Statistical inference for stochastic coefficient regression models (Technical Report)

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Abstract

The classical multiple regression model plays a very important role in statistical analysis. The typical assumption is that changes in the response variable, due to a small change in a given regressor, is constant over time. In other words, the rate of change is not influenced by any unforeseen external variables and remains the same over the entire time period of observation. This strong assumption may, sometimes, be unrealistic, for example, in areas like social sciences, environmental sciences etc. In view of this, we propose stochastic coefficient regression (SCR) models with stationary, correlated random errors and consider their statistical inference. We assume that the coefficients are stationary processes, where each admits a linear process representation. We propose a frequency domain method of estimation, the advantage of this method is that no assumptions on the distribution of the coefficients are necessary. These models are fitted to two real data sets and their predictive performance are also examined.

Keywords and phrases Gaussian maximum likelihood, frequency domain, locally stationary time series, multiple linear regression, nonstationarity, stochastic coefficients.

1 Introduction

The classical multiple linear regression model is ubiquitous in many fields of research. However, in situations where the response variable $\{Y_t\}$ is observed over time, it is not always possible to assume that the influence the regressors $\{x_{t,j}\}$ exert on the response Y_t is constant over time. A classical example, given in Burnett and Guthrie (1970), is when predicting air quality as a function of pollution emission. The influence the emissions have on air quality on any given day may depend on various factors such as the meterological conditions on the current and previous days. Modelling the variable influence in a deterministic way can be too complex and a simpler method could be

to treat the regression coefficients as stochastic. In order to allow for the influence of the previous regression coefficient on the current coefficient, it is often reasonable to assume that the underlying unobservable regression coefficients are stationary processes and each coefficient admits a linear process representation. In other words, a plausible model for modelling the varying influence of regressors on the response variable is

$$Y_t = \sum_{j=1}^{n}(a_j + \alpha_{t,j})x_{t,j} + \varepsilon_t = \sum_{j=1}^{n} a_j x_{t,j} + X_t, \tag{1}$$

where $\{x_{t,j}\}$ are the deterministic regressors, $\{a_j\}$ are the mean regressor coefficients, $\mathbb{E}(X_t) = 0$ and satisfies $X_t = \sum_{j=1}^{n}\alpha_{t,j}x_{t,j} + \varepsilon_t$, $\{\varepsilon_t\}$ and $\{\alpha_{t,j}\}$ are jointly stationary linear time series with $\mathbb{E}(\alpha_{t,j}) = 0$, $\mathbb{E}(\varepsilon_t) = 0$, $\mathbb{E}(\alpha_{t,j}^2) < \infty$ and $\mathbb{E}(\varepsilon_t^2) < \infty$. We observe that this model includes the classical multiple regression model as a special case, with $\mathbb{E}(\alpha_{t,j}) = 0$ and $\text{var}(\alpha_{t,j}) = 0$. The above model is often refered to as a stochastic coefficient regression (SCR) model. Such models have a long history in statistics (see Hildreth and Houck (1968) and Swamy (1970, 1971), Burnett and Guthrie (1970), Rosenberg (1972, 1973), Duncan and Horn (1972), Fama (1977), Bruesch and Pagan (1980), Swamy and Tinsley (1980), Synder (1985), Pfeffermann (1984), Newbold and Bos (1985), Stoffer and Wall (1991) and Franke and Gründer (1995). For a review of this model the reader is refered to Newbold and Bos (1985). In recent years several other statistical models have been proposed to model temporal changes, examples of such models include varying coefficient models and locally stationary processes. In this article, our aim is to revisit the SCR model, comparing the SCR model with these alternative models, we show that there is a close relationship between them (see Section 2). Having demonstrated that SCR models can model a wide range of time-varying behaviours, in the rest of the article we consider methods of testing for randomness of the regression coefficients and develop parameter estimation methods which do not require any assumptions on the distribution of the stochastic coefficients.

In the aforementioned literature, it is usually assumed that $\{\alpha_{t,j}\}$ satisfies a linear process with structure specified by a finite number of parameters estimated by Gaussian maximum likelihood (GML). In the case $\{Y_t\}$ is Gaussian, the estimators are asymptotically normal and the variance of these estimators can be obtained from the inverse of the Fisher Information matrix. Even in the situation $\{Y_t\}$ is non-Gaussian, the Gaussian likelihood is usually used as the objective function to be maximised, in this case the objective function is often called the quasi-Gaussian likelihood (quasi-GML). The quasi-GML estimator is a consistent estimate of the parameters (see Ljung and Caines (1979), Caines (1988), Chapter 8.6 and Shumway and Stoffer (2006)), but when $\{Y_t\}$ is non-Gaussian, obtaining an expression for the standard errors of the quasi-GML estimators seems

to be extremely difficult. Therefore implicitly it is usually assumed that $\{Y_t\}$ is Gaussian, and most statistical inference is based on the assumption of Gaussianity. In several situations the assumption of Gaussianity may not be plausible, and there is a need for estimators which are free of distributional assumptions. In this paper we address this issue.

In Section 3, two methods to estimate the mean regression parameters and the finite number of parameters which are characterising the impulse response sequences of the linear processes are considered. The suggested methods are based on taking the Fourier transform of the observations, this is because spectral methods usually don't require distributional assumptions, are computationally fast, and can be analysed asymptotically (see Whittle (1962), Walker (1964), Dzhapharidze (1971), Hannan (1971, 1973), Dunsmuir (1979), Taniguchi (1983), Giraitis and Robinson (2001), Dahlhaus (2000) and Shumway and Stoffer (2006)). Both of the proposed methods offer an alternative perspective of the SCR model and are free of any distributional assumptions. In Section 5.2 and 5.3 we consider the asymptotic properties of the proposed estimators. A theoretical comparison of our estimators with the GML estimator, in most cases, is not possible, because it is usually not possible to obtain the asymptotic variance of the GML estimator. However, if we consider a subclass of SRC models, where the regressors are smooth, then the asymptotic variance of the GML estimator can be derived. Thus in Section 5.4 we compare our estimator with the GML estimator, for the subclass SCR models with smooth regressors, and show that both estimators have asymptotically equivalent distributions.

