frac{\lambda}{2\pi}\right)^d \sum_{\boldsymbol{j}_1,\boldsymbol{j}_2 \in \mathbb{Z}^d} \gamma_{\boldsymbol{j}_1}\gamma_{\boldsymbol{j}_2} \int_{\mathbb{R}^d} f(\boldsymbol{\omega}) \text{Sinc}\left(\frac{\lambda}{2}\left[\boldsymbol{\omega} + \frac{2\pi \boldsymbol{j}_1}{\lambda} + \frac{2\pi \boldsymbol{k}_1}{\Omega}\right]\right) \\
&\times \text{Sinc}\left(\frac{\lambda}{2}\left[\boldsymbol{\omega} - \frac{2\pi \boldsymbol{j}_2}{\lambda} + \frac{2\pi \boldsymbol{k}_2}{\Omega}\right]\right) d\boldsymbol{\omega} + \frac{c(0)}{n} \int_{[-\lambda/2,\lambda/2]^d} h\left(\frac{\boldsymbol{s}}{\lambda}\right) \exp\left(i\frac{2\pi \boldsymbol{s}'}{\Omega}(\boldsymbol{k}_1 - \boldsymbol{k}_2)\right) d\boldsymbol{s}.
\end{aligned}$$

(ii) Under Assumption 2.3 (uniform sampling of locations) we have

$$\begin{aligned}
&\operatorname{cov}\left[J_n\left(\frac{2\pi\boldsymbol{k}_1}{\Omega}\right), J_n\left(\frac{2\pi\boldsymbol{k}_2}{\Omega}\right)\right] \\
= &\left(\frac{\lambda}{2\pi}\right)^d \int_{\mathbb{R}^d} f(\boldsymbol{\omega})\mathrm{Sinc}\left(\frac{\lambda}{2}\left[\boldsymbol{\omega}+\frac{2\pi\boldsymbol{k}_1}{\Omega}\right]\right)\mathrm{Sinc}\left(\frac{\lambda}{2}\left[\boldsymbol{\omega}+\frac{2\pi\boldsymbol{k}_2}{\Omega}\right]\right)d\boldsymbol{\omega} \\
&+\frac{c(0)\lambda^d}{n}\mathrm{Sinc}\left(\frac{2\pi\lambda}{\Omega}(\boldsymbol{k}_1-\boldsymbol{k}_2)\right).
\end{aligned}$$

(iii) Let $A_\lambda(\cdot)$ be defined as in (2.5), then under Assumption 2.3 (uniform sampling of locations) we have

$$\operatorname{var}\left[J_n\left(\frac{2\pi\boldsymbol{k}}{\Omega}\right)\right] = A_\lambda\left(\frac{\boldsymbol{k}}{\Omega}\right)+\frac{c(0)\lambda^d}{n}.$$

(iv) Under Assumptions 2.4 (near lattice) and 2.6(e) we have

$$\begin{aligned}
&\operatorname{cov}\left[J_n\left(\frac{2\pi k_1}{\Omega}\right), J_n\left(\frac{2\pi k_2}{\Omega}\right)\right] \\
= &\left(\frac{\lambda}{2\pi}\right)\int_{\mathbb{R}} f(\omega)\mathrm{Sinc}\left(\frac{\lambda}{2}\left[\omega+\frac{2\pi k_1}{\Omega}\right]\right)\mathrm{Sinc}\left(\frac{\lambda}{2}\left[\omega+\frac{2\pi k_2}{\Omega}\right]\right)d\omega + \\
&O\left(\frac{\lambda}{n}\left(\frac{|k_1|_1+|k_2|_1}{\Omega}+1\right)\right)
\end{aligned}$$

PROOF The proof of (i) immediately follows from the proof of Theorem 2.1, equation (2.4). (ii) follows from (i) using that for a uniform density $\gamma_0 = 1$ and $\gamma_{\boldsymbol{k}} = 0$ for $\boldsymbol{k} \neq 0$. (iii) follows from (ii) with $\boldsymbol{k}_1 = \boldsymbol{k}_2 = (\boldsymbol{k})$ and that

$$A_\lambda\left(\frac{\boldsymbol{k}}{\Omega}\right) = \left(\frac{\lambda}{2\pi}\right)^d\int_{\mathbb{R}^d} f(\boldsymbol{\omega})\mathrm{Sinc}^2\left(\frac{\lambda}{2}\left[\boldsymbol{\omega}+\frac{2\pi\boldsymbol{k}}{\Omega}\right]\right)d\boldsymbol{\omega}.$$

Finally we prove (iv) for $d = 1$. We suppose the process is observed at the locations $\{s_j\}_{j=1}^n$ which satisfy Assumption 2.4 and we denote the ordered locations as $s_{(j)}$. Expanding $\operatorname{cov}\left(J_n(\frac{2\pi k_1}{\Omega}), J_n(\frac{2\pi k_2}{\Omega})\right)$ gives

$$\begin{aligned}
\operatorname{cov}\left[J_n\left(\frac{2\pi k_1}{\Omega}\right), J\left(\frac{2\pi k_2}{\Omega}\right)\right] &= \frac{\lambda}{n^2}\sum_{j_1,j_2=1}^{n} c\left(s_{j_1}-s_{j_1}\right)\exp\left(\frac{2\pi i}{\Omega}\left[s_{j_1}k_1 - s_{j_2}k_2\right]\right) \\
= \frac{1}{2\pi\lambda}\int_{-\infty}^{\infty} f(\omega)&\left[\frac{\lambda}{n}\sum_{j_1=1}^{n}\exp\left(is_{j_1}\left[\frac{2\pi k_1}{\Omega}+\omega\right]\right)\right]\left[\frac{\lambda}{n}\sum_{j_2=1}^{n}\exp\left(-is_{j_2}\left[\frac{2\pi k_2}{\Omega}+\omega\right]\right)\right]d\omega.
\end{aligned} \quad (G.1)$$

.

Replacing the two summands with integrals such that s_{j_1} and s_{j_2} become s_1 and s_2

$$
\begin{aligned}
& \operatorname{cov}\left[J_n\left(\frac{2\pi k_1}{\Omega}\right), J\left(\frac{2\pi k_2}{\Omega}\right)\right] \\
= & \frac{1}{2\pi\lambda}\int_{-\infty}^{\infty} f(\omega)\int_{-\lambda/2}^{\lambda/2}\int_{-\lambda/2}^{\lambda/2}\exp\left(is_1\left[\frac{2\pi k_1}{\Omega}+\omega\right]\right)\exp\left(-is_2\left[\frac{2\pi k_2}{\Omega}+\omega\right]\right)ds_1 ds_2 d\omega + R_n,
\end{aligned}
$$

where

$$
\begin{aligned}
R_n &= \frac{1}{2\pi\lambda}\int_{-\infty}^{\infty} f(\omega)\left[\frac{\lambda}{n}\sum_{j_1=1}^{n}\exp\left(is_{j_1}\left[\frac{2\pi k_1}{\Omega}+\omega\right]\right)\right]\overline{D_{k_2}(\omega)}d\omega \\
& \frac{1}{2\pi\lambda}\int_{-\infty}^{\infty} f(\omega)\int_{-\lambda/2}^{\lambda/2}\exp\left(-is_2\left[\frac{2\pi k_2}{\Omega}+\omega\right]\right)D_{k_1}(\omega)dsd\omega,
\end{aligned}
$$

and

$$
D_k(\omega) = \frac{\lambda}{n}\sum_{j=1}^{n}\exp\left(is_j\left[\omega+\frac{2\pi k}{\Omega}\right]\right) - \int_{-\lambda/2}^{\lambda/2}\exp\left(is\left[\frac{2\pi k}{\Omega}+\omega\right]\right)ds.
$$

We now bound $D_k(\omega)$. Applying the mean value theorem (on the integral) we have

$$
\begin{aligned}
D_k(\omega) &= \frac{\lambda\exp(is_{(n)}[\frac{2\pi k}{\Omega}+\omega])}{n} + \\
& \sum_{j=1}^{n-1}\left(\frac{\lambda}{n}\exp\left(is_{(j)}\left[\frac{2\pi k}{\Omega}+\omega\right]\right) - \int_{-s_{(j)}}^{s_{(j+1)}}\exp\left(is\left[\frac{2\pi k}{\Omega}+\omega\right]\right)ds\right) \\
&= \frac{\lambda\exp(is_{(n)}[\frac{2\pi k}{\Omega}+\omega])}{n} + \\
& \sum_{j=1}^{n-1}\left\{\frac{\lambda}{n}\exp\left(is_{(j)}\left[\frac{2\pi k}{\Omega}+\omega\right]\right) - (s_{(j+1)}-s_{(j)})\exp\left(i\tilde{s}_j\left[\frac{2\pi k}{\Omega}+\omega\right]\right)\right\},
\end{aligned}
$$

where $\tilde{s}_j \in [s_{(j)}, s_{(j+1)}]$. Adding and subtracting $(s_{(j+1)}-s_{(j)})\exp\left(is_{(j)}\left[\frac{2\pi k}{\Omega}+\omega\right]\right)$ to the above gives

$$
\begin{aligned}
D_k(\omega) &= \frac{\lambda\exp(is_{(n)}[\frac{2\pi k}{\Omega}+\omega])}{n} + \sum_{j=1}^{n-1}\left(\frac{\lambda}{n}-(s_{(j+1)}-s_{(j)})\right)\exp\left(is_{(j)}\left[\frac{2\pi k}{\Omega}+\omega\right]\right) \\
& + \sum_{j=1}^{n-1}(s_{(j+1)}-s_{(j)})\left\{\exp\left(is_{(j)}\left[\frac{2\pi k}{\Omega}+\omega\right]\right) - \exp\left(i\tilde{s}_{(j)}\left[\frac{2\pi k}{\Omega}+\omega\right]\right)\right\}.
\end{aligned}
$$

By applying the mean value theorem on the second term we have

$$
\begin{aligned}
D_k(\omega) &= \frac{\lambda\exp(is_{(n)}[\omega+\frac{2\pi k}{\Omega}])}{n} + \sum_{j=1}^{n-1}\left(\frac{\lambda}{n}-(s_{(j+1)}-s_{(j)})\right)\exp\left(is_{(j)}\left[\frac{2\pi k}{\Omega}+\omega\right]\right) \\
& + i\sum_{j=1}^{n-1}(s_{(j+1)}-s_{(j)})(s_{(j)}-\tilde{s}_{(j)})\left[\frac{2\pi k}{\Omega}+\omega\right]\exp\left(i\hat{s}_{(j)}\left[\frac{2\pi k}{\Omega}+\omega\right]\right),
\end{aligned}
$$

where $\hat{s}_{(j)} \in [s_{(j)}, s_{(j+1)}]$. Taking absolute and using Assumption 2.4 we have

$$D_k(\omega) \leq \frac{\lambda}{n} + \left|\frac{2\pi k}{\Omega} + \omega\right| \sum_{j=1}^{n-1} (s_{(j+1)} - s_{(j)})^2 + \sum_{j=1}^{n-1} \left|\frac{\lambda}{n} - (s_{(j+1)} - s_{(j)})\right| = O\left(\frac{\lambda}{n}\left[1 + \omega + \frac{2\pi k}{\Omega}\right]\right).$$

We use the above to bound R_n, using that $\int_{\mathbb{R}} f(\omega)|\omega| d\omega < \infty$ (which is true under Assumption 2.6(e)) we have

$$|R_n| \leq C \int_{-\infty}^{\infty} f(\omega) \frac{\lambda}{n}\left[1 + \omega + \frac{2\pi(|k_1| + |k_2|)}{\Omega}\right] d\omega = O\left(\frac{\lambda}{n}\left(\frac{|k_1| + |k_2|}{\Omega} + 1\right)\right).$$

Thus substituting the bound for $|R_n|$ into (G.2) we have proved (iv). □

The results below are used to prove the results in Section 4.4, in particular Theorem 4.7. We focus on the case $d = 1$. We first consider the Fourier transform of the continuous time anologue $\{Z(s); s \in [-\lambda/2, \lambda/2]\}$. It is straightforward to show

$$\begin{aligned} & \mathrm{cov}\left[\mathcal{J}_\lambda\left(\frac{k_1}{\Omega}\right), \mathcal{J}_\lambda\left(\frac{k_2}{\Omega}\right)\right] && \text{(G.2)} \\ = & \frac{2}{\lambda\pi} \int_{-\infty}^{\infty} f(\omega) \mathrm{sinc}\left(\frac{\lambda}{2}\left[\omega + \frac{2\pi k_1}{\Omega}\right]\right) \mathrm{sinc}\left(\frac{\lambda}{2}\left[\omega + \frac{2\pi k_2}{\Omega}\right]\right) d\omega \\ = & \frac{1}{\lambda} \int_{-\lambda/2}^{\lambda/2} \int_{-\lambda/2}^{\lambda/2} c(s_1 - s_2) \exp\left(\frac{2\pi i s_1 k_1}{\Omega}\right) \exp\left(-\frac{2\pi i s_2 k_2}{\Omega}\right) ds_1 ds_2 && \text{(G.3)} \end{aligned}$$

where

$$\mathcal{J}_\lambda\left(\frac{k}{\Omega}\right) = \frac{1}{\lambda^{1/2}} \int_{-\lambda/2}^{\lambda/2} Z(s) \exp\left(\frac{2\pi i k s}{\Omega}\right) ds. \qquad \text{(G.4)}$$

Thus comparing the above with Theorem G.1 we have

$$\mathrm{cov}\left[J_n\left(\frac{2\pi k_1}{\Omega}\right), J_n\left(\frac{2\pi k_2}{\Omega}\right)\right] = \mathrm{cov}\left[\mathcal{J}_\lambda\left(\frac{k_1}{\Omega}\right), \mathcal{J}_\lambda\left(\frac{k_2}{\Omega}\right)\right] + o(1),$$

where the error in the approximation depends on the sampling of the locations and are given in Theorem G.1. We now obtain an explicit expression for the above, which helps us quantify the dependence.