In Section 6 we consider two real data sets. The two time series are taken from the field of economics and evironmental sciences. In the first case, the SCR model is used to examine the relationship between monthly inflation and nominal T-bills interest rates, where monthly inflation is the response and the T-bills rate is the regressor. We confirm the findings off Newbold and Bos (1985), who observe that the regression coefficient is stochastic. In the second case, we consider the influence man made emissions (the regressors) have on particulate matter (the reponse variable) in Shenandoah National Park, U.S.A. Typically, it is assumed that man-made emissions linearly influence the amount of particulate matter and a multiple linear regression model is fitted to the data. We show that there is clear evidence to suggest that the regression coefficients are random, hence the dependence between man-made emissions and particulate matter is more complicated than previously thought.

The proofs can be found in the technical report.

2 The stochastic coefficient regression model

2.1 The model

Throughout this paper we will assume that the response variable $\{Y_t\}$ satisfies (1), where the regressors $\{x_{t,j}\}$ are observed and the following assumptions are satisfied.

Assumption 2.1 *(i) The stationary time series $\{\alpha_{t,j}\}$ and $\{\varepsilon_t\}$ satisfy the following $MA(\infty)$ representations*

$$\alpha_{t,j} = \sum_{i=0}^{\infty} \psi_{i,j} \eta_{t-i,j}, \quad \text{for } j = 1, \ldots, n, \quad \varepsilon_t = \sum_{i=0}^{\infty} \psi_{i,n+1} \eta_{t-i,n+1}, \tag{2}$$

where for all $1 \le j \le n+1$, $\sum_{i=0}^{\infty} |\psi_{i,j}| < \infty$, $\sum_{i=0}^{\infty} |\psi_{i,j}|^2 = 1$, $\mathbb{E}(\eta_{t,j}) = 0$, $\mathbb{E}(\eta_{t,j}^2) = \sigma_{j,0}^2 < \infty$, for each j, $\{\eta_{t,j}\}$ are independent, identically distributed (iid) random variables, they are also independent over j.

The parameters $\{\psi_{i,j}\}$ are unknown but have a parametric form, that is there is a known function $\psi_{i,j}(\cdot)$, such that for some vector $\boldsymbol{\theta}_0 = (\boldsymbol{\vartheta}_0, \boldsymbol{\Sigma}_0)$, $\psi_{i,j}(\boldsymbol{\vartheta}_0) = \psi_{i,j}$ and $\boldsymbol{\Sigma}_0 = diag(\sigma_{1,0}^2, \ldots, \sigma_{n+1,0}^2) = var(\underline{\eta}_t)$, where $\underline{\eta}_t = (\eta_{t,1}, \ldots, \eta_{t,n+1})$.

(ii) We define the compact parameter spaces $\Omega \subset \mathbb{R}^n$, $\Theta_1 \subset \mathbb{R}^q$ and $\Theta_2 \subset diag(\mathbb{R}^{n+1})$, we have $\sum_{i=0}^{\infty} |\psi_{i,j}(\boldsymbol{\vartheta})|^2 = 1$. We shall assume that $\boldsymbol{a}_0 = (a_1, \ldots, a_n)$, $\boldsymbol{\vartheta}_0$ and $\boldsymbol{\Sigma}_0$ (which are defined in (i), above) lie in the interior of Ω, Θ_1 and Θ_2 respectively.

Define the transfer function $A_j(\boldsymbol{\vartheta}, \omega) = (2\pi)^{-1/2} \sum_{k=0}^{\infty} \psi_{k,j}(\boldsymbol{\vartheta}) \exp(ik\omega)$, and the spectral density $f_j(\boldsymbol{\vartheta}, \omega) = |A_j(\boldsymbol{\vartheta}, \omega)|^2$. Using this notation the spectral density of the time series $\{\alpha_{t,j}\}$ is $\sigma_{j,0}^2 f_j(\boldsymbol{\vartheta}_0, \omega)$. Let $c_j(\boldsymbol{\theta}, t-\tau) = \sigma_j^2 \int f_j(\boldsymbol{\vartheta}, \omega) \exp(i(t-\tau)\omega) d\omega$ (hence $cov(\alpha_{t,j}, \alpha_{\tau,j}) = c_j(\boldsymbol{\theta}_0, t - \tau)$).

It should be noted that it is straightforward to generalise (2), such that the vector time series $\{\boldsymbol{\alpha}_t = (\alpha_{t,1}, \ldots, \alpha_{t,n})\}_t$ has a vector $MA(\infty)$ representation. However, using this generalisation makes the notation quite cumbersome. For this reason, we have considered the simpler case (2).

Example 2.1 *If $\{\alpha_{t,j}\}$ and ε_t are autoregressive processes, then Assumption 2.1 is satisfied. That is $\{\alpha_{t,j}\}$ and ε_t satisfy*

$$\alpha_{t,j} = \sum_{k=1}^{p_j} \phi_{k,j} \alpha_{t-k,j} + \eta_{t,j} \quad j = 1, \ldots, n \quad \text{and} \quad \varepsilon_t = \sum_{k=1}^{p_{n+1}} \phi_{k,n+1} \varepsilon_{t-k} + \eta_{t,n+1},$$

4

where $\eta_{t,j}$ are iid random variables with $\mathbb{E}(\eta_{t,j}) = 0$ and $var(\eta_{t,j}) = \sigma_j^2$ and the roots of the characteristics polynomial $1 - \sum_{k=1}^{p_j} \phi_{k,j} z^k$ lie outside unit circle. In this case, the true parameters are $\vartheta_0 = (\phi_{1,1}, \ldots, \phi_{p_{n+1}, n+1})$ and $\Sigma_0 = diag(\frac{\sigma_1^2}{(\int g_1(\omega)d\omega)}, \ldots, \frac{\sigma_{n+1}^2}{(\int g_{n+1}(\omega)d\omega)})$, where $g_j(\omega) = \frac{1}{2\pi}|1 - \sum_{k=1}^{p_j} \phi_{k,j} \exp(ik\omega)|^{-2}$.

2.2 A comparision of the SCR model with other statistical models

In this section we show that the SCR model is closely related to several popular statistical models. Of course, the SCR model includes the multiple linear regression model as a special case, with $var(\alpha_{t,j}) = 0$ and $\mathbb{E}(\alpha_{t,j}) = 0$.