Theorem G.2 *Suppose $\{Z(u)\}$ is a spatial second order stationary process and $\mathcal{J}_\lambda\left(\frac{k}{\Omega}\right)$ is defined in (G.4). Then*

$$\mathrm{cov}\left[\mathcal{J}_\lambda\left(\frac{k_1}{\Omega}\right), \mathcal{J}_\lambda\left(\frac{k_2}{\Omega}\right)\right] = \begin{cases} A_\lambda\left(\frac{k}{\Omega}\right) & k_1 = k_2 (= k) \\ B_\lambda\left(\frac{k_1}{\Omega}, \frac{k_2}{\Omega}\right) & k_1 \leq k_2 \end{cases}$$

where

$$A_\lambda\left(\frac{k}{\Omega}\right) = \int_{-\lambda}^{\lambda} T\left(\frac{u}{\lambda}\right) c(u) \exp\left(\frac{2i\pi k u}{\Omega}\right) du$$

and

$$B_\lambda\left(\frac{k_1}{\Omega},\frac{k_2}{\Omega}\right) = \frac{\Omega}{(k_1-k_2)\pi\lambda}\sin\left(\frac{2\pi\lambda(k_1-k_2)}{\Omega}\right)\int_{-\lambda}^{\lambda} c(v)\exp\left(\frac{2\pi i k_1 v}{\Omega}\right)dv$$
$$+\frac{\Omega}{(k_1-k_2)\pi\lambda}\Im\left[e^{-\lambda i\pi(k_1-k_2)/\Omega}\left(\int_0^{\lambda} c(v)e^{2\pi i k_2 v/\Omega}dv - \int_0^{\lambda} c(v)e^{2\pi i k_1 v/\Omega}dv\right)\right].$$

PROOF For the case $k_1 = k_2 (= k)$ and (G.3) we have

$$\begin{aligned}\text{var}\left[\mathcal{J}_\lambda\left(\frac{k}{\Omega}\right)\right] &= \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2}\int_{-\lambda/2}^{\lambda/2} c(s_1-s_2)\exp\left(\frac{2\pi i(s_1-s_2)k}{\Omega}\right)ds_1 ds_2 \\ &= \frac{1}{\lambda}\int_{-\lambda}^{\lambda}(\lambda-|u|)\,c(u)\exp\left(\frac{2\pi i u k}{\Omega}\right)du = \int_{-\lambda}^{\lambda}\left(1-\frac{|u|}{\lambda}\right)c(u)\exp\left(\frac{2\pi i u k}{\Omega}\right)du.\end{aligned}$$

This gives $A_\lambda(\frac{k}{\Omega})$. For $k_1 \neq k_2$ we have

$$\begin{aligned}&\text{cov}\left[\mathcal{J}_\lambda\left(\frac{k_1}{\Omega}\right),\mathcal{J}_\lambda\left(\frac{k_2}{\Omega}\right)\right] \\ =\ & \frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2}\int_{-\lambda/2}^{\lambda/2} c(u_1-u_2)\exp\left(\frac{2\pi i k_1(u_1-u_2)}{\Omega}\right)\exp\left(\frac{2\pi i u_2(k_1-k_2)}{\Omega}\right)du_1 du_2 \\ =\ & \frac{1}{\lambda}\int_{-\lambda}^{\lambda} c(v)\exp\left(\frac{2\pi i k_1 v}{\Omega}\right)\int_{\max(-\lambda/2,-\lambda/2-v)}^{\min(\lambda/2,\lambda/2-v)}\exp\left(\frac{2\pi i u_2(k_1-k_2)}{\Omega}\right)du_2 dv \\ =\ & A_1 + A_2 + A_3,\end{aligned}$$

where

$$\begin{aligned}A_1 &= \frac{1}{\lambda}\int_{-\lambda}^{\lambda} c(v)\exp\left(\frac{2\pi i k_1 v}{\Omega}\right)\int_{-\lambda/2}^{\lambda/2}\exp\left(\frac{2\pi i u_2(k_1-k_2)}{\Omega}\right)du_2 dv \\ A_2 &= -\frac{1}{\lambda}\int_{-\lambda}^{0} c(v)\exp\left(\frac{2\pi i k_1 v}{\Omega}\right)\int_{-\lambda/2}^{-\lambda/2-v}\exp\left(\frac{2\pi i u_2(k_1-k_2)}{\Omega}\right)du_2 dv \\ A_3 &= -\frac{1}{\lambda}\int_{0}^{\lambda} c(v)\exp\left(\frac{2\pi i k_1 v}{\Omega}\right)\int_{\lambda/2-v}^{\lambda/2}\exp\left(\frac{2\pi i u_2(k_1-k_2)}{\Omega}\right)du_2 dv.\end{aligned}$$

It is easily seen that

$$A_1 = \frac{\Omega}{\lambda(k_1-k_2)}\sin\left(\frac{2\pi\lambda(k_1-k_2)}{\Omega}\right)\int_{-\lambda}^{\lambda} c(v)\exp\left(\frac{2\pi i k_1 v}{\Omega}\right)dv.$$

We obtain expressions for A_2 and A_3

$$
\begin{aligned}
A_2 &= -\frac{1}{\lambda}\int_{-\lambda}^{0} c(v)\exp\left(\frac{2\pi i k_1 v}{\Omega}\right)\left[\frac{e^{2\pi i u(k_1-k_2)/\Omega}}{2\pi i(k_1-k_2)/\Omega}\right]_{u=-\lambda/2}^{-v-\lambda/2} \\
&= \frac{-\Omega}{\lambda 2\pi i(k_1-k_2)}\int_{-\lambda}^{0} c(v)e^{2\pi i k_1 v/\lambda}\left(e^{-\frac{2\pi i(k_1-k_2)}{\Omega}(\frac{\lambda}{2}+v)} - e^{-i\pi\lambda(k_1-k_2)/\Omega}\right) \\
&= \frac{-\Omega e^{-\lambda i\pi(k_1-k_2)/\Omega}}{\lambda 2\pi i(k_1-k_2)}\int_{-\lambda}^{0} c(v)\left(e^{2\pi i k_2 v/\Omega} - e^{2\pi i k_1 v/\Omega}\right)dv \\
&= \frac{-\Omega e^{-\lambda i\pi(k_1-k_2)/\Omega}}{\lambda 2\pi i(k_1-k_2)}\int_{0}^{\lambda} c(v)\left(-e^{-2\pi i k_1 v/\Omega} + e^{-2\pi i k_1 v/\Omega}\right)dv
\end{aligned}
$$

Similarly

$$
\begin{aligned}
A_3 &= -\frac{1}{\lambda}\int_{0}^{\lambda} c(v)\exp\left(\frac{2\pi i k_1 v}{\Omega}\right)\left[\frac{e^{2\pi i u(k_1-k_2)/\Omega}}{2\pi i(k_1-k_2)/\Omega}\right]_{u=\lambda/2-v}^{\lambda/2} \\
&= \frac{-\Omega}{\lambda 2\pi i(k_1-k_2)}\int_{0}^{\lambda} c(v)e^{2\pi i k_1 v/\Omega}\left(e^{\lambda i\pi(k_1-k_2)/\Omega} - e^{\frac{2\pi i(k_1-k_2)}{\Omega}(\frac{\lambda}{2}-v)}\right) \\
&= \frac{-\Omega e^{\lambda i\pi(k_1-k_2)/\Omega}}{\lambda 2\pi i(k_1-k_2)}\int_{0}^{\lambda} c(v)\left(e^{2\pi i k_1 v/\Omega} - e^{2\pi i k_2 v/\Omega}\right)dv.
\end{aligned}
$$

This gives

$$
A_2 + A_3 = \frac{\Omega}{(k_1-k_2)\pi\lambda}\Im\left[e^{-\lambda i\pi(k_1-k_2)/\Omega}\left(\int_0^{\lambda} c(v)e^{2\pi i k_2 v/\Omega}dv - \int_0^{\lambda} c(v)e^{2\pi i k_1 v/\Omega}dv\right)\right].
$$

Thus

$$
\mathrm{cov}\left[\mathcal{J}_\lambda\left(\frac{k_1}{\Omega}\right), \mathcal{J}_\lambda\left(\frac{k_2}{\Omega}\right)\right] = \frac{\Omega}{\lambda(k_1-k_2)}\sin\left(\frac{2\pi\lambda(k_1-k_2)}{\Omega}\right)\int_{-\lambda}^{\lambda} c(v)\exp\left(\frac{2\pi i k_1 v}{\Omega}\right)dv +
\frac{\Omega}{(k_1-k_2)\pi\lambda}\Im\left[e^{-\lambda i\pi(k_1-k_2)/\Omega}\left(\int_0^{\lambda} c(v)e^{2\pi i k_2 v/\Omega}dv - \int_0^{\lambda} c(v)e^{2\pi i k_1 v/\Omega}dv\right)\right].
$$

giving the required result. □

In the following lemma we focus on the Fourier transforms $\mathcal{J}_\lambda\left(\frac{k}{\lambda}\right)$ (frequency grid is $\Omega = \lambda$).

Corollary G.1 *Suppose $\{Z(u)\}$ is a spatial second order stationary process and $\mathcal{J}_\lambda\left(\frac{k}{\lambda}\right)$ is defined in (G.4). Then*

$$
\mathrm{cov}\left[\mathcal{J}_\lambda\left(\frac{k_1}{\lambda}\right), \mathcal{J}_\lambda\left(\frac{k_2}{\lambda}\right)\right] = \begin{cases} A_\lambda\left(\frac{k}{\lambda}\right) & k_1 = k_2 (= k) \\ \frac{(-1)^{k_1-k_2+1}}{\pi(k_1-k_2)}\left[B_\lambda(\frac{k_1}{\lambda}) - B_\lambda(\frac{k_2}{\lambda})\right] & k_1 \le k_2 \end{cases}
$$

if in addition the process is Gaussian then

$$\text{cov}\left[\left|\mathcal{J}_\lambda\left(\frac{k_1}{\lambda}\right)\right|^2, \left|\mathcal{J}_\lambda\left(\frac{k_2}{\lambda}\right)\right|^2\right]$$
$$= \begin{cases} A_\lambda\left(\frac{k}{\lambda}\right)^2 + \frac{\lambda^2}{\pi^2 k^2 \lambda^2} B_\lambda(\frac{k}{\lambda})^2 & k_1 = k_2 (= k) \\ \frac{\lambda^2}{\lambda^2\pi^2(k_1-k_2)^2}\left[B_\lambda(\frac{k_1}{\lambda}) - B_\lambda(\frac{k_2}{\lambda})\right]^2 + \frac{\lambda^2}{\lambda^2\pi^2(k_1+k_2)^2}\left[B_\lambda(\frac{k_1}{\lambda}) + B_\lambda(\frac{k_2}{\lambda})\right]^2 & k_1 \neq k_2 \end{cases}$$

where

$$A_\lambda\left(\frac{k}{\lambda}\right) = \int_{-\lambda}^{\lambda} T\left(\frac{u}{\lambda}\right) c(u) \exp\left(\frac{2i\pi k u}{\lambda}\right) du$$

and

$$B_\lambda\left(\frac{k}{\lambda}\right) = \frac{1}{\lambda}\int_0^{\lambda} c(u) \sin\left(\frac{2\pi k u}{\lambda}\right) du.$$

PROOF. The proof for $k_1 = k_2$ is clear. The proof for $k_1 \neq k_2$ follows from Theorem G.2 and using that $e^{i\pi(k_1-k_2)} = (-1)^{k_1-k_2}$ and $\sin(2\pi(k_1-k_2)) = 0$, this gives

$$\begin{aligned} \text{cov}\left[\mathcal{J}_\lambda\left(\frac{k_1}{\lambda}\right), \mathcal{J}_\lambda\left(\frac{k_2}{\lambda}\right)\right] &= \frac{(-1)^{(k_1-k_2-1)}}{\lambda 2\pi(k_1-k_2)}\left[\int_0^{\lambda} c(v)\left(\sin\left(\frac{2\pi k_1}{\lambda}\right) - \sin\left(\frac{2\pi k_2}{\lambda}\right)\right) dv\right] \\ &= \frac{(-1)^{k_1-k_2+1}}{\pi(k_1-k_2)}\left[B_\lambda(\frac{k_1}{\lambda}) - B_\lambda(\frac{k_2}{\lambda})\right]. \end{aligned}$$

The result for $\text{cov}\left[|\mathcal{J}_\lambda\left(\frac{k_1}{\lambda}\right)|^2, |\mathcal{J}_\lambda\left(\frac{k_2}{\lambda}\right)|^2\right]$ follows from the above, the covariance expansion in terms of cumulants and Gaussianity of the process. □

PROOF of Theorem 4.7 The proof of (4.7) immediately follows from Lemma 3.1(ii) with $\Omega = \lambda$.

To prove (4.8) we use Remark 2.4 and Corollary G.1 which immediately gives the result. □

H Sample properties of $Q_{a,\Omega,\lambda}(g; \boldsymbol{r})$

In this section we summarize the sampling properties of $Q_{a,\Omega,\lambda}(g; 0)$ (general frequency grid when the bias is not removed) and $Q_{a,\lambda,\lambda}(g; \boldsymbol{r})$ (frequency grid $\Omega = \lambda$ and general $\boldsymbol{r}$). We do not consider $Q_{a,\Omega,\lambda}(g; \boldsymbol{r})$ when $\boldsymbol{r} \neq 0$, since for the general frequency grid $Q_{a,\Omega,\lambda}(g; \boldsymbol{r})$ does not appear to have any useful sampling properties (we recall that when the locations are uniformly sampled $\widetilde{Q}_{a,\lambda,\lambda}(g; \boldsymbol{r})$ can be used to estimate the variance of $\widetilde{Q}_{a,\lambda,\lambda}(g; 0)$). We recall that

$$Q_{a,\Omega,\lambda}(g; \boldsymbol{r}) = \widetilde{Q}_{a,\Omega,\lambda}(g; \boldsymbol{r}) + G_\Omega V_{\Omega,\boldsymbol{r}} \tag{H.1}$$

where $V_{\Omega,\boldsymbol{r}} = \frac{1}{n}\sum_{j=1}^{n} Z(\boldsymbol{s}_j)^2 \exp(-i\boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\boldsymbol{s}_j)$, $G_{\Omega} = \frac{\lambda^d}{n\Omega^d}\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})$ and $\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r})$ is defined in (3.6). To simplify notation we define $Q_{a,\lambda}(g;\boldsymbol{r}) := Q_{a,\Omega,\lambda}(g;\boldsymbol{r})$. The sampling properties are derived under the assumption the spatial process is Gaussian and the locations are uniformly distributed.