2.2.1 Varying coefficient models

In several applications, linear regression models with time-dependent parameters are fitted to the data. Examples include the varying-coefficient models considered by Martinussen and Scheike (2000), where $\{Y_t\}$ satisfies

$$Y_t = \sum_{j=1}^n \alpha_j(\frac{t}{T}) x_{t,j} + \varepsilon_t, \quad t = 1, \ldots, T \tag{3}$$

and $\{\alpha_j(\cdot)\}$ are smooth, unknown functions and $\{\varepsilon_t\}_t$ are iid random variables with $\mathbb{E}(\varepsilon_t) = 0$ and $var(\varepsilon_t) < \infty$. Comparing this model with the SCR model, we observe that the difference between the two models lies in the modelling of the time-dependent coefficients of the regressors. In (3) the coefficient of the regressor is assumed to be deterministic, whereas the SCR model treats the coefficient as a stationary stochastic process. In some sense, one can suppose that the correlation in $\{\alpha_{t,j}\}$ determines the 'smoothness' of $\{\alpha_{t,j}\}$. The higher the correlation of $\{\alpha_{t,j}\}$, the smoother the coefficients are likely to be. Thus the SCR model could be used as an alternative to varying coefficient models for modelling 'rougher' changes.

2.2.2 Locally stationary time series

In this section we show that a subclass of SCR models and the class of locally stationary linear processes defined in Dahlhaus (1996) are closely related. We restrict the regressors to be smooth $x_{t,j}$, and we will suppose that there exist smooth functions $\{x_j(\cdot)\}$ such that the regressors satisfy $x_{t,j} = x_j(\frac{t}{N})$ for some value N (setting $\frac{1}{T}\sum_t x_{t,j}^2 = 1$) and $\{Y_{t,N}\}$ satisfies

$$Y_{t,N} = \sum_{j=1}^n a_j x_j(\frac{t}{N}) + X_{t,N}, \quad \text{where} \quad X_{t,N} = \sum_{j=1}^n \alpha_{t,j} x_j(\frac{t}{N}) + \varepsilon_t \quad t = 1, \ldots, T. \tag{4}$$

A nonstationary process can be considered locally stationary process, if in any neighbourhood of t, the process can be approximated by a stationary process. We now show that $\{X_{t,N}\}$ (defined in (4)) can be considered as a locally stationary process.

Proposition 2.1 *Suppose Assumption 2.1(i,ii) is satisfied, and the regressors are bounded ($\sup_{j,v} |x_j(v)| < \infty$), let $X_{t,N}$ be defined as in (4) and define the unobserved stationary process $X_t(v) = \sum_{j=1}^{n} \alpha_{t,j} x_j(v) + \varepsilon_t$. Then we have*

$$|X_{t,N} - X_t(v)| = O_p(|\frac{t}{N} - v|).$$

PROOF. Using the continuity of the regressors the proof is straightforward, hence we omit the details. □

The above result shows that in the neighbourhood of t, $\{X_{t,N}\}$ can locally be approximated by a stationary process. Therefore the SCR model with slowly varying regressors can be considered as a 'locally stationary' process.

We now show the converse, that is the class of locally stationary linear processes defined in Dahlhaus (1996), can be approximated to any order by an SCR model with slowly varying regressors. Dahlhaus (1996) defines locally stationary process as the stochastic process $\{X_{t,N}\}$ which satisfies the representation

$$X_{t,N} = \int A_{t,N}(\omega) \exp(it\omega) dZ(\omega) \tag{5}$$

where $\{Z(\omega)\}$ is a complex valued orthogonal process on $[0, 2\pi]$ with $Z(\lambda+\pi) = Z(\lambda)$, $\mathbb{E}(Z(\lambda)) = 0$, and $\mathbb{E}\{dZ(\lambda)dZ(\mu)\} = \eta(\lambda + \nu)d\lambda d\mu$, $\eta(\lambda) = \sum_{j=-\infty}^{\infty} \delta(\lambda + 2\pi j)$ is the periodic extension of the Dirac delta function. Furthermore, there exists a Lipschitz continuous function $A(\cdot)$, such that $\sup_{\omega,t} |A(\frac{t}{N}, \omega) - A_{t,N}(\omega)| \leq KN^{-1}$, where K is a finite constant which does not depend on N.

In the following lemma we show that there always exists a SCR model which can approximate a locally stationary process to any degree.

Proposition 2.2 *Let us suppose that $\{\tilde{X}_{t,N}\}$ is a locally stationary process, which satisfies (5) and $\sup_u \int |A(u, \lambda)|^2 d\lambda < \infty$. Then for any basis $\{x_j(\cdot)\}$ of $L_2[0,1]$, and for every δ there exists an $n_\delta \in \mathbb{Z}$, such that $\tilde{X}_{t,N}$ can be represented as*

$$\tilde{X}_{t,N} = \sum_{j=1}^{n_\delta} \alpha_{t,j} x_j(\frac{t}{N}) + O_p(\delta + N^{-1}), \tag{6}$$

where $\{\boldsymbol{\alpha}_t\} = \{(\alpha_1, \dots, \alpha_{t,n_\delta})\}_t$ is a second order stationary vector process.

PROOF. In the technical report. □

One application of the above result is that if the covariance structure of a time series is believed to change smoothly over time, then a SCR model can be fitted to the observations.

In the sections below we will propose a method of estimating the parameters in the SCR model and use the SCR model with slowly varying parameters as a means of comparing the proposed method with existing methods.

3 The estimators

We now consider two methods to estimate the mean regression parameters $\boldsymbol{a}_0 = (a_{1,0}, \ldots, a_{n,0})$ and the parameters $\boldsymbol{\theta}_0$ in the time series model (defined in Assumption 2.1).