The expectation of $Q_{a,\Omega,\lambda}(g;0)$ (for a general frequency grid) is given in Lemma 3.1. Below we consider the expectation of $Q_{a,\lambda}(g;\boldsymbol{r})$ (the frequency grid $\Omega = \lambda$ and general $\boldsymbol{r}$).

Theorem H.1 *Suppose Assumptions 2.1(i), 2.3, $b = b(\boldsymbol{r})$ denotes the number of zero elements in the vector $\boldsymbol{r} \in \mathbb{Z}^d$ and*

(i) Assumptions 2.5(i) and 2.6(a,c) hold. Then we have

$$\begin{aligned} & \mathrm{E}\left[Q_{a,\lambda}(g;\boldsymbol{r})\right] \\ = & \begin{cases} O(\frac{1}{\lambda^{d-b}}) & \boldsymbol{r} \in \mathbf{Z}^d/\{0\} \\ \frac{1}{(2\pi)^d}\int_{\boldsymbol{\omega}\in 2\pi[-C,C]^d} f(\boldsymbol{\omega})g(\boldsymbol{\omega})d\boldsymbol{\omega} + O(\frac{1}{\lambda} + \frac{\lambda^d}{n}) & \boldsymbol{r} = 0 \end{cases} \end{aligned}$$

(ii) Suppose the Assumptions 2.5(ii) and Assumption 2.6(b,c) hold and $\{m_1, \ldots, m_{d-b}\}$ is the subset of non-zero values in $\boldsymbol{r} = (r_1, \ldots, r_d)$, then we have

$$\begin{aligned} & \mathrm{E}\left[Q_{a,\lambda}(g;\boldsymbol{r})\right] \\ = & \begin{cases} O\left(\frac{1}{\lambda^{d-b}}\prod_{j=1}^{d-b}(\log\lambda + \log|m_j|)\right) & \boldsymbol{r} \in \mathbf{Z}^d/\{0\} \\ \frac{1}{(2\pi)^d}\int_{\boldsymbol{\omega}\in\mathbb{R}^d} f(\boldsymbol{\omega})g(\boldsymbol{\omega})d\boldsymbol{\omega} + \frac{c(0)}{n}\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega}_{\boldsymbol{k}}) + O\left(\frac{\log\lambda}{\lambda} + \frac{1}{n}\right) & \boldsymbol{r} = 0 \end{cases}. \end{aligned}$$

PROOF The proof of (i) immediately follows from Theorem 2.1.

The proof of (ii) follows from writing $Q_{a,\lambda}(g;\boldsymbol{r})$ as a quadratic form and taking expectations

$$\begin{aligned} & \mathrm{E}\left[Q_{a,\lambda}(g;\boldsymbol{r})\right] \\ = & c_2 \sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega}_{\boldsymbol{k}}) \frac{1}{\lambda^d}\int_{[-\lambda/2,\lambda/2]^d} c(\boldsymbol{s}_1 - \boldsymbol{s}_2)\exp(i\boldsymbol{\omega}_k'(\boldsymbol{s}_1 - \boldsymbol{s}_2) - i\boldsymbol{s}_2'\boldsymbol{\omega}_{\boldsymbol{r}})d\boldsymbol{s}_1 d\boldsymbol{s}_2 + W_{\boldsymbol{r}}, \end{aligned}$$

where $c_2 = n(n-1)/n^2$ and $W_{\boldsymbol{r}} = \frac{c(0)I(\boldsymbol{r}=0)}{n}\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega}_{\boldsymbol{k}})$ (where $I(\boldsymbol{r}=0) = 1$ if $\boldsymbol{r} = 0$ else $I(\boldsymbol{r}=0) = 1$). We then follow the same proof used to prove Theorem 4.1. □

We now obtain an expression for the variance of $Q_{a,\lambda}(g;\boldsymbol{r})$ and $Q_{a,\Omega,\lambda}(g;0)$. All the calculations for $\widetilde{Q}_{a,\Omega,\lambda}(g;0)$ have been done in previous sections, therefore from (H.1) it is we only need expressions for $\mathrm{cov}[V_{\Omega,\boldsymbol{r}_1}, V_{\Omega,\boldsymbol{r}_2}]$ and $\mathrm{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_1), V_{\Omega,\boldsymbol{r}_2}\right]$.

Lemma H.1 *Suppose Assumptions 2.1, 2.3, 2.5 2.6(b,c) hold. Define*

$$f_2(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} f(\boldsymbol{\lambda})f(\boldsymbol{\omega} - \boldsymbol{\lambda})d\boldsymbol{\lambda}.$$

If $\Omega > \lambda$*, then*

$$\lambda^d \mathrm{cov}[V_{\Omega,\boldsymbol{r}_1}, V_{\Omega,\boldsymbol{r}_2}] = 2c_2 f_2\left(\boldsymbol{\omega}_{\Omega,\boldsymbol{r}_2}\right)\mathrm{Sinc}\left(\frac{\lambda\pi}{\Omega}(\boldsymbol{r}_1-\boldsymbol{r}_2)\right) + \frac{3\lambda^d c(0)^2}{n}\mathrm{Sinc}\left(\frac{\lambda\pi}{\Omega}(\boldsymbol{r}_1-\boldsymbol{r}_2)\right) + O\left(\frac{\log\lambda + \log\left(1+\frac{\lambda}{\Omega}\|\boldsymbol{r}_2-\boldsymbol{r}_1\|_1\right)}{\lambda}\right) \quad (H.2)$$

If $\Omega \leq \lambda$*, then*

$$\Omega^d \mathrm{cov}[V_{\Omega,\boldsymbol{r}_1}, V_{\Omega,\boldsymbol{r}_2}] = \frac{2c_2\Omega^d}{\lambda^d} f_2\left(\boldsymbol{\omega}_{\Omega,\boldsymbol{r}_2}\right)\mathrm{Sinc}\left(\frac{\lambda\pi}{\Omega}(\boldsymbol{r}_1-\boldsymbol{r}_2)\right) + \frac{3\Omega^d c(0)^2}{n}\mathrm{Sinc}\left(\frac{\lambda\pi}{\Omega}(\boldsymbol{r}_1-\boldsymbol{r}_2)\right) + O\left(\frac{\log\lambda + \log\left(1+\frac{\lambda}{\Omega}\|\boldsymbol{r}_2-\boldsymbol{r}_1\|_1\right)}{\lambda}\right) \quad (H.3)$$

If $\Omega > \lambda$*, then*

$$\begin{aligned}
&\lambda^d \mathrm{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_1), V_{\Omega,\boldsymbol{r}_2}\right] \\
&= \frac{c_3}{\pi^d}\mathrm{Sinc}\left(\frac{\lambda\pi}{\Omega}(\boldsymbol{r}_1-\boldsymbol{r}_2)\right)\int_{[-2\pi a/\Omega, 2\pi a/\Omega]^d} g(\omega) f\left(\boldsymbol{\omega}\right) f\left(\boldsymbol{\omega}+\boldsymbol{\omega}_{\Omega,\boldsymbol{r}_1}\right) d\boldsymbol{\omega} + \\
&O\left(\frac{\lambda^d}{n} + \frac{1}{\Omega} + \frac{\left[\log\lambda + \log\left(1+\frac{\lambda}{\Omega}\|\boldsymbol{r}_1-\boldsymbol{r}_2\|_1\right)\right]^3}{\lambda}\right). \quad (H.4)
\end{aligned}$$

If $\Omega \leq \lambda$*, then*

$$\begin{aligned}
&\Omega^d \mathrm{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_1), V_{\Omega,\boldsymbol{r}_2}\right] \\
&= \frac{c_3\Omega^d}{\pi^d\lambda^d}\mathrm{Sinc}\left(\frac{\lambda\pi}{\Omega}(\boldsymbol{r}_1-\boldsymbol{r}_2)\right)\int_{[-2\pi a/\Omega, 2\pi a/\Omega]^d} g(\boldsymbol{\omega}) f\left(\boldsymbol{\omega}\right) f\left(\boldsymbol{\omega}+\boldsymbol{\omega}_{\Omega,\boldsymbol{r}_1}\right) d\boldsymbol{\omega} + \\
&O\left(\frac{\lambda^d}{n} + \frac{1}{\Omega} + \frac{\left[\log\lambda + \log\left(1+\frac{\lambda}{\Omega}\|\boldsymbol{r}_1-\boldsymbol{r}_2\|_1\right)\right]^3}{\lambda}\right). \quad (H.5)
\end{aligned}$$

PROOF. We prove the result for $d = 1$.

To shorten the proof, we prove (H.2) and (H.3) together. We define

$$\overline{\Omega} = \begin{cases} \lambda & \Omega > \lambda \\ \Omega & \Omega \leq \lambda \end{cases}$$

Conditioning on location and using that $\mathrm{cov}[Z(s_1)^2, Z(s_2)^2 | s_1, s_2] = 2c(s_1 - s_2)^2$ (in the case the spatial process is Gaussian) we have

$$
\begin{aligned}
\overline{\Omega}\mathrm{cov}[V_{\Omega,r_1},V_{\Omega,r_2}] &= \frac{\overline{\Omega}}{n^2}\sum_{j_1,j_2=1}^{n}\mathrm{cov}[Z(s_{j_1})^2e^{-is_{j_1}\omega_{\Omega,r_1}},Z(s_{j_2})^2e^{-is_{j_2}\omega_{\Omega,r_2}}] \\
&= \frac{\overline{\Omega}}{n^2}\sum_{j_1,j_2=1}^{n}\Big\{\mathrm{E}\left(e^{-is_{j_1}\omega_{\Omega,r_1}+is_{j_2}\omega_{\Omega,r_2}}\mathrm{cov}[Z(s_{j_1})^2,Z(s_{j_2})^2|s_{j_1},s_{j_2}]\right) \\
&\quad +\mathrm{cov}[c(0)e^{-is_{j_1}\omega_{\Omega,r_1}},c(0)e^{-is_{j_2}\omega_{\Omega,r_2}}]\Big\} \\
&= \frac{\overline{\Omega}}{n^2}\sum_{j_1\neq j_2=1}^{n}\mathrm{E}\left(2e^{-is_{j_1}\omega_{\Omega,r_1}+is_{j_2}\omega_{\Omega,r_2}}c(s_{j_1}-s_{j_2})^2\right) \\
&\quad +\frac{\overline{\Omega}}{n^2}\sum_{j=1}^{n}\mathrm{E}\left(2e^{-is_j(\omega_{\Omega,r_1}-\omega_{\Omega,r_2})}c(0)^2\right)+\frac{\overline{\Omega}^2}{n}\sum_{j=1}^{n}c(0)^2\mathrm{E}[e^{-is_j(\omega_{\Omega,r_1}-\omega_{\Omega,r_2})}] \\
&= A_1+A_2
\end{aligned}
$$

where

$$
A_1 = 2\overline{\Omega}c_2\mathrm{E}\left[c(s_1-s_2)^2\exp(-is_1\omega_{\Omega,r_1}+is_2\omega_{\Omega r_2})\right] \text{ and } A_2=\frac{3\overline{\Omega}c(0)^2}{n}\mathrm{E}\left[\exp(-is(\omega_{\Omega,r_1-r_2}))\right].
$$

It is clear that

$$
A_2 = \frac{3\overline{\Omega}c(0)^2}{n}\mathrm{sinc}\left(\frac{\lambda\pi}{\Omega}(r_1-r_2)\right).
$$

Integrating out the location in A_1 and using that $c(u)^2=\frac{1}{2\pi}\int_{\mathbb{R}}f_2(\omega)\exp(-iu\omega)du$ we have

$$
\begin{aligned}
A_1 &= 2c_2\frac{\overline{\Omega}}{\lambda^2}\int_{[-\lambda/2,\lambda/2]^s}c(s_1-s_2)^2\exp(-is_1\omega_{\Omega,r_1}+is_2\omega_{\Omega,r_2})ds_1ds_2 \\
&= \frac{2c_2\overline{\Omega}}{2\pi}\int_{\mathbb{R}}f_2(\omega)\frac{1}{\lambda^2}\int_{[-\lambda/2,\lambda/2]^s}\exp(-is_1(\omega+\omega_{\Omega,r_1})+is_2(\omega+\omega_{\Omega,r_2}))ds_1ds_2d\omega \\
&= \frac{2c_2\overline{\Omega}}{2\pi}\int_{\mathbb{R}}f_2(\omega)\mathrm{sinc}\left(\frac{\lambda}{2}\left[\omega+\omega_{\Omega,r_1}\right]\right)\mathrm{sinc}\left(\frac{\lambda}{2}\left[\omega+\omega_{\Omega,r_2}\right]\right)d\omega.
\end{aligned}
$$

With the change of variables $u=\frac{\lambda}{2}(\omega+\omega_{\Omega,r_1})$ we have

$$
A_1 = \frac{2c_2\overline{\Omega}}{\pi\lambda}\int_{\mathbb{R}}f_2\left(\frac{2u}{\lambda}-\omega_{\Omega,r_2}\right)\mathrm{sinc}\,(u)\,\mathrm{sinc}\left(u+\frac{\lambda}{2}\omega_{\Omega,r_2-r_1}\right)du
$$

We replace $f_2\left(\frac{2u}{\lambda}-\omega_{\Omega,r_2}\right)$ with $f_2\left(-\omega_{\Omega,r_2}\right)$ and using Lemma F.2 gives

$$
\begin{aligned}
A_1 &= \frac{2c_2\overline{\Omega}}{\pi\lambda}f_2\left(\omega_{\Omega,r_2}\right)\int_{\mathbb{R}}\mathrm{sinc}\,(u)\,\mathrm{sinc}\left(u+\frac{\lambda}{2}\left[\omega_{\Omega,r_2-r_1}\right]\right)du+O\left(\frac{\log\lambda+\log\left(1+\frac{\lambda}{\Omega}(r_2-r_1)\right)}{\lambda}\right) \\
&= \frac{2c_2\overline{\Omega}}{\lambda}f_2\left(\omega_{\Omega,r_2}\right)\mathrm{sinc}\left(\frac{\lambda\pi}{\Omega}(r_1-r_2)\right)+O\left(\frac{\log\lambda+\log\left(1+\frac{\lambda}{\Omega}(r_2-r_1)\right)}{\lambda}\right).
\end{aligned}
$$

This proves (H.2) and (H.3).

We now prove (H.4), where the frequency grid is fine (we recall that $\Omega > \lambda$). Using that $\mathrm{cov}[A,B] = \mathrm{cov}[\mathrm{E}[A|C],\mathrm{E}[B|C]] + \mathrm{E}[\mathrm{cov}(A,B|C)]$ we have

$$
\begin{aligned}
& \lambda \mathrm{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;r_1), V_{\Omega,r_2}\right] \\
= & \frac{\lambda^2}{n^3\Omega}\sum_{k=-a}^{a} g(\omega_{\Omega,k}) \sum_{\substack{j_1,j_2,j_3=1 \\ j_1\neq j_2}}^{n} \mathrm{cov}\left[Z(s_{j_1})Z(s_{j_2})e^{is_{j_1}\omega_{\Omega,k}-is_{j_2}\omega_{\Omega,k+r_1}}, Z(s_{j_3})^2 e^{-is_{j_3}\omega_{\omega,r_2}}\right] \\
= & \frac{\lambda^2}{n^3\Omega}\sum_{k=-a}^{a} g(\omega_{\Omega,k}) \sum_{\substack{j_1,j_2,j_3=1 \\ j_1\neq j_2}}^{n} \mathrm{E}\left[2e^{is_{j_1}\omega_{\Omega,k}-is_{j_2}\omega_{\Omega,k+r_1}} e^{is_{j_3}\omega_{\Omega,r_2}} c(s_{j_1}-s_{j_3})c(s_{j_2}-s_{j_3})\right] + \\
& \frac{\lambda^2}{n^3\Omega}\sum_{k=-a}^{a} g(\omega_{\Omega,k}) \sum_{\substack{j_1,j_2,j_3=1 \\ j_1\neq j_2}}^{n} \underbrace{\mathrm{cov}\left[e^{is_{j_1}\omega_{\Omega,k}-is_{j_2}\omega_{\Omega,k+r_1}} c(s_{j_1}-s_{j_2}), e^{is_{j_3}\omega_{\Omega,r_2}}c(0)\right]}_{=0 \text{ unless } j_1=j_3 \text{ or } j_2=j_3} \\
= & \frac{\lambda^2}{n^3\Omega}\sum_{k=-a}^{a} g(\omega_{\Omega,k}) \sum_{j_1,j_2,j_3\in\mathcal{B}_3}^{n} \mathrm{E}\left[2e^{is_{j_1}\omega_{\Omega,k}-is_{j_2}\omega_{\Omega,k+r_1}} e^{is_{j_3}\omega_{\Omega,r_2}} c(s_{j_1}-s_{j_3})c(s_{j_2}-s_{j_3})\right] + R_1
\end{aligned} \tag{H.6}
$$

$\mathcal{B}_3 = \{j_1,j_2,j_3; 1\leq j_1,j_2,j_3\leq n;\ \text{and all different}\}$ where

$$
\begin{aligned}
R_1 = & \frac{\lambda^2}{n^3\Omega}\sum_{k=-a}^{a} g(\omega_{\Omega,k}) \sum_{\substack{j_1,j_2,j_3=1, j_1\neq j_2 \\ j_1=j_3 \text{ or } j_2=j_3}}^{n} \mathrm{E}\left[2e^{is_{j_1}\omega_{\Omega,k}-is_{j_2}\omega_{\Omega,k+r_1}} e^{is_{j_3}\omega_{\Omega,r_2}} c(s_{j_1}-s_{j_3})c(s_{j_2}-s_{j_3})\right] + \\
& \frac{\lambda^2}{n^3\Omega}\sum_{k=-a}^{a} g(\omega_{\Omega,k}) \sum_{\substack{j_1,j_2,j_3=1, j_1\neq j_2 \\ j_1=j_3}}^{n} \mathrm{E}\left[e^{is_{j_1}\omega_{\Omega,k}-is_{j_2}\omega_{\Omega,k+r_1}} e^{is_{j_1}\omega_{\Omega,r_2}} c(0)c(s_{j_1}-s_{j_2})\right] + \\
& \frac{\lambda^2}{n^3\Omega}\sum_{k=-a}^{a} g(\omega_{\Omega,k}) \sum_{\substack{j_1,j_2,j_3=1, j_1\neq j_2 \\ j_2=j_3}}^{n} \mathrm{E}\left[e^{is_{j_1}\omega_{\Omega,k}-is_{j_2}\omega_{\Omega,k+r_1}} e^{is_{j_2}\omega_{\Omega,r_2}} c(0)c(s_{j_1}-s_{j_2})\right] - \\
& \frac{\lambda^2}{n^3\Omega}\sum_{k=-a}^{a} g(\omega_{\Omega,k}) \sum_{\substack{j_1,j_2,j_3=1, j_1\neq j_2 \\ j_1=j_3}}^{n} \mathrm{E}\left[e^{is_{j_1}\omega_{\Omega,k}-is_{j_2}\omega_{\Omega,k+r_1}} c(s_{j_1}-s_{j_2})\right] \mathrm{E}\left[e^{is_{j_1}\omega_{\Omega,r_2}}c(0)\right] - \\
& \frac{\lambda^2}{n^3\Omega}\sum_{k=-a}^{a} g(\omega_{\Omega,k}) \sum_{\substack{j_1,j_2,j_3=1, j_1\neq j_2 \\ j_2=j_3}}^{n} \mathrm{E}\left[e^{is_{j_1}\omega_{\Omega,k}-is_{j_2}\omega_{\Omega,k+r_1}} c(s_{j_1}-s_{j_2})\right] \mathrm{E}\left[e^{is_{j_2}\omega_{\Omega,r_2}}c(0)\right]
\end{aligned}
$$

Following the same methods used to prove Lemma E.2 we have $|R_1| = O(\lambda/n)$. To find approximations to the lead term in (H.6) we use similar arguments to those in the proof of

Lemma B.1 to obtain

$$
\begin{aligned}
M &= \frac{\lambda^2}{n^3\Omega}\sum_{k=-a}^{a} g(\omega_{\Omega,k}) \sum_{\substack{j_1,j_2,j_3=1\\ j_1\neq j_2\neq j_3}}^{n} \mathrm{E}\left[2e^{is_{j_1}\omega_{\Omega,k}-is_{j_2}\omega_{\Omega,k+r_1}}e^{is_{j_3}\omega_{\Omega,r_2}}c(s_{j_1}-s_{j_3})c(s_{j_2}-s_{j_3})\right] \\
&= \frac{2c_3\lambda^2}{(2\pi)^2\Omega}\sum_{k=-a}^{a} g(\omega_{\Omega,k})\int_{\mathbb{R}^2} f(x)f(y)\times \\
&\quad \mathrm{sinc}\left(\frac{\lambda}{2}(x+\omega_{\Omega,k})\right)\mathrm{sinc}\left(\frac{\lambda}{2}(y-\omega_{\Omega,k+r_1})\right)\mathrm{sinc}\left(\frac{\lambda}{2}(x+y-\omega_{\Omega,r_2})\right)dxdy,
\end{aligned}
$$

where $c_3 = n(n-1)(n-2)/n^3$. Now using a change of variables $u = \frac{\lambda}{2}(x+\omega_{\Omega,k})$ and $v = \frac{\lambda}{2}(y-\omega_{\Omega,k+r_1})$ we have

$$
\begin{aligned}
M &= \frac{2c_3}{\Omega\pi^2}\sum_{k=-a}^{a} g(\omega_{\Omega,k})\int_{\mathbb{R}^2} f\left(\frac{2u}{\lambda}-\omega_{\Omega,k}\right) f\left(\frac{2v}{\lambda}+\omega_{\Omega,k+r_1}\right)\times \\
&\quad \mathrm{sinc}(u)\,\mathrm{sinc}(v)\,\mathrm{sinc}\left(u+v+\frac{\lambda\pi}{\Omega}(r_1-r_2)\right)dudv.
\end{aligned}
$$

Replacing the sum $\frac{1}{\Omega}\sum_k$ with an integral gives

$$
\begin{aligned}
M &= \frac{c_3}{\pi^3}\int_{-2\pi a/\Omega}^{2\pi a/\Omega} g(\omega)\int_{\mathbb{R}^2} f\left(\frac{2u}{\lambda}-\omega\right) f\left(\frac{2v}{\lambda}+\omega+\omega_{\Omega,r_1}\right)\times \\
&\quad \mathrm{sinc}(u)\,\mathrm{sinc}(v)\,\mathrm{sinc}\left(u+v+\frac{\lambda\pi}{\Omega}(r_1-r_2)\right)dudvd\omega + O\left(\frac{1}{\Omega}\right).
\end{aligned}
$$

Finally replacing $f\left(\frac{2u}{\lambda}-\omega\right) f\left(\frac{2v}{\lambda}+\omega+\omega_{\Omega,r_1}\right)$ with $f(-\omega)\,f(\omega+\omega_{\Omega,r_1})$ and using Lemma F.3 gives

$$
\begin{aligned}
M &= \frac{c_3}{\pi^3}\int_{-2\pi a/\Omega}^{2\pi a/\Omega} g(\omega)f(\omega)\,f(\omega+\omega_{\Omega,r_1})\int_{\mathbb{R}^2}\mathrm{sinc}(u)\,\mathrm{sinc}(v)\,\mathrm{sinc}\left(u+v+\frac{\lambda\pi}{\Omega}(r_1-r_2)\right)dudvd\omega + \\
&\quad O\left(\frac{1}{\Omega}+\frac{[\log\lambda+\log\left(1+\frac{\lambda}{\Omega}|r_1-r_2|\right)]^3}{\lambda}\right) \\
&= \frac{c_3}{\pi}\int_{-2\pi a/\Omega}^{2\pi a/\Omega} g(\omega)f(\omega)\,f(\omega+\omega_{\Omega,r_1})\,\mathrm{sinc}\left(\frac{\lambda\pi}{\Omega}(r_1-r_2)\right)d\omega + \\
&\quad O\left(\frac{1}{\Omega}+\frac{[\log\lambda+\log\left(1+\frac{\lambda}{\Omega}|r_1-r_2|\right)]^3}{\lambda}\right).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\lambda\mathrm{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;r_1),V_{\Omega,r_2}\right] \\
&= \frac{c_3}{\pi}\int_{-2\pi a/\Omega}^{2\pi a/\Omega} g(\omega)f(\omega)\,f(\omega+\omega_{\Omega,r_1})\,\mathrm{sinc}\left(\frac{\lambda\pi}{\Omega}(r_1-r_2)\right)d\omega + \\
&\quad O\left(\frac{\lambda}{n}+\frac{1}{\Omega}+\frac{[\log\lambda+\log\left(1+\frac{\lambda}{\Omega}|r_1-r_2|\right)]^3}{\lambda}\right)
\end{aligned}
$$

which proves (H.4). A similar proof can be used to prove (H.5), we omit the details. □

Using the result above and Corollary D.1 (under the assumption the spatial random field is Gaussian) one can easily deduce an expression for $\mathrm{cov}[Q_{a,\Omega,\lambda}(g;\boldsymbol{r}_1),Q_{a,\Omega,\lambda}(g;\boldsymbol{r}_2)]$. However, the expression is long and not so instructive. Thus in the following lemma we restrict outselves to some special cases which are of interest. Define

$$
\begin{aligned}
C_1\left(\frac{a}{\Omega},\boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right) &= \frac{1}{(2\pi)^d}\int_{2\pi[-a/\Omega,a/\Omega]^d} f(\boldsymbol{\omega})f(\boldsymbol{\omega}+\boldsymbol{\omega}_{\Omega,\boldsymbol{r}})|g(\boldsymbol{\omega})|^2 d\boldsymbol{\omega} + \\
&\frac{1}{(2\pi)^d}\int_{\mathcal{D}_{\Omega,\boldsymbol{r}}} f(\boldsymbol{\omega})f(\boldsymbol{\omega}+\boldsymbol{\omega}_{\Omega,\boldsymbol{r}})g(\boldsymbol{\omega})\overline{g(-\boldsymbol{\omega}-\boldsymbol{\omega}_{\Omega,\boldsymbol{r}})}d\boldsymbol{\omega} \\
C_2\left(\frac{a}{\Omega},\boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right) &= \frac{1}{(2\pi)^d}\int_{2\pi[-a/\Omega,a/\Omega]^d} f(\boldsymbol{\omega})f(\boldsymbol{\omega}+\boldsymbol{\omega}_{\Omega,\boldsymbol{r}})g(\boldsymbol{\omega})g(-\boldsymbol{\omega}) d\boldsymbol{\omega} + \\
&\frac{1}{(2\pi)^d}\int_{\mathcal{D}_{\Omega,\boldsymbol{r}}} f(\boldsymbol{\omega})f(\boldsymbol{\omega}+\boldsymbol{\omega}_{\Omega,\boldsymbol{r}})g(\boldsymbol{\omega})g(\boldsymbol{\omega}+\boldsymbol{\omega}_{\Omega,\boldsymbol{r}})d\boldsymbol{\omega}, \\
C_3\left(\frac{a}{\Omega},\boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right) &= \frac{1}{\pi^d}\int_{2\pi[-a/\Omega,a/\Omega]^d} g(\boldsymbol{\omega})f(\boldsymbol{\omega})f(\boldsymbol{\omega}+\boldsymbol{\omega}_{\Omega,\boldsymbol{r}})d\boldsymbol{\omega} \\
C_4\left(\boldsymbol{\omega}_{\Omega,\boldsymbol{r}}\right) &= 2f_2(\boldsymbol{\omega}_{\Omega,\boldsymbol{r}}) \qquad \text{(H.7)}
\end{aligned}
$$

where

$$
f_2(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} f(\boldsymbol{\lambda})f(\boldsymbol{\omega}-\boldsymbol{\lambda})d\boldsymbol{\lambda}
$$

and $\int_{\mathcal{D}_{\Omega,\boldsymbol{r}}} = \int_{2\pi\max(-a,-a-r_1)/\Omega}^{2\pi\min(a,a-r_1)/\Omega}\ldots\int_{2\pi\max(-a,-a-r_d)/\Omega}^{2\pi\min(a,a-r_d)/\Omega}$.