3.1 Motivating the objective function

To motivate the objective function consider the 'localised' finite Fourier transform of $\{Y_t\}_t$ centered at t, that is $J_{Y,t}(\omega) = \frac{1}{\sqrt{2\pi m}} \sum_{k=1}^{m} Y_{t-m/2+k} \exp(ik\omega)$ (where m is even). We can partition $J_{Y,t}(\omega)$ into the sum of deterministic and stochastic terms, $J_{Y,t}(\omega) = \sum_{j=1}^{n} a_{j,0} J_{t,m}^{(j)}(\omega) + J_{X,t}(\omega)$, where $J_{t,m}^{(j)}(\omega) = \frac{1}{\sqrt{2\pi m}} \sum_{k=1}^{m} x_{t-m/2+k,j} \exp(ik\omega)$, $J_{X,t}(\omega) = \frac{1}{\sqrt{2\pi m}} \sum_{k=1}^{m} X_{t-m/2+k} \exp(ik\omega)$ and (Y_t, X_t) are defined in (1). Let us consider the Fourier transform $J_{Y,t}(\omega)$ at the fundamental frequencies $\omega_k = \frac{2\pi k}{m}$ and define the $m(T-m)$-dimensional vectors $\mathcal{J}_{Y,T} = (J_{Y,m/2}(\omega_1), \ldots, J_{Y,T-m/2}(\omega_m))$ and $\mathcal{J}_{x,T}(\boldsymbol{a}_0) = (\sum_{j=1}^{n} a_{j,0} J_{m/2,m}^{(j)}(\omega_1), \ldots, \sum_{j=1}^{n} a_{j,0} J_{T-m/2,m}^{(j)}(\omega_m))$. In order to derive the objective function, we observe that $\mathcal{J}_{Y,T}$ is a linear transformation of the observations $\underline{Y}$, thus there exists a $m(T-m) \times T$-dimensional complex matrix, A, such that $\mathcal{J}_{Y,T} = A\underline{Y}$. Regardless of whether $\{Y_t\}$ is Gaussian or not, we treat $\mathcal{J}_{Y,T}$ as if it were a multivariate complex normal and define the quantity which is proportional to the quasi-likelihood of $\mathcal{J}_{Y,T}$ as

$$\ell_T(\boldsymbol{\theta}_0) = ((\mathcal{J}_{Y,T} - \mathcal{J}_{x,T}(\boldsymbol{a}_0))^H) \Delta_T(\boldsymbol{\theta}_0)^{-1}((\mathcal{J}_{Y,T} - \mathcal{J}_{x,T}(\boldsymbol{a}_0)) + \log(\det\Delta_T(\boldsymbol{\theta}_0)),$$

where $\Delta_T(\boldsymbol{\theta}_0) = \mathbb{E}((\mathcal{J}_{Y,T} - \mathcal{J}_{x,T}(\boldsymbol{a}_0)(\mathcal{J}_{Y,T} - \mathcal{J}_{x,T}(\boldsymbol{a}_0))^H) = A\mathbb{E}(\underline{X}_T\underline{X}_T')A^H$ ($\underline{X}_T = (X_1, \ldots, X_T)'$ and H denotes the transpose and complex conjugate (see Picinbono (1996), equation (17)). Evaluating $\ell_T(\boldsymbol{\theta}_0)$ involves inverting the singular matrix $\Delta_T(\boldsymbol{\theta}_0)$. Hence it is an unsuitable criterion for estimating the parameters $\boldsymbol{a}_0$ and $\boldsymbol{\theta}_0$. Instead let us consider a related criterion, where we ignore the off-diagonal covariances in $\Delta_T(\boldsymbol{\theta}_0)$ and replace it with a diagonal matrix which shares the same diagonal as $\Delta_T(\boldsymbol{\theta}_0)$. Straightforward calculations show that when the $\Delta_T(\boldsymbol{\theta}_0)$ in $\ell_T(\boldsymbol{\theta}_0)$ is replaced

by its diagonal, what remains is proportional to

$$\tilde{\ell}_T(\boldsymbol{\theta}_0) = \frac{1}{T_m m} \sum_{t=m/2}^{T-m/2} \sum_{k=1}^{m} \left(\frac{|J_{Y,t}(\omega_k) - \sum_{j=1}^{n} a_{j,0} J_{t,m}^{(j)}(\omega_k)|^2}{\mathcal{F}_{t,m}(\boldsymbol{\theta}_0, \omega_k)} + \log \mathcal{F}_{t,m}(\boldsymbol{\theta}_0, \omega_k) \right), \tag{7}$$

where $T_m = (T - m)$,

$$
\begin{aligned}
\mathcal{F}_{t,m}(\boldsymbol{\theta}_0, \omega) &= \frac{1}{2\pi} \sum_{r=-(m-1)}^{m-1} \exp(ir\omega) \sum_{j=1}^{n+1} c_j(\boldsymbol{\theta}_0, r) \frac{1}{m} \sum_{k=1}^{m-|r|} x_{t-m/2+k,j} x_{t-m/2+k+r,j} \\
&= \sum_{j=1}^{n} \sigma_{j,0}^2 \int_{-\pi}^{\pi} I_{t,m}^{(j)}(\lambda) f_j(\boldsymbol{\vartheta}_0, \omega - \lambda) d\lambda + \sigma_{n+1,0}^2 \int_{-\pi}^{\pi} I_m^{(n+1)}(\lambda) f_{n+1}(\boldsymbol{\vartheta}_0, \omega - \lambda) d\lambda, \tag{8}
\end{aligned}
$$

letting $x_{t,n+1} = 1$ for all t, $I_{t,m}^{(j)}(\omega) = |J_{t,m}^{(j)}(\omega)|^2$ and $I_m^{(n+1)}(\omega) = \frac{1}{2\pi m}|\sum_{k=1}^{m} \exp(ik\omega)|^2$. It is worth noting that if $\{Y_t\}$ were a second order stationary time series then its DFT is almost uncorrelated and $\Delta_T(\boldsymbol{\theta}_0)$ would be close to a diagonal matrix (this property was used as the basis of a test for second order stationarity in ? (?)).