Theorem H.2 *[Asymptotic expression for variance] Suppose Assumptions 2.1, 2.3, 2.5 2.6(b,c) hold. Let* $C_1(\frac{a}{\Omega},\boldsymbol{\omega}_{\boldsymbol{r}})$, $C_2(\frac{a}{\Omega},\boldsymbol{\omega}_{\boldsymbol{r}})$, $C_3(\frac{a}{\Omega},\boldsymbol{\omega}_{\boldsymbol{r}})$, $C_4(\frac{a}{\Omega},\boldsymbol{\omega}_{\boldsymbol{r}})$ *be defined in (H.7). Let* $G_\lambda = \frac{1}{n}\sum_{\boldsymbol{k}=-a}^{a} g(\boldsymbol{\omega}_{\boldsymbol{k}})$, $a^d = O(n)$. *Then we have*

$$
\begin{aligned}
&\lambda^d \mathrm{cov}\left[Q_{a,\lambda}(g;\boldsymbol{r}_1),Q_{a,\lambda}(g;\boldsymbol{r}_2)\right] \\
&= \begin{cases} C_1(\frac{a}{\lambda},\boldsymbol{\omega}_{\lambda,\boldsymbol{r}}) + 2\Re[\overline{G_\lambda}C_3(\frac{a}{\lambda},\boldsymbol{\omega}_{\lambda,\boldsymbol{r}})] + G_4\left(\boldsymbol{\omega}_{\lambda,\boldsymbol{r}}\right)|G_\lambda|^2 + O(\ell_{\lambda,a,n}) & \boldsymbol{r}_1=\boldsymbol{r}_2(=\boldsymbol{r}) \\ O(\ell_{\lambda,a,n}) & \boldsymbol{r}_1\neq\boldsymbol{r}_2 \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&\lambda^d \mathrm{cov}\left[Q_{a,\lambda}(g;\boldsymbol{r}_1),\overline{Q_{a,\lambda}(g;\boldsymbol{r}_2)}\right] \\
&= \begin{cases} C_2(\frac{a}{\lambda},\boldsymbol{\omega}_{\lambda,\boldsymbol{r}}) + 2G_\lambda\Re[C_3(\frac{a}{\lambda},\boldsymbol{\omega}_{\lambda,\boldsymbol{r}})] + G_4(\boldsymbol{\omega}_{\lambda,\boldsymbol{r}})G_\lambda^2 + O(\ell_{\lambda,a,n}) & \boldsymbol{r}_1=-\boldsymbol{r}_2(=\boldsymbol{r}) \\ O(\ell_{\lambda,a,n}) & \boldsymbol{r}_1\neq-\boldsymbol{r}_2 \end{cases},
\end{aligned}
$$

Further, if $\Omega>\lambda$ *(fine frequency grid), then*

$$
\begin{aligned}
&\lambda^d \mathrm{var}\left[Q_{a,\Omega,\lambda}(g;0)\right] \\
&= C_1\left(\frac{a}{\Omega},0\right)\frac{\lambda^d}{\Omega^d}\sum_{\boldsymbol{m}=-2a}^{2a}\mathrm{Sinc}^2\left(\frac{\lambda}{\Omega}\boldsymbol{m}\pi\right) + 2\Re\left[C_3\left(\frac{a}{\Omega},0\right)G_\Omega\right] + G_4(0)|G_\Omega|^2 + O(\widetilde{\ell}_{a,\Omega,\lambda}).
\end{aligned}
$$

If $\Omega \leq \lambda$, then

$$\begin{aligned} & \Omega^d \mathrm{var}\left[Q_{a,\Omega,\lambda}(g;0)\right] \\ = \; & C_1\left(\frac{a}{\Omega},0\right)\sum_{\boldsymbol{m}=-2a}^{2a}\mathrm{Sinc}^2\left(\frac{\lambda}{\Omega}\boldsymbol{m}\pi\right)+\frac{2\Omega^d}{\lambda^d}\Re\left[C_3\left(\frac{a}{\Omega},0\right)G_\Omega\right]+\frac{\Omega^d}{\lambda^d}G_4\left(\frac{a}{\Omega},0\right)|G_\Omega|^2 \\ & +O(\widetilde{\ell}_{a,\Omega,\lambda}), \end{aligned}$$

where $\widetilde{\ell}_{a,\Omega,\lambda}$ is defined in (D.14).

PROOF To prove the result we use (H.1). Expanding $\mathrm{cov}[Q_{a,\Omega,\lambda}(g;\boldsymbol{r}_1), Q_{a,\Omega,\lambda}(g;\Omega,\boldsymbol{r}_2)]$ in terms of the covariances between $\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r})$ and $V_{\Omega,\boldsymbol{r}}$ gives

$$\begin{aligned} & \mathrm{cov}\left[Q_{a,\Omega,\lambda}(g;\boldsymbol{r}_1),Q_{a,\Omega,\lambda}(g;\boldsymbol{r}_2)\right] \\ = \; & \mathrm{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_1),\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_2)\right]+|G_\Omega|^2\mathrm{cov}\left[V_{\Omega,\boldsymbol{r}_1},V_{\Omega,\boldsymbol{r}_2}\right]+ \\ & \overline{G_\Omega}\mathrm{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_1),V_{\Omega,\boldsymbol{r}_2}\right]+G_\Omega\mathrm{cov}\left[V_{\Omega,\boldsymbol{r}_1},\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_2)\right] \end{aligned}$$

$$\begin{aligned} & \mathrm{cov}\left[Q_{a,\Omega,\lambda}(g;\boldsymbol{r}_1),\overline{Q_{a,\Omega,\lambda}(g;\boldsymbol{r}_2)}\right] \\ = \; & \mathrm{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_1),\overline{\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_2)}\right]+G_\Omega^2\mathrm{cov}\left[V_{\Omega,\boldsymbol{r}_1},\overline{V_{\Omega,\boldsymbol{r}_2}}\right]+ \\ & G_\Omega\mathrm{cov}\left[\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_1),\overline{V_{\Omega,\boldsymbol{r}_2}}\right]+G_\Omega\mathrm{cov}\left[V_{\Omega,\boldsymbol{r}_1},\overline{\widetilde{Q}_{a,\Omega,\lambda}(g;\boldsymbol{r}_2)}\right]. \end{aligned}$$

In Corollary D.1 and Lemma H.1 expressions for each of the terms above are given. By substituting these expressions into the above we obtain the result. □

In the following lemma we show that we further simplify the expression for the asymptotic variance (in the case $\boldsymbol{r}$ is fixed).

Corollary H.1 *Suppose Assumption 2.5, 2.6(a,c) or 2.6(b,c) holds, and $\boldsymbol{r}$ is fixed. Let $C_3(\frac{a}{\Omega},\boldsymbol{\omega})$ and $C_4(\boldsymbol{\omega})$ be defined in (H.7). Then we have*

$$C_3(\frac{a}{\lambda},\boldsymbol{\omega}_{\lambda,\boldsymbol{r}}) = C_3 + O\left(\frac{\|\boldsymbol{r}\|_1}{\lambda}\right),$$

and $C_4(\boldsymbol{\omega}_{\lambda,\boldsymbol{r}}) = C_4 + O\left(\frac{\|\boldsymbol{r}\|_1}{\lambda}\right)$, where

$$C_3 \;=\; \frac{2}{(2\pi)^d}\int_{2\pi[-a/\lambda,a/\lambda]^d} g(\boldsymbol{\omega})f(\boldsymbol{\omega})^2 d\boldsymbol{\omega}$$

and $C_4 = C_4(0)$.

PROOF The proof is the same as the proof of Corollary 4.1. □

Theorem H.3 *[CLT on real and imaginary parts] Suppose Assumptions 2.1, 2.3, 2.5 and 2.6(b,c) hold. Let C_1 and C_2 be defined as in Corollary 4.1 and C_3 and C_4 be defined as in Corollary H.1. We define the m-dimension complex random vectors $\boldsymbol{Q}_m = (Q_{a,\lambda}(g,\boldsymbol{r}_1),\ldots,Q_{a,\lambda}(g,\boldsymbol{r}_m))$, where $\boldsymbol{r}_1,\ldots,\boldsymbol{r}_m$ are such that $\boldsymbol{r}_i \neq -\boldsymbol{r}_j$ and $\boldsymbol{r}_i \neq 0$. Under these conditions we have*

$$\frac{2\lambda^{d/2}}{E_1}\left(\frac{E_1}{E_1+\Re E_2}\Re Q_{a,\lambda}(g,0),\Re\boldsymbol{Q}_m,\Im\boldsymbol{Q}_m\right)\overset{\mathcal{P}}{\to}\mathcal{N}\big(0,I_{2m+1}\big), \tag{H.8}$$

where $E_1 = C_1 + 2\Re[G_\lambda C_3] + C_4|G_\lambda|^2$ and $E_2 = C_2 + 2G_\lambda\Re C_3 + C_4G_\lambda^2$ with $\frac{\lambda^d}{n}\to 0$ and $\frac{\log^2(a)}{\lambda^{1/2}}\to 0$ as $\lambda\to\infty$, $n\to\infty$ and $a\to\infty$.

PROOF Using the same method used to prove Lemma E.5, and analogous results can be derived for the cumulants of $Q_{a,\lambda}(g;\boldsymbol{r})$. Asymptotic normality follows from this. We omit the details. □

Application to nonparametric covariance estimator

In this section we apply the results to the nonparametric estimator considered in Section 2.3.3. We define

$$\widetilde{c}_{\Omega,n}(\boldsymbol{u}) = \frac{1}{\Omega^d}\sum_{\boldsymbol{k}=-a}^{a}|J_n(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})|^2\exp(i\boldsymbol{u}'\boldsymbol{\omega}_{\Omega,\boldsymbol{k}}) \text{ and } \widehat{c}_{,\Omega,n}(\boldsymbol{u}) = T\left(\frac{\boldsymbol{u}}{\widehat{\Omega}}\right)\widetilde{c}_{\Omega,n}(\boldsymbol{u})$$

where T is the d-dimensional triangle kernel. It is clear the asymptotic sampling properties of $\widehat{c}_{\Omega,n}(\boldsymbol{u})$ are determined by $\widetilde{c}_{\Omega,n}(\boldsymbol{u})$. Therefore, we first derive the asymptotic sampling properties of $\widetilde{c}_{\Omega,n}(\boldsymbol{u})$. We observe that $\widetilde{c}_{\Omega,n}(\boldsymbol{u}) = Q_{a,\Omega,\lambda}(e^{i\boldsymbol{v}'\cdot};0)$, thus we use the results in Section 4 to derive the asymptotic sampling properties of $\widetilde{c}_{\Omega,n}(\boldsymbol{u})$. By using Theorem 3.1 and under Assumptions 2.1, 2.3 and 2.5(ii) we have

$$\begin{aligned}\mathrm{E}[\widetilde{c}_{\Omega,n}(\boldsymbol{u})] &= \frac{1}{(2\pi)^d}\int_{2\pi[-a/\Omega,a/\Omega]^d}f(\boldsymbol{\omega})\exp(i\boldsymbol{u}'\boldsymbol{\omega})d\boldsymbol{\omega}+O\left(\frac{\log\lambda}{\lambda}\right)\\ &= c(\boldsymbol{u})+O\left(\left(\frac{\Omega}{a}\right)^{\delta}+\frac{\log\lambda}{\lambda}\right),\end{aligned} \tag{H.9}$$

for $\boldsymbol{u}\in[-\lambda/2,\lambda/2]^d$ (if $\Omega=\lambda$) else for $\boldsymbol{u}\in[-\lambda,\lambda]^d$ if $\Omega\geq 2\lambda$. We recall that δ is such that it satisfies Assumption 2.5(ii)(a). Using Theorem H.2, we have $\lambda^d\mathrm{var}\left[\widetilde{Q}_{a,\lambda,\Omega}(e^{i\boldsymbol{u}'\cdot};0)\right] = \Sigma(\frac{a}{\Omega};\boldsymbol{u})+O\left(\widetilde{\ell}_{a,\lambda,\Omega}\right)$ where

$$\begin{aligned}\Sigma\left(\frac{a}{\Omega};\boldsymbol{u}\right) &= \left[\frac{1}{(2\pi)^d}\int_{[-2\pi a/\Omega,2\pi a/\Omega]^d}f(\boldsymbol{\omega})^2\left[1+\exp(2i\boldsymbol{u}'\boldsymbol{\omega})\right]d\boldsymbol{\omega}\right]\left[\frac{\lambda^d}{\Omega^d}\sum_{\boldsymbol{k}=-2a}^{2a}\mathrm{Sinc}^2\left(\frac{\lambda}{\Omega}\boldsymbol{k}\pi\right)\right]+\\ &\quad\frac{2}{\pi^d}\int_{[-2\pi a/\Omega,2\pi a/\Omega]^d}\exp(i\boldsymbol{u}'\boldsymbol{\omega})f(\boldsymbol{\omega})^2d\boldsymbol{\omega}+2f_2(0)\left|\frac{\lambda^d}{\Omega^d n}\sum_{\boldsymbol{k}=-a}^{a}\exp(i\boldsymbol{u}'\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})\right|^2.\end{aligned}$$

Therefore, if $\Omega^{\delta}\lambda^{d/2}/a^{\delta} \to 0$ as $a \to \infty$ and $\lambda \to \infty$, then by using Theorem H.3 we have

$$\lambda^{d/2}\left[\widetilde{c}_{\Omega,n}(\boldsymbol{u}) - c(\boldsymbol{u})\right] \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \Sigma\left(\frac{a}{\Omega}; \boldsymbol{u}\right)\right), \tag{H.10}$$

$\frac{\lambda^d}{n} \to 0$ and $\frac{\log^2(a)}{\lambda^{1/2}} \to 0$ as $n \to \infty$, $a \to \infty$ and $\lambda \to \infty$. By using (H.9) and (H.10) we have

$$\mathrm{E}[\widehat{c}_{\Omega,n}(\boldsymbol{u})] = T\left(\frac{\boldsymbol{u}}{\widehat{\Omega}}\right) c(\boldsymbol{u}) + O\left(\widetilde{\ell}_{a,\lambda,\Omega}\right).$$

Application to parameter estimation using an L_2 criterion

In this section we consider the asymptotic sampling properties of $\widehat{\theta}_n = \arg\min_{\theta\in\Theta} L_n(\theta)$, where $L_n(\cdot)$ is defined in Section 2.3.4 and Θ is a compact set. We will assume that there exists a $\theta_0 \in \Theta$, such that for all $\boldsymbol{\omega} \in \mathbb{R}^d$, $f_{\theta_0}(\boldsymbol{\omega}) = f(\boldsymbol{\omega})$ and there does not exist another $\theta \in \Theta$ such that for all $\boldsymbol{\omega} \in \mathbb{R}^d$ $f_{\theta_0}(\boldsymbol{\omega}) = f_\theta(\boldsymbol{\omega})$ and in addition $\int_{\mathbb{R}^d} \|\nabla_\theta f(\boldsymbol{\omega};\theta_0)\|_1^2 d\boldsymbol{\omega} < \infty$. Furthermore, we will assume that $\widehat{\theta}_n \xrightarrow{\mathcal{P}} \theta_0$ as $\lambda \to \infty$.

Making the usual Taylor expansion we have $\lambda^{d/2}(\widehat{\theta}_n - \theta_0) = A^{-1}\lambda^{d/2}\frac{1}{2}\nabla L_n(\theta_0) + o_p(1)$, where

$$A = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} [\nabla_\theta f_{\theta_0}(\boldsymbol{\omega})][\nabla_{\theta_0} f(\boldsymbol{\omega})]' d\boldsymbol{\omega}, \tag{H.11}$$

and it is clear the asymptotic sampling properties of $\widehat{\theta}_n$ are determined by $\nabla_\theta L_n(\theta_0)$, which we see from (2.7) can be written as

$$\begin{aligned} &\frac{1}{2}\nabla_\theta L_n(\theta_0) \\ = \;& \widetilde{Q}_{a,\lambda}(-\nabla_\theta f_{\theta_0}(\cdot);0) - V_\lambda \underbrace{\frac{1}{n}\sum_{\boldsymbol{k}=-a}^{a}\nabla_\theta f_{\theta_0}(\boldsymbol{\omega}_{\boldsymbol{k}})}_{G_\lambda} + \frac{1}{\lambda^d}\sum_{\boldsymbol{k}=-a}^{a} f_{\theta_0}(\boldsymbol{\omega}_{\boldsymbol{k}})\nabla_\theta f_{\theta_0}(\boldsymbol{\omega}_{\boldsymbol{k}}). \end{aligned}$$

Thus by using Theorem H.3 we have $\lambda^{d/2}\nabla_\theta \frac{1}{2}L_n(\theta_0) \xrightarrow{\mathcal{D}} \mathcal{N}(0,B)$, where

$$\begin{aligned} B \;=\; & \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} f(\boldsymbol{\omega})^2 \left[\nabla_\theta f_\theta(\boldsymbol{\omega})\right]\left[\nabla_\theta f_\theta(\boldsymbol{\omega})\right]'\rfloor_{\theta=\theta_0} d\boldsymbol{\omega} \\ & + \frac{2G_\lambda}{\pi^d}\int_{\mathbb{R}^d} f(\boldsymbol{\omega})^2 \nabla_\theta f_{\theta_0}(\boldsymbol{\omega})' d\boldsymbol{\omega} + 2G_\lambda G_\lambda' f_2(0) \end{aligned}$$

and $G_\lambda = \frac{1}{n}\sum_{\boldsymbol{k}=-a}^{a}\nabla_\theta f_{\theta_0}(\boldsymbol{\omega}_{\boldsymbol{k}})$. Therefore, by using the above we have

$$\lambda^{d/2}(\widehat{\theta}_n - \theta_0) \xrightarrow{\mathcal{P}} \mathcal{N}(0, A^{-1}BA^{-1})$$

with $\frac{\lambda^d}{n} \to 0$ and $\frac{\log^2(a)}{\lambda^{1/2}} \to 0$ as $a \to \infty$ and $\lambda \to \infty$.

I Additional proofs

In this section we prove the remaining results required in this paper.

For example, Theorems 4.1, 4.2, 4.3(i,iii), 4.6 and B.2, involve replacing sums with integrals. In the case that the frequency grid is unbounded stronger assumptions are required than in the case the frequency grid is fixed. We state the required result in the following lemma.

Lemma I.1 *Let us suppose the function g_1, g_2 are bounded ($\sup_{\boldsymbol{\omega}\in\mathbb{R}^d}|g_1(\boldsymbol{\omega})| < \infty$ and $\sup_{\boldsymbol{\omega}\in\mathbb{R}^d}|g_2(\boldsymbol{\omega})| < \infty$) and for all $1 \leq j \leq d$, $\sup_{\boldsymbol{\omega}\in\mathbb{R}^d}|\frac{\partial g_1(\boldsymbol{\omega})}{\partial \omega_j}| < \infty$ and $\sup_{\boldsymbol{\omega}\in\mathbb{R}^d}|\frac{\partial g_2(\boldsymbol{\omega})}{\partial \omega_j}| < \infty$.*

(i) Suppose $a/\Omega = C$ (where C is a fixed finite constant) and h is a bounded function whose first partial derivative $1 \leq j \leq d$, $\sup_{\boldsymbol{\omega}\in\mathbb{R}^d}|\frac{\partial h(\boldsymbol{\omega})}{\partial \omega_j}| < \infty$. Then we have

$$\left|\frac{1}{\Omega^d}\sum_{\boldsymbol{k}=-a}^{a} g_1(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})h(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}}) - \frac{1}{(2\pi)^d}\int_{2\pi[-C,C]^d} g_1(\boldsymbol{\omega})h(\boldsymbol{\omega})\boldsymbol{\omega}\right| \leq K\Omega^{-1},$$

where K is a finite constant independent of Ω.

(ii) Suppose $a/\Omega \to \infty$ as $\Omega \to \infty$. Furthermore, $h(\boldsymbol{\omega}) \leq \beta_{1+\delta}(\boldsymbol{\omega})$ and for all $1 \leq j \leq d$ the partial derivatives satisfy $|\frac{\partial h(\boldsymbol{\omega})}{\partial \omega_j}| \leq \beta_{1+\delta}(\boldsymbol{\omega})$. Then uniformly over a we have

$$\left|\frac{1}{\Omega^d}\sum_{k=-a}^{a} g_1(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}})h\left(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}}\right) - \frac{1}{(2\pi)^d}\int_{2\pi[-a/\Omega,a/\Omega]^d} g_1(\boldsymbol{\omega})h\left(\boldsymbol{\omega}\right)d\boldsymbol{\omega}\right| \leq K\Omega^{-1}$$

(iii) Suppose $a/\Omega \to \infty$ as $\Omega \to \infty$. Furthermore, $f_4(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3) \leq \beta_{1+\delta}(\boldsymbol{\omega}_1)\beta_{1+\delta}(\boldsymbol{\omega}_2)\beta_{1+\delta}(\boldsymbol{\omega}_3)$ and for all $1 \leq j \leq 3d$ the partial derivatives satisfy $|\frac{\partial f_4(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\boldsymbol{\omega}_3)}{\partial \omega_j}| \leq \beta_{1+\delta}(\boldsymbol{\omega})$.

$$\begin{aligned}&\left|\frac{1}{\Omega^{2d}}\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2=-a}^{a} g_1(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}_1})g_2(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}_2})f_4\left(\boldsymbol{\omega}_{\Omega,\boldsymbol{k}_1+\boldsymbol{r}_1}, \boldsymbol{\omega}_{\Omega,\boldsymbol{k}_2}, \boldsymbol{\omega}_{\Omega,\boldsymbol{k}_2+\boldsymbol{r}_2}\right) - \right.\\ &\left.\quad\frac{1}{(2\pi)^{2d}}\int_{2\pi[-a/\Omega,a/\Omega]^d}\int_{2\pi[-a/\Omega,a/\Omega]^d} g_1(\boldsymbol{\omega}_1)g_2(\boldsymbol{\omega}_2)f_4\left(\boldsymbol{\omega}_1+\boldsymbol{\omega}_{\boldsymbol{r}_1}, \boldsymbol{\omega}_2, \boldsymbol{\omega}_2+\boldsymbol{\omega}_{\boldsymbol{r}_2}\right)d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2\right|\\ &\leq K\Omega^{-1}.\end{aligned}$$

PROOF. We first prove the result in the univariate case. We expand the difference between sum and integral

$$\begin{aligned}&\frac{1}{\Omega}\sum_{k=-a}^{a} g_1(\omega_{\Omega,k})h(\omega_{\Omega,k}) - \frac{1}{2\pi}\int_{-2\pi a/\Omega}^{2\pi a/\Omega} g_1(\omega)h(\omega)d\omega\\ =\;&\frac{1}{\Omega}\sum_{k=-a}^{a} g_1(\omega_{\Omega,k})h(\omega_{\Omega,k}) - \frac{1}{2\pi}\sum_{k=-a}^{a-1}\int_{0}^{2\pi/\Omega} g_1(\omega_{\Omega,k}+\omega)h(\omega_{\Omega,k}+\omega)d\omega.\end{aligned}$$

By applying the mean value theorem for integrals to the integral above we have

$$\begin{aligned}
&= \frac{1}{\Omega}\sum_{k=-a}^{a} g_1(\omega_{\Omega,k})h(\omega_{\Omega,k}) - \frac{1}{2\pi}\sum_{k=-a}^{a-1}\int_0^{2\pi/\Omega} g_1(\omega_{\Omega,k}+\omega)h(\omega_{\Omega,k}+\omega)d\omega \\
&= \frac{1}{\Omega}\sum_{k=-a}^{a-1}\left[g_1(\omega_{\Omega,k})h(\omega_{\Omega,k}) - g_1(\omega_{\Omega,k}+\overline{\omega_{\Omega,k}})h(\omega_{\Omega,k}+\overline{\omega_{\Omega,k}})\right] + \frac{1}{\Omega}g_1(\omega_{\Omega,a})h(\omega_{\Omega,a})
\end{aligned}$$

where $\overline{\omega_{\Omega,k}} \in [0, \frac{2\pi}{\Omega}]$. Next, by applying the mean value theorem to the difference above we have

$$\begin{aligned}
&\left|\frac{1}{\Omega}\sum_{k=-a}^{a}\left[g_1(\omega_{\Omega,k})h(\omega_{\Omega,k}) - g_1(\omega_{\Omega,k}+\overline{\omega_{\Omega,k}})h(\omega_{\Omega,k}+\overline{\omega_{\Omega,k}})\right]\right| \\
\leq\ & \frac{1}{\Omega^2}\sum_{k=-a}^{a}\left[g_1'(\widetilde{\omega}_{\Omega,k})h(\widetilde{\omega}_{\Omega,k}) + g_1(\widetilde{\omega}_{\Omega,k})h'(\widetilde{\omega}_{\Omega,k})\right] \qquad \text{(I.1)}
\end{aligned}$$

where $\widetilde{\omega}_{\Omega,k} \in [\omega_{\Omega,k}, \omega_{\Omega,k}+\omega]$ (note this is analogous to the expression given in (Brillinger, 1981), Exercise 1.7.14).

Under the condition that $a = C\Omega$, $g_1(\omega)$ and $h(\omega)$ are bounded and using (I.1) it is clear that

$$\begin{aligned}
&\left|\frac{1}{\Omega}\sum_{k=-a}^{a} g_1(\omega_{\Omega,k})h(\omega_{\Omega,k}) - \frac{1}{2\pi}\int_{-2\pi a/\Omega}^{2\pi a/\Omega} g_1(\omega)h(\omega)d\omega\right| \\
\leq\ & \frac{\sup_\omega|h'(\omega)g_1(\omega)| + \sup_\omega|h(\omega)g_1'(\omega)|}{\Omega^2}\sum_{k=-a}^{a} 1 = C\Omega^{-1}.
\end{aligned}$$

For $d = 1$, this proves (i).

In the case that $a/\Omega \to \infty$ as $\Omega \to \infty$, we use that h and h' are dominated by a montonic function and that g_1 is bounded. Thus by using (I.1) we have

$$\begin{aligned}
&\left|\frac{1}{\Omega}\sum_{k=-a}^{a} g_1(\omega_{\Omega,k})h(\omega_{\Omega,k}) - \frac{1}{2\pi}\int_{-2\pi a/\Omega}^{2\pi a/\Omega} g_1(\omega)h(\omega)d\omega\right| \\
\leq\ & \frac{1}{\Omega^2}\sum_{k=-a}^{a}\sup_{\omega_{\Omega,k}\leq\omega\leq\omega_{\Omega,k+1}}\left(|g_1'(\omega)h(\omega)| + |g_1(\omega)h'(\omega)|\right) \\
\leq\ & \frac{2(\sup_\omega|g_1(\omega)| + \sup_\omega|g_1'(\omega)|)}{\Omega^2}\sum_{k=0}^{a}\beta_{1+\delta}(\omega_{\Omega,k}) \leq \frac{C}{\Omega}\int_0^\infty \beta_{1+\delta}(\omega)d\omega = O(\Omega^{-1}).
\end{aligned}$$

For $d = 1$, this proves (ii).