3.2 Estimator 1

We use (7) to motivate the objective function of the estimator. Replacing the summand $\frac{1}{m}\sum_{k=1}^{m}$ in (7) with an integral yields the objective function

$$\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta}) = \frac{1}{T_m} \sum_{t=m/2}^{T-m/2} \int_{-\pi}^{\pi} \left\{ \frac{\mathcal{I}_{t,m}(\boldsymbol{a}, \omega)}{\mathcal{F}_{t,m}(\boldsymbol{\theta}, \omega)} + \log \mathcal{F}_{t,m}(\boldsymbol{\theta}, \omega) \right\} d\omega, \tag{9}$$

where m is even and

$$\mathcal{I}_{t,m}(\boldsymbol{a}, \omega) = \frac{1}{2\pi m} \left| \sum_{k=1}^{m} (Y_{t-m/2+k} - \sum_{j=1}^{n} a_j x_{t-m/2+k,j}) \exp(ik\omega) \right|^2. \tag{10}$$

We recall that $\boldsymbol{\theta} = (\boldsymbol{\vartheta}, \boldsymbol{\Sigma})$, hence $\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta}) = \mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\vartheta}, \boldsymbol{\Sigma})$. Let $\boldsymbol{a} \in \Omega \subset \mathbb{R}^n$ and $\boldsymbol{\theta} \in \Theta_1 \otimes \Theta_2 \subset \mathbb{R}^{n+q+1}$. We use $\hat{\boldsymbol{a}}_T$ and $\hat{\boldsymbol{\theta}}_T = (\hat{\boldsymbol{\vartheta}}_T, \hat{\boldsymbol{\Sigma}}_T)$ as an estimator of $\boldsymbol{a}_0$ and $\boldsymbol{\theta}_0 = (\boldsymbol{\vartheta}_0, \boldsymbol{\Sigma}_0)$ where

$$(\hat{\boldsymbol{a}}_T, \hat{\boldsymbol{\vartheta}}_T, \hat{\boldsymbol{\Sigma}}_T) = \arg \inf_{\boldsymbol{a} \in \Omega, \boldsymbol{\vartheta} \in \Theta_1, \boldsymbol{\Sigma} \in \Theta_2} \mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\vartheta}, \boldsymbol{\Sigma}). \tag{11}$$

We choose m, such that $T_m/T \to 1$ as $T \to \infty$ (thus m can be fixed, or grow at a rate slower than T).

3.3 Estimator 2

In the case that the number of regressors is relatively large, the minimisation of $\mathcal{L}_T^{(m)}$ is computationally slow and has a tendency of converging to local minimums rather than the global minimum.

8

We now suggest a second estimator, which is based on Estimator 1, but estimates the parameters $\boldsymbol{a}, \boldsymbol{\theta} = (\boldsymbol{\Sigma}, \boldsymbol{\vartheta})$ in two steps. Empirical studies suggest that it is less sensitive to initial values than Estimator 1. In the first step of the scheme we estimate $\boldsymbol{\Sigma}_0$ and in the second step obtain an estimator of $\boldsymbol{a}_0$ and $\boldsymbol{\vartheta}_0$, thereby reducing the total number parameters to be estimated at each step. An additional advantage of estimating the variance of the coefficients in the first stage is that we can use it to determine whether a coefficient of a regressor is fixed or random.

The two-step parameter estimation scheme

(i) **Step 1** In the first step we ignore the correlation of the stochastic coefficients $\{\alpha_{t,j}\}$ and errors $\{\varepsilon_t\}$ and estimate the mean regressors $\{a_{t,j}\}$ and variance of the innovations $\boldsymbol{\Sigma}_T = \{\sigma_{j,0}^2\}$ using weighted least squares with $(\bar{\boldsymbol{a}}_T, \tilde{\boldsymbol{\Sigma}}_T) = \arg\min_{\boldsymbol{a} \in \Omega, \boldsymbol{\Sigma} \in \Theta_2} \mathcal{L}_T(\boldsymbol{a}, \boldsymbol{\Sigma})$, where

$$\mathcal{L}_T(\boldsymbol{a}, \boldsymbol{\Sigma}) = \frac{1}{T} \sum_{t=1}^{T} \left(\frac{(Y_t - \sum_{j=1}^{n} a_j x_{t,j})^2}{\sigma_t(\boldsymbol{\Sigma})} + \log \sigma_t(\boldsymbol{\Sigma}) \right). \tag{12}$$

and $\sigma_t(\boldsymbol{\Sigma}) = \sum_{j=1}^{n} \sigma_j^2 x_{t,j}^2 + \sigma_{n+1}^2$.

(ii) **Step 2** We now use $\tilde{\boldsymbol{\Sigma}}_T$ to estimate $\tilde{\boldsymbol{a}}_T$ and $\boldsymbol{\vartheta}_0$. We substitute $\tilde{\boldsymbol{\Sigma}}_T$ into $\mathcal{L}_T^{(m)}$, keep $\tilde{\boldsymbol{\Sigma}}_T$ fixed and minimise $\mathcal{L}_T^{(m)}$ with respect to $(\boldsymbol{a}, \boldsymbol{\vartheta})$. We use $(\tilde{\boldsymbol{a}}_T, \tilde{\boldsymbol{\vartheta}}_T)$ as an estimator of $(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0)$ where

$$(\tilde{\boldsymbol{a}}_T, \tilde{\boldsymbol{\vartheta}}_T) = \arg\min_{\boldsymbol{a} \in \Omega, \boldsymbol{\vartheta} \in \Theta_1} \mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\vartheta}, \tilde{\boldsymbol{\Sigma}}_T). \tag{13}$$

We choose m, such that $T_m/T \to 1$ as $T \to \infty$.

4   Testing for randomness of the coefficients in the SCR model

Before fitting a SCR model to the data, it is of interest to check whether there is any evidence to suggest the coefficients are random. Bruesch and Pagan (1980) have proposed a test, based on the score statistic, to test the possibility that the parameters of a regression model are fixed against the alternative that they are random. Their test statistic is constructed under the assumption that the errors in the regression model are Gaussian and are identically distributed. Further, Newbold and Bos (1985), Chapter 3, argue that the test proposed in Bruesch and Pagan (1980) can be viewed as the sample correlation between the squared residuals and the regressors (under the assumption of Gaussianity). In this section, we suggest a distribution free version of the test given in Newbold and Bos (1985), to test the hypothesis that the parameters are fixed against the alternative that they are random. Further, we propose a test to test the hypothesis the parameters are random (iid) against the alternative that they are stochastic (and correlated).