To prove the result for $d=2$ we take differences

$$
\begin{aligned}
&\frac{1}{\Omega^2}\sum_{k_1=-a}^{a}\sum_{k_2=-a}^{a} g(\omega_{\Omega,k_1},\omega_{\Omega,k_2})h(\omega_{\Omega,k_1},\omega_{\Omega,k_2}) - \frac{1}{(2\pi)^2}\int_{-2\pi a/\Omega}^{a/\Omega}\int_{-2\pi a/\Omega}^{2\pi a/\Omega} g(\omega_1,\omega_2)h(\omega_1,\omega_2)d\omega_1 d\omega_2 \\
= &\frac{1}{\Omega}\sum_{k_1=-a}^{a}\left(\frac{1}{\Omega}\sum_{k_2=-a}^{a} g(\omega_{\Omega,k_1},\omega_{\Omega,k_2})h(\omega_{\Omega,k_1},\omega_{\Omega,k_2}) - \frac{1}{2\pi}\int_{-2\pi a/\Omega}^{2\pi a/\Omega} g(\omega_{k_1},\omega_2)h(\omega_{k_1},\omega_2)d\omega_2\right) \\
&+\frac{1}{2\pi}\int_{-2\pi a/\Omega}^{2\pi a/\Omega}\left(\frac{1}{\lambda}\sum_{k_1=-a}^{a} g(\omega_{k_1},\omega_2)h(\omega_{k_1},\omega_2) - \frac{1}{2\pi}\int_{-2\pi a/\Omega}^{2\pi a/\Omega} g(\omega_1,\omega_2)h(\omega_1,\omega_2)d\omega_2\right)d\omega_1.
\end{aligned}
$$

For each of the terms above we apply the method described for the case $d=1$; for the first term we take the partial derivative over ω_2 and the for the second term we take the partial derivative over ω_1. This method can easily be generalized to the case $d>2$. The proof of (iii) is identical to the proof of (ii).

We mention that the assumptions on the derivatives (used replace sum with integral) can be relaxed to that of bounded variation of the function. However, since we require the bounded derivatives to decay at certain rates (to prove other results) we do not relax the assumption here. □

PROOF of Theorem 5.1 Making the classical variance-bias decomposition we have

$$
\mathrm{E}\left(\widetilde{V}_{\mathcal{S}} - \lambda^d \mathrm{var}[\widetilde{Q}_{a,\lambda}(g;0)]\right)^2 = \mathrm{var}[\widetilde{V}_{\mathcal{S}}] + \left(\mathrm{E}[\widetilde{V}_{\mathcal{S}}] - \lambda^d\mathrm{var}[\widetilde{Q}_{a,\lambda}(g;0)]\right)^2.
$$

We first analysis the bias term, in particular $\mathrm{E}[\widetilde{V}_{\mathcal{S}}]$. We note that by using the expectation and variance result in Theorems 4.1 and B.1 respectively, we have

$$
\begin{aligned}
\mathrm{E}[\widetilde{V}_{\mathcal{S}}] &= \frac{\lambda^d}{|\mathcal{S}|}\sum_{\boldsymbol{r}\in\mathcal{S}}\mathrm{var}[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})] + \frac{\lambda^d}{|\mathcal{S}|}\sum_{\boldsymbol{r}\in\mathcal{S}}\underbrace{\left|\mathrm{E}[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})]\right|^2}_{=O(\lambda^{-2d}\prod_{j=1}^{d}(\log\lambda+\log|\boldsymbol{r}_j|)^2)} \\
&= \frac{1}{|\mathcal{S}|}\sum_{\boldsymbol{r}\in\mathcal{S}} C_1(\boldsymbol{\omega_r}) + O\left(\ell_{\lambda,a,n} + \frac{[\log\lambda+\log M]^d}{\lambda^d}\right) \\
&= C_1 + O\left(\ell_{a,\lambda,n} + \frac{[\log\lambda+\log M]^d}{\lambda^d} + \frac{|M|}{\lambda}\right).
\end{aligned}
$$

Next we consider $\mathrm{var}[\widetilde{V}_{\mathcal{S}}]$, by using the classical cumulant decomposition we have

$$
\begin{aligned}
\mathrm{var}[\widetilde{V}_{\mathcal{S}}] &= \frac{\lambda^{2d}}{|\mathcal{S}|^2}\sum_{\boldsymbol{r}_1,\boldsymbol{r}_2\in\mathcal{S}}\left(\left|\mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1),\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)\right]\right|^2 + \left|\mathrm{cov}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1),\overline{\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)}\right]\right|^2\right. \\
&\quad +\frac{\lambda^{2d}}{|\mathcal{S}|^2}\sum_{\boldsymbol{r}_1,\boldsymbol{r}_2\in\mathcal{S}}\mathrm{cum}\left(\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1),\overline{\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1)},\overline{\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)},\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)\right) \\
&\quad +O\left(\frac{\lambda^{2d}}{|\mathcal{S}|^2}\sum_{\boldsymbol{r}_1,\boldsymbol{r}_2\in\mathcal{S}}\left|\mathrm{cum}\left(\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1),\overline{\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_1)},\overline{\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)}\right)\mathrm{E}\left(\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r}_2)\right)\right|\right).
\end{aligned}
$$

By substituting the variance/covariance results for $\widetilde{Q}_{a,\lambda}(\cdot)$ based on uniformly sampled locations in Theorem B.1 and the cumulant bounds in Lemma E.5 into the above we have

$$\mathrm{var}[\widetilde{V}_{\mathcal{S}}] = \frac{\lambda^{2d}}{|\mathcal{S}|^2}\sum_{\boldsymbol{r}\in\mathcal{S}}\left|\mathrm{var}\left[\widetilde{Q}_{a,\lambda}(g;\boldsymbol{r})\right]\right|^2 + O\left(\ell_{\lambda,a,n}+\frac{\log^{4d}(a)}{\lambda^d}\right) = O\left(\frac{1}{|\mathcal{S}|}+\ell_{\lambda,a,n}+\frac{\log^{4d}(a)}{\lambda^d}\right).$$

Thus altogether we have the result. □

J Simulations

In this section we illustrate the performance of the nonparametric non-negative definite estimator of the spatial covariance defined in Section 2.3.3. We compare our method to the nonparametric estimator proposed in (Hall et al., 1994). We conduct all the simulations for $d = 1$ and the observations are observed over the spatial domain $[-20, 20]$ ($\lambda = 40$).

We use the estimator

$$\widehat{c}_{a,\Omega,n}(u) = T\left(\frac{u}{\widehat{\Omega}}\right)\widetilde{c}_{\Omega,n}(u),$$

where

$$\widetilde{c}_{a,\Omega,n}(u) = \frac{1}{\Omega}\sum_{k=-a}^{a}\left|J_n\left(\omega_{k,\Omega}\right)\right|^2\exp(iu\omega_{k,\Omega}).$$

Since $\lambda = 40$ we use $\Omega = 80$ and $\widehat{\Omega} = 40$ to construct the estimator.

To evaluate the covariance estimator proposed in (Hall et al., 1994) (from now on referred to as the HFH estimator) we use the kernel method proposed in (Hall et al., 1994) to estimate the covariance $c(u)$ at $u = 0, 0.5, 1, 1.5, \ldots, 30$, we denote this estimator as $\widetilde{c}_{HRH}(u)$. As suggested by (Hall et al., 1994) for $u = 30, 30.5\ldots, 35$ we taper the covariance to zero and let $\widetilde{c}_{HRH}(u) = (1-\frac{|u|-30}{5})\widehat{c}_{HRH}(30)$ for $u \in [35, 40]$ and let $\widetilde{c}_{HRH}(u) = 0$ for $u \in [35, 40]$. To ensure the estimator is non-negative definite we evaluate the Fourier transform of $\{\widetilde{c}_{HRH}(u); u = 0, 0.5, 1, \ldots, 40\}$ and set negative values to zero and invert the Fourier transform. The result is a non-negative covariance estimator of the spatial covariance sequence. We denote this estimator as $\widehat{c}_{HRH}(u)$.

In the simulations we simulate from a spatial Gaussian random field with autocovariance $c(u) = \exp(-|u|/R)$ (R denotes the range parameter) and use $R = 2$ (range is 5% of the field), $R = 5$ (range is $R = 12.5$% of the field) and $R = 10$ (range is 25% of the field). We also compare sample sizes $n = 1000$ and $n = 2000$. In Figure 1 we make a plot of $\{|J_n(\omega_{80,k})|^2\}_{k=0}^{150}$ against $\{\omega_{80,k}\}_{k=0}^{150}$ for $R = 2, 5, 10$ and $n = 1000, 2000$. The locations $\{u_j\}_{j=1}^{n}$ are sampled from a uniform distribution. We observe that since $J_n(\omega)$ is being sampled on a fine frequency grid ($2\pi/(2\times 40)$ compared with $2\pi/40$) adjacent values of $J_n(\omega_{80,k})$ are highly correlated.

Furthermore, as expected, when $R = 10$ the periodogram drops close to zero "faster" than when $R = 2$. However, in all cases we see that the amplitude of $|J_n(\omega_{80,k})|^2$ drops to close zero when a is larger than 100. A plot of the periodogram can be used to determine a and in Figure 2 we give a plot of the estimator $\widehat{c}_{a,80,k}(u)$ for $a = 50$ and $a = 150$ together with the (Hall et al., 1994) estimator, $\widehat{c}_{HRH}(u)$. The plots are for $R = 2, 5, 10$ and $n = 1000$ and $n = 2000$. All the estimators are evaluated at $u = 0, 0.5, \ldots, 20$.

In order to compare the estimators we conduct a simulation study using the specifications given above, where 500 replications are made for each (R, n)-pair. To understand how the choice of a influences the estimator we evaluate $\widehat{c}_{a,\Omega,n}(u)$ for $a = 50, 100, 150, 200$. We also evaluate $\widehat{c}_{HFH}(u)$ (defined in (Hall et al., 1994)). The simulations are done over 500 replications. For each simulation (for a given range parameter and sample size) we calculate the square root average squared error (SASE), the average (Ave) and the (g)lobal SASE (for $u = 0, 0.5, \ldots, 20$). They are defined as

$$\begin{aligned}
\text{SASE}(u) &= \sqrt{\frac{1}{N}\sum_{j=1}^{N}\{\widehat{c}^{(j)}(u) - c(u)\}^2}, \text{Ave}(u) = \frac{1}{N}\sum_{j=1}^{N}\widehat{c}^{(j)}(u) \\
\text{and gSASE} &= \frac{1}{41}\sum_{i=0}^{40}\text{SASE}\left(\frac{i}{2}\right),
\end{aligned}$$

where $\widehat{c}^{(j)}(u)$ denotes the estimator based on the jth replication.

The results of the simulations are reported in Tables 1-6. We observe that for most values of a the estimator $\widehat{c}_{a,\Omega,n}$ seems to perform better than the HRH estimator. The sensitivity of the estimator to the choice of a depends on the location the covariance is estimating. We observe that when $n = 2000$ the SASE is to roughly the same for all choices of a and all range parameters. However, when $n = 1000$ for the estimator at $u = 0$ the SASE is larger for larger choices of a (this is *not* seen for estimators at other values of u, except when the range parameter is $R = 10$, when something similar is also seen at $u = 2$). An explanation can be found from the way in which $\widetilde{c}_{a,\Omega,n}(u)$ is defined. We recall that

$$\widetilde{c}_{a,\Omega,n}(u) = \widetilde{Q}_{a,\Omega,\lambda}(e^{iu\cdot}) + \left(\frac{1}{n}\sum_{j=1}^{n} Z(\boldsymbol{s}_j)^2\right) \times \left(\frac{1}{n}\sum_{k=-a}^{a} e^{iu2\pi k/\Omega}\right).$$

The second term on the right hand side of the above is the "so called" finite bias its expectation is approximately equal to

$$\frac{\sigma^2\Omega}{2\pi n}\int_{-2\pi a/\lambda}^{2\pi a/\lambda} e^{iu\omega} d\omega.$$

If n is large or $u \neq 0$ the above bias will be close to zero, however for small n and $u = 0$ then the above may be quite large which may explain the effect that we see. Despite this the

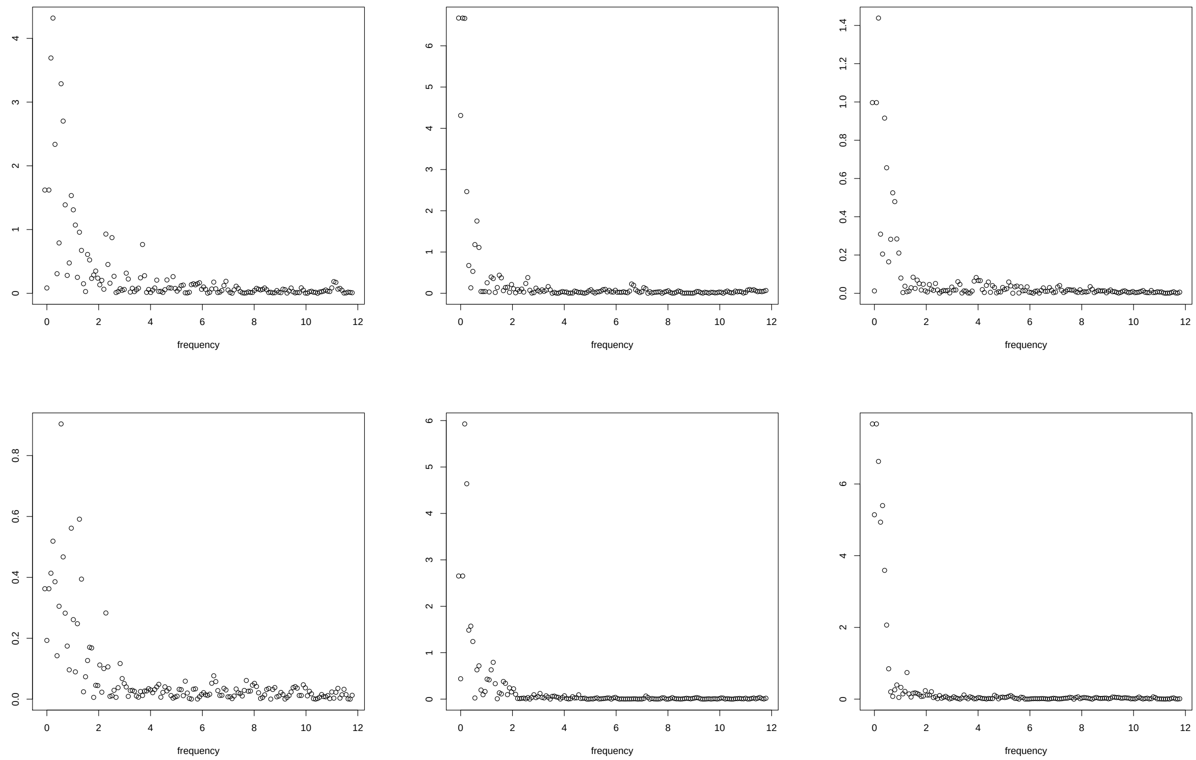

Figure 1: Plot of $|J_n(\omega_{80,k})|^2$. Top for $n = 1000$ and Left to Right $R = 2, 5$ and 10. Bottom for $n = 2000$ and Left to Right $R = 2, 5$ and 10.

estimator is not overly sensitive to the choice of a, since we are comparing the performance of the estimator over a very wide range of a ($a = 50 - 200$). It seems that the best way to choose a is simply to make a plot of the periodogram (similar to Figure 1) and select the a where most of the amplitudes drop close to zero.