To simplify notation we will consider simple regression models with just one regressor, the discussion below can be generalised to the multiple regression case. Let us consider the null hypothesis $H_0 : Y_t = a_0 + a_1 x_t + \epsilon_t$ where $\{\epsilon_t\}$ are iid random variables with $\mathbb{E}(\varepsilon_t) = 0$ and $\text{var}(\varepsilon_t) = \sigma_\epsilon^2 < \infty$ against the alternative $H_A : Y_t = a_0 + a_1 x_t + \epsilon_t$, where $\epsilon_t = \alpha_t x_t + \varepsilon_t$ and $\{\alpha_t\}$ and $\{\varepsilon_t\}$ are iid random variables with $\mathbb{E}(\alpha_t) = 0$, $\mathbb{E}(\varepsilon_t) = 0$, $\text{var}(\alpha_t) = \sigma_\alpha^2 < \infty$ and $\text{var}(\varepsilon_t) = \sigma_\varepsilon^2 < \infty$. If the alternative were true, then $\text{var}(\epsilon_t) = x_t^2 \sigma_\alpha^2 + \sigma_\varepsilon^2$, hence plotting $\text{var}(\epsilon_t)$ against x_t should give a clear positive slope. The following test is based on this observation. Suppose we observe $\{(Y_t, x_t)\}$ and use OLS to fit the model $a_0 + a_1 x_t$ to Y_t, let $\hat{\epsilon}_t$ denote the residuals. We use as the test statistic the sample correlation between $\{x_t^2\}$ and $\{\hat{\epsilon}_t^2\}$

$$\mathcal{S}_1 = \frac{1}{T} \sum_{t=1}^{T} x_t^2 \hat{\epsilon}_t^2 - \Big(\frac{1}{T} \sum_{t=1}^{T} x_t^2\Big)\Big(\frac{1}{T} \sum_{t=1}^{T} \hat{\epsilon}_t^2\Big). \tag{14}$$

To understand how $\mathcal{S}_1$ behaves under the null and alternative, we rewrite $\mathcal{S}_1$ as

$$\mathcal{S}_1 = \underbrace{\frac{1}{T} \sum_{t=1}^{T} \big(\hat{\epsilon}_t^2 - \mathbb{E}(\epsilon_t^2)\big)\big(x_t^2 - \frac{1}{T} \sum_{s=1}^{T} x_s^2\big)}_{o_p(1)} + R_1, \text{ where } R_1 = \frac{1}{T} \sum_{t=1}^{T} x_t^2 \Big(\mathbb{E}(\epsilon_t^2) - \frac{1}{T} \sum_{s=1}^{T} \mathbb{E}(\epsilon_s^2)\Big).$$

We observe that in the case that the null is true, then $\mathbb{E}(\epsilon_t^2)$ is constant for all t and $\mathcal{S}_1 = o_p(1)$. On the other hand when the alternative is true we have $\mathcal{S}_1 \to \mathbb{E}(R_1)$, noting that in this case

$$R_1 = \frac{1}{T} \sum_{t=1}^{T} x_t^2 \sigma_\alpha^2 \Big(x_t^2 - \frac{1}{T} \sum_{t=1}^{T} x_s^2\Big). \tag{15}$$

In the proposition below we derive the distribution of the test statistic $\mathcal{S}_1$, under the null and the alternative.

Proposition 4.1 *Let $\mathcal{S}_1$ be defined in (14), and suppose the null is true and $\mathbb{E}(|\epsilon_t|^{4+\delta}) < \infty$ (for some $\delta > 0$) then we have $\sqrt{T}\Gamma_{1,T}^{-1/2}\mathcal{S}_1 \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$, where $\Gamma_{1,T} = \frac{\text{var}(\epsilon_t^2)}{T} \sum_{t=1}^{T}(x_t^2 - \frac{1}{T}\sum_{s=1}^{T} x_s^2)^2$.*

Suppose the alternative is true and $\{\varepsilon_t\}$ and $\{\epsilon_t\}$ are iid random variables, $\mathbb{E}(|\varepsilon_t|^{4+\delta}) < \infty$ and $\mathbb{E}(|\alpha_t|^{4+\delta}) < \infty$ (for some $\delta > 0$) then we have

$$\sqrt{T}\Gamma_{2,T}^{-1/2}\big(\mathcal{S}_1 - R_1\big) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1),$$

where R_1 is defined as in (15) and $\Gamma_{2,T} = \frac{1}{T}\sum_{t=1}^{T} \text{var}((\alpha_t x_t + \epsilon_t)^2)(x_t^2 - \frac{1}{T}\sum_{s=1}^{T} x_s^2)^2$.

PROOF. In the Technical report. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We mention that in the case that the parameters in the regression model are fixed, but the variance of the errors vary over time (independent of x_t) the test statistic, $\mathcal{S}_1$, may mistakenly lead

to the conclusion that the alternative were true (because in this case R_1 will be non-zero). However, if the variance varies slowly over time, it is possible to modify the test statistic $\mathcal{S}_1$ to allow for a time-dependent variance, we omit the details.

We now adapt the test above to determine whether the parameters in the regressor model are random against the alternative there is correlation. More precisely, consider the null hypothesis that the coefficients are random $H_0 : Y_t = a_0 + a_1 x_t + \epsilon_t$, where $\epsilon_t = \alpha_t x_t + \varepsilon_t$ and $\{\alpha_t\}$ are iid random variables and $\{\varepsilon_t\}$ is a stationary time series with $\mathrm{cov}(\varepsilon_0, \varepsilon_k) = c(k)$ against the alternative that the coefficients are stochastic and correlated $H_A : Y_t = a_0 + a_1 x_t + \epsilon_t$, where $\epsilon_t = \alpha_t x_t + \varepsilon_t$ and $\{\alpha_t\}$ and $\{\varepsilon_t\}$ are stationary random variables with $\mathrm{cov}(\varepsilon_0, \varepsilon_k) = c(k)$ and $\mathrm{cov}(\epsilon_0, \epsilon_k) = \rho(k)$. We observe that if the null were true $\mathbb{E}(\epsilon_t \epsilon_{t-1}) = c(1)$, whereas if the alternative were true then $\mathbb{E}(\epsilon_t \epsilon_{t-1}) = x_t x_{t-1} \rho(1) + c(1)$, hence plotting $\epsilon_t \epsilon_{t-1}$ against $x_t x_{t-1}$ should give a clear line with a slope. Therefore the empirical correlation between $\{\hat{\epsilon}_t \hat{\epsilon}_{t-1}\}$ and $\{x_t x_{t-1}\}$ can be used as the test statistic, and we define

$$\mathcal{S}_2 = \frac{1}{T} \sum_{t=2}^{T} x_t x_{t-1} \hat{\epsilon}_t \hat{\epsilon}_{t-1} - \Big(\frac{1}{T} \sum_{t=2}^{T} x_t x_{t-1}\Big)\Big(\frac{1}{T} \sum_{t=2}^{T} \hat{\epsilon}_t \hat{\epsilon}_{t-1}\Big). \tag{16}$$

Rewriting $\mathcal{S}_2$ we have

$$\mathcal{S}_2 = \underbrace{\frac{1}{T} \sum_{t=2}^{T} \big(\hat{\epsilon}_t \hat{\epsilon}_{t-1} - \mathbb{E}(\epsilon_t \epsilon_{t-1})\big)\big(x_t x_{t-1} - \frac{1}{T} \sum_{s=2}^{T} x_s x_{s-1}\big)}_{o_p(1)} + R_2, \tag{17}$$

where $R_2 = \frac{1}{T} \sum_{t=2}^{T} x_t x_{t-1}\Big(\mathbb{E}(\epsilon_t \epsilon_{t-1}) - \frac{1}{T} \sum_{s=2}^{T} \mathbb{E}(\epsilon_s \epsilon_{s-1})\Big)$. It is straightforward to see that if the null were true $\mathcal{S}_2 = o_p(1)$, but if the alternative were true then $\mathbb{E}(\mathcal{S}_2) \xrightarrow{\mathcal{P}} R_2$, noting that $R_2 = \frac{1}{T} \sum_{t=2}^{T} x_t x_{t-1}\Big(x_t x_{t-1} \mathrm{cov}(\alpha_t, \alpha_{t-1}) - \frac{1}{T} \sum_{s=2}^{T} x_s x_{s-1} \mathrm{cov}(\alpha_s, \alpha_{s-1})\Big)$. Below we derive the distribution of $\mathcal{S}_2$ under the null and alternative.

Proposition 4.2 *Let $\mathcal{S}_2$ be defined in (16), and suppose the null is true, that is $Y_t = a_0 + a_1 x_t + \epsilon_t$, where $\epsilon_t = \alpha_t x_t + \varepsilon_t$, and $\{\alpha_t\}$ are iid random variables $\mathbb{E}(|\alpha_t|^8) < \infty$ and $\{\varepsilon_t\}$ is a stationary time series which satisfies $\varepsilon_t = \sum_{j=0}^{\infty} \psi_j \eta_{t-j}$ and $\sum_j |\psi_j| < \infty$ and $\mathbb{E}(|\eta_j|^8) < \infty$. Then we have $\sqrt{T} \Delta_{1,T}^{-1/2} \mathcal{S}_2 \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$, where $\Delta_{1,T} = \frac{1}{T^2} \sum_{t_1,t_2=1}^{n} \mathrm{cov}(\varepsilon_{t_1} \varepsilon_{t_1-1}, \varepsilon_{t_2} \varepsilon_{t_2-1}) v_{t_1} v_{t_2}$ and $v_t = \big(x_t^2 - \frac{1}{T} \sum_{s=2}^{T} x_s^2\big)^2$.*

On the other hand suppose the alternative were true, that is $Y_t = a_0 + a_1 x_t + \epsilon_t$, where $\epsilon_t = \alpha_t x_t + \varepsilon_t$, and $\{\alpha_t\}$ and $\{\varepsilon_t\}$ are stationary time series which satisfies $\varepsilon_t = \sum_{j=0}^{\infty} \psi_j \eta_{t-j}$, $\alpha_t = \sum_{j=0}^{\infty} \psi_{j,1} \eta_{t-j,1}$, $\sum_j |\psi_j| < \infty$, $\sum_j |\psi_{j,1}| < \infty$, $\mathbb{E}(|\eta_j|^8) < \infty$ and $\mathbb{E}(|\eta_{j,1}|^8) < \infty$. Then we have $\sqrt{T} \Delta_{2,T}^{-1/2} (\mathcal{S}_2 - R_2) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$, where $\Delta_{2,T} = var(\mathcal{S}_2)$.

11

PROOF. Similar to Proposition 4.1. □

It is worth noting, it is not necessary to limit the test statistic for testing for correlation at lag one (as was done in $\mathcal{S}_2$). It is straightforward to generalise $\mathcal{S}_2$ to test for correlations at larger lags.

5 Asymptotic properties of the estimators

5.1 Some assumptions

We now consider the asymptotic sampling properties of estimators 1 and 2. We need the following assumptions on the stochastic coefficients and the regressors, which we use to show consistency and the sampling distribution.

Let $|\cdot|$ denote the Euclidean norm of a vector or matrix respectively.

Assumption 5.1 (On the stochastic coefficients) *(i) The parameter spaces Θ_1 and Θ_2 are such that, there exists a $\delta > 0$, where $\inf_{\boldsymbol{\Sigma}\in\Theta_2} \sigma_{n+1}^2 \geq \delta$ and $\inf_{\boldsymbol{\vartheta}\in\Theta_1} \int_{-\pi}^{\pi} \left(\sum_{r=-(m-1)}^{m-1} (\frac{m-|r|}{m}) \exp(ir\lambda) \right) \cdot f_0(\boldsymbol{\vartheta}, \omega - \lambda) d\lambda \geq \delta$.*

(ii) The parameter spaces Ω, Θ_1 and Θ_2 are compact.

(iii) The coefficients $\psi_{i,j}$ of the $MA(\infty)$ representation given in Assumption 2.1, satisfy $\sup_{\boldsymbol{\vartheta}\in\Theta_1} \sum_{i=0}^{\infty} |i| \cdot |\nabla_{\boldsymbol{\vartheta}}^k \psi_{i,j}(\boldsymbol{\vartheta})| < \infty$ (for all $0 \leq k \leq 3$ and $1 \leq j \leq n+1$).

(iv) The innovation sequences $\{\eta_{t,j}\}$ satisfy $\sup_{1\leq j\leq n+1} \mathbb{E}(\eta_{t,j}^8) < \infty$.