To understand the effect sample size, n, has on the estimation scheme simulations were conducted for $R = 2$ and $n = 1000, 2000, 4000, 6000$. We focussed on $a = 50$ and 100 and also evaluated the HFH estimator. 100 replications were done for each (R, n)-pair. The results are reported in Table 7. As the sample size increases there does not seem to any real change in the SASE. This observations is supported by the theory developed in this paper.

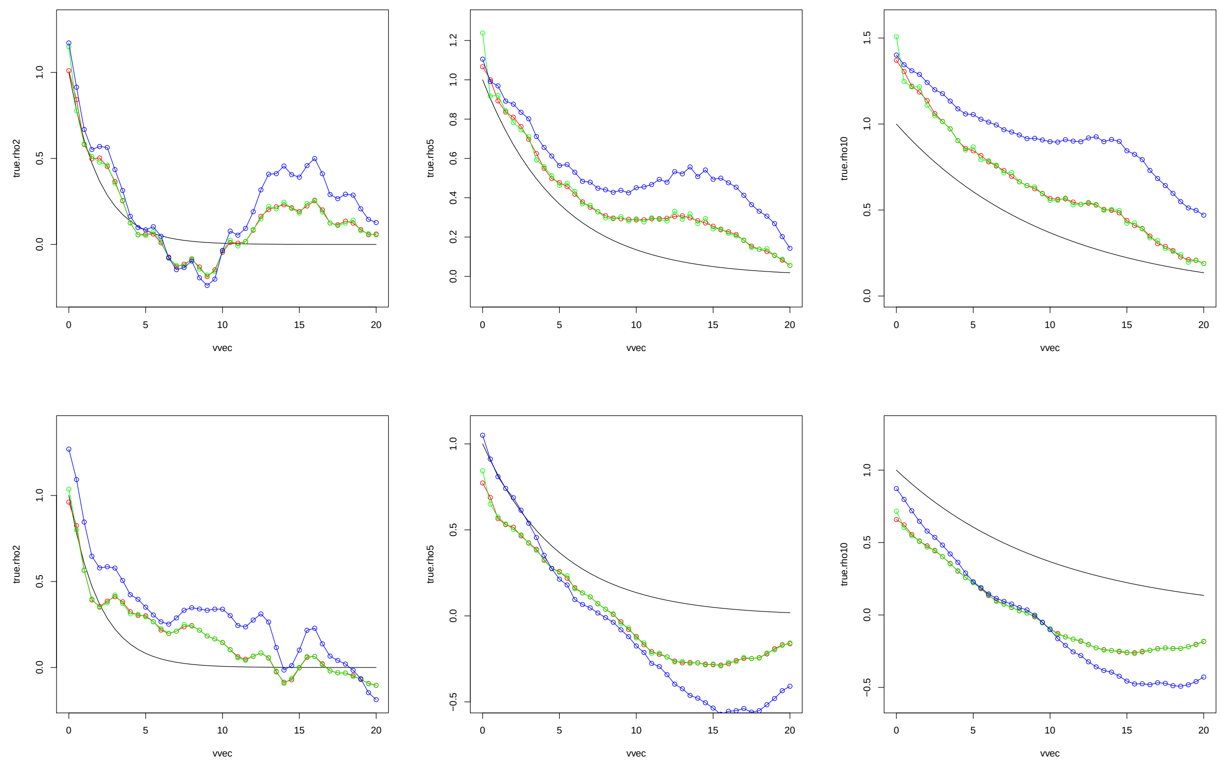

Figure 2: Plot of $\widehat{c}_{a,80,n}(u)$ and $\widehat{c}_{HFH}(u)$ evaluated at $u = 0, 0.5, 1, \ldots, 20$. The Black line is the true autocovariance $\exp(-|u|/R)$, Red line is $\widehat{c}_{50,80,n}(u)$ $(a = 50)$. Green line is $\widehat{c}_{150,80,n}(u)$ $(a = 150)$. Blue line is $\widehat{c}_{HRH}(u)$. Left to Right $R = 2, 5$ and 10. Top for $n = 1000$ and bottom for $n = 2000$.

Method		$c(0) = 1$	$c(2) = 0.3678$	$c(4) = 0.1353$	$c(6) = 0.0497$	$c(8) = 0.0183$	$c(10) = 0.0067$	gSASE
$a = 50$	SASE	0.365	0.274	0.243	0.217	0.186	0.151	0.189
	Ave	0.964	0.366	0.124	0.032	0.009	0.005	
$a = 100$	SASE	0.388	0.274	0.244	0.218	0.188	0.154	0.190
	Ave	1.056	0.350	0.126	0.036	0.008	0.001	
$a = 150$	SASE	0.423	0.273	0.244	0.220	0.188	0.153	0.191
	Ave	1.119	0.342	0.125	0.039	0.008	0.001	
$a = 200$	SASE	0.463	0.275	0.246	0.220	0.188	0.153	0.192
	Ave	1.177	0.350	0.126	0.037	0.007	0.003	
HFH	SASE	0.465	0.350	0.301	0.286	0.254	0.223	0.280
	Ave	1.149	0.437	0.165	0.055	0.027	0.027	

Table 1: $n = 1000$, $R = 2$. Estimates evaluated at $u = 0, 2, \ldots, 10$ and globally.

Method		$c(0)=1$	$c(2)=0.3678$	$c(4)=0.1353$	$c(6)=0.0497$	$c(8)=0.0183$	$c(10)=0.0067$	gSASE
$a=50$	SASE	0.330	0.224	0.197	0.164	0.168	0.164	0.171
	Ave	0.950	0.362	0.123	0.043	0.030	0.019	
$a=100$	SASE	0.333	0.225	0.197	0.164	0.169	0.166	0.171
	Ave	1.017	0.349	0.124	0.047	0.029	0.016	
$a=150$	SASE	0.343	0.224	0.199	0.166	0.168	0.166	0.172
	Ave	1.055	0.346	0.123	0.049	0.029	0.016	
$a=200$	SASE	0.357	0.226	0.200	0.165	0.168	0.166	0.172
	Ave	1.087	0.350	0.124	0.048	0.029	0.017	
HFH	SASE	0.406	0.281	0.241	0.202	0.226	0.241	0.253
	Ave	1.141	0.436	0.158	0.062	0.047	0.041	

Table 2: $n=2000,\ R=2$. Estimates evaluated at $u=0,2,\ldots,10$ and globally.

Method		$c(0)=1$	$c(2)=0.670$	$c(4)=0.449$	$c(6)=0.301$	$c(8)=0.202$	$c(10)=0.135$	gSASE
$a=50$	SASE	0.564	0.475	0.417	0.376	0.314	0.266	0.312
	Ave	1.045	0.655	0.394	0.237	0.159	0.106	
$a=100$	SASE	0.575	0.443	0.401	0.364	0.306	0.266	0.303
	Ave	1.114	0.645	0.395	0.240	0.160	0.103	
$a=150$	SASE	0.617	0.441	0.401	0.366	0.304	0.265	0.304
	Ave	1.171	0.637	0.395	0.242	0.160	0.102	
$a=200$	SASE	0.660	0.444	0.402	0.368	0.305	0.266	0.306
	Ave	1.227	0.645	0.396	0.241	0.160	0.104	
HFH	SASE	0.747	0.609	0.519	0.471	0.422	0.384	0.456
	Ave	1.236	0.790	0.490	0.308	0.227	0.176	

Table 3: $n=1000,\ R=5$. Estimates evaluated at $u=0,2,\ldots,10$ and globally.

Method		$c(0)=1$	$c(2)=0.670$	$c(4)=0.449$	$c(6)=0.301$	$c(8)=0.202$	$c(10)=0.135$	gSASE
$a=50$	SASE	0.492	0.445	0.403	0.363	0.327	0.299	0.308
	Ave	0.965	0.605	0.363	0.217	0.127	0.074	
$a=100$	SASE	0.508	0.445	0.402	0.364	0.328	0.297	0.308
	Ave	1.007	0.598	0.364	0.219	0.127	0.073	
$a=150$	SASE	0.525	0.443	0.402	0.364	0.327	0.297	0.308
	Ave	1.038	0.594	0.363	0.220	0.127	0.073	
$a=200$	SASE	0.540	0.443	0.402	0.364	0.327	0.298	0.309
	Ave	1.065	0.598	0.363	0.219	0.127	0.074	
HFH	SASE	0.630	0.547	0.483	0.450	0.430	0.421	0.445
	Ave	1.163	0.741	0.455	0.282	0.187	0.136	

Table 4: $n=2000,\ R=5$. Estimates evaluated at $u=0,2,\ldots,10$ and globally.

Method		$c(0)=1$	$c(2)=0.819$	$c(4)=0.670$	$c(6)=0.549$	$c(8)=0.449$	$c(10)=0.368$	gSASE
$a=50$	SASE	0.605	0.540	0.495	0.457	0.424	0.385	0.397
	Ave	1.056	0.789	0.594	0.447	0.340	0.253	
$a=100$	SASE	0.634	0.539	0.495	0.456	0.423	0.386	0.397
	Ave	1.117	0.780	0.596	0.450	0.340	0.251	
$a=150$	SASE	0.668	0.535	0.495	0.457	0.424	0.386	0.398
	Ave	1.172	0.774	0.594	0.452	0.339	0.250	
$a=200$	SASE	0.705	0.536	0.495	0.457	0.423	0.386	0.399
	Ave	1.227	0.781	0.596	0.451	0.339	0.251	
HFH	SASE	0.828	0.696	0.601	0.550	0.529	0.496	0.549
	Ave	1.275	0.964	0.728	0.561	0.462	0.388	

Table 5: $n=1000$, $R=10$. Estimates evaluated at $u=0,2,\ldots,10$ and globally.

Method		$c(0)=1$	$c(2)=0.819$	$c(4)=0.670$	$c(6)=0.549$	$c(8)=0.449$	$c(10)=0.368$	gSASE
$a=50$	SASE	0.737	0.674	0.608	0.551	0.502	0.458	0.471
	Ave	0.987	0.742	0.551	0.408	0.308	0.229	
$a=100$	SASE	0.757	0.673	0.609	0.553	0.504	0.460	0.473
	Ave	1.021	0.737	0.552	0.409	0.308	0.227	
$a=150$	SASE	0.779	0.669	0.609	0.554	0.503	0.461	0.473
	Ave	1.050	0.733	0.552	0.410	0.308	0.227	
$a=200$	SASE	0.803	0.673	0.608	0.554	0.504	0.459	0.475
	Ave	1.076	0.737	0.552	0.409	0.308	0.228	
HFH	SASE	0.964	0.853	0.751	0.692	0.663	0.646	0.676
	Ave	1.206	0.910	0.671	0.504	0.416	0.353	

Table 6: $n=2000$, $R=10$. Estimates evaluated at $u=0,2,\ldots,10$ and globally.

Method		$c(0) = 1$	$c(2) = 0.3678$	$c(4) = 0.1353$	$c(6) = 0.0497$	$c(8) = 0.0183$	$c(10) = 0.0067$	gSASE
n=1000								
$a = 50$	SASE	0.334	0.249	0.203	0.184	0.188	0.170	0.180
	Ave	0.968	0.355	0.100	0.001	0.011	0.011	
$a = 100$	SASE	0.388	0.274	0.244	0.218	0.188	0.154	0.190
	Ave	1.056	0.350	0.126	0.036	0.008	0.001	
HFH	SASE	0.465	0.350	0.301	0.286	0.254	0.223	0.280
	Ave	1.149	0.437	0.165	0.055	0.027	0.027	
n=2000								
$a = 50$	SASE	0.321	0.235	0.209	0.212	0.227	0.209	0.192
	Ave	0.943	0.356	0.118	0.026	-0.004	-0.004	
$a = 100$	SASE	0.323	0.236	0.208	0.212	0.226	0.210	0.192
	Ave	1.010	0.343	0.119	0.031	-0.005	-0.005	
HFH	SASE	0.400	0.295	0.259	0.267	0.305	0.313	0.287
	A	1.132	0.431	0.153	0.039	0.001	0.006	
n=4000								
$a = 50$	SASE	0.323	0.242	0.202	0.197	0.184	0.144	0.174
	Ave	0.899	0.317	0.095	0.033	0.014	0.027	
$a = 100$	SASE	0.313	0.245	0.202	0.197	0.185	0.144	0.174
	Ave	0.950	0.307	0.097	0.034	0.014	0.025	
HFH	SASE	0.368	0.288	0.239	0.240	0.241	0.217	0.251
	Ave	1.076	0.382	0.128	0.049	0.027	0.053	
n=6000								
$a = 50$	SASE	0.322	0.235	0.211	0.195	0.180	0.173	0.181
	A	0.951	0.363	0.128	0.045	0.002	-0.001	
$a = 100$	SASE	0.319	0.236	0.211	0.195	0.180	0.172	0.181
	Ave	1.001	0.352	0.128	0.049	0.001	-0.003	
HFH	SASE	0.429	0.312	0.259	0.237	0.234	0.242	0.273
	Ave	1.150	0.443	0.174	0.075	0.023	0.018	

Table 7: $n = 1000$, 2000, 4000 and 6000; $R = 2$. Estimates evaluated at $u = 0, 2, \ldots, 10$ and globally. Simulations conducted over 100 replications.