Assumption 5.2 (On the regressors) *(i) $\sup_{t,j} |x_{t,j}| < \infty$ and $\frac{1}{T} \sum_{t=1}^{T} \frac{\boldsymbol{X}_t \boldsymbol{X}_t'}{var(Y_t)}$ is non-singular for all T $(\boldsymbol{X}_t' = (x_{t,1}, \ldots, x_{t,n}))$.*

(ii) Suppose that $\boldsymbol{\theta}_0$ is the true parameter. There does not exist another $\boldsymbol{\theta}^* \in \Theta_1 \otimes \Theta_2$ such that for all $0 \leq r \leq m-1$ and infinite number of t we have

$$\sum_{j=1}^{n} \left(c_j(\boldsymbol{\theta}_0, r) - c_j(\boldsymbol{\theta}^*, r)\right) \sum_{k=0}^{m-|r|} x_{t-m/2+k,j} x_{t-m/2+k+r,j} = 0.$$

Let $\mathcal{J}_{T,m}(\boldsymbol{\vartheta}, \omega)' = \sum_{t=m/2}^{T-m/2} \mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \omega)^{-1} (J_{t,m}^{(1)}(\omega), \ldots, J_{t,m}^{(n)}(\omega))$. For all T and $\boldsymbol{\vartheta} \in \Theta_1$, $\int \mathcal{J}_{T,m}(\boldsymbol{\vartheta}, \omega) \mathcal{J}_{T,m}(\boldsymbol{\vartheta}, \omega)' d\omega$ is nonsingular and the smallest eigenvalue is bounded away from zero.

(iii) For all T, $\mathbb{E}(\nabla_{\boldsymbol{\theta}}^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0))$ and $\mathbb{E}(\nabla_{\boldsymbol{a}}^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0))$ are nonsingular matrices and the smallest eigenvalue is bounded away from zero.

Assumption 5.1(i) ensures that $\mathcal{F}_{t,m}$ is bounded away from zero and thus $\mathbb{E}|\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta})| < \infty$, similarly Assumption 5.1(iii) implies that $\sup_j \sum_{r=-\infty}^{\infty} |r \cdot \nabla^k c_j(\boldsymbol{\theta}, r)| < \infty$, therefore $\mathbb{E}|\nabla_{\boldsymbol{\theta}}^k \mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta})| < \infty$. Assumption 5.2(i,ii) ensures that the estimators converge to the true parameters.

5.2 Sampling properties of Estimator 1: $\hat{\boldsymbol{\alpha}}_T$ and $\hat{\boldsymbol{\theta}}_T$

We first show consistency of $(\hat{\boldsymbol{a}}_T, \hat{\boldsymbol{\theta}}_T)$.

Proposition 5.1 *Suppose Assumptions 2.1, 5.1(i,ii) and 5.2 are satisfied and the estimators $\hat{\boldsymbol{a}}_T$, $\hat{\boldsymbol{\theta}}_T$ are defined as in (11). Then we have $\hat{\boldsymbol{a}}_T \overset{\mathcal{P}}{\to} \boldsymbol{a}_0$ and $\hat{\boldsymbol{\theta}}_T \overset{\mathcal{P}}{\to} \boldsymbol{\theta}_0$, as $T_m \to \infty$ and $T \to \infty$.*

PROOF. In the Technical report. $\qquad\square$

We now obtain the rate of convergence and asymptotic normality of the estimator, this requires a bound for the variance of $\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta})$ and its derivatives. To do this we rewrite $\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta})$ and its derivatives as a quadratic form. Substituting $J_{X,T}(\omega) = J_{Y,T}(\omega) - \sum_{j=1}^n a_j J_{t,m}^{(j)}(\omega)$ into $\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta})$ gives

$$
\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta}) = \frac{1}{T_m} \left\{ V_T(\mathcal{F}_{\boldsymbol{\theta}}^{-1}) + 2 \sum_{j=1}^n (a_{j,0} - a_j) D_T^{(j)}(\mathcal{F}_{\boldsymbol{\theta}}^{-1}) + \right.
$$
$$
\left. \sum_{j_1, j_2 = 1}^n (a_{j_1,0} - a_{j_1})(a_{j_2,0} - a_{j_2}) H_T^{(j_1, j_2)}(\mathcal{F}_{\boldsymbol{\theta}}^{-1}) + \sum_{t=m/2}^{T-m/2} \int \log \mathcal{F}_{t,m}(\boldsymbol{\theta}, \omega) d\omega \right\},
$$

where for a general function $\{G_t(\omega)\}_t$ we define

$$
V_T(\mathcal{G}) = \sum_{t=m/2}^{T-m/2} \int G_t(\omega) |J_{X,t}(\omega)|^2 d\omega = \frac{1}{2\pi m} \sum_{k=1}^m \sum_{s=k}^{T-m+k} \sum_{r=-k}^{m-k} X_s X_{s+r} g_{s+m/2-k}(r), \quad (18)
$$

$$
H_T^{(j_1, j_2)}(\mathcal{G}) = \sum_{t=m/2}^{T-m/2} \int G_t(\omega) J_{t,m}^{(j_1)}(\omega) J_{t,m}^{(j_2)}(-\omega) d\omega = \frac{1}{2\pi m} \sum_{k=1}^m \sum_{s=k}^{T-m+k} \sum_{r=-k}^{m-k} x_{s,j_1} x_{s+r,j_2} g_{s+m/2-k}(r),
$$

$$
D_T^{(j)}(\mathcal{G}) = \sum_{t=m/2}^{T-m/2} \int G_t(\omega) \Re\{J_{X,t}(\omega) J_{t,m}^{(j)}(-\omega)\} d\omega = \frac{1}{2\pi m} \sum_{k=1}^m \sum_{s=k}^{T-m+k} \sum_{r=-k}^{m-k} X_s x_{s+r,j} \tilde{g}_{s+m/2-k}(r),
$$

$g_s(r) = \int G_s(\omega) \exp(ir\omega) d\omega$ and $\tilde{g}_s(r) = \int G_s(\omega) \cos(r\omega) d\omega$. A similar expansion also holds for the derivatives of $\mathcal{L}_T^{(m)}$ (which can be used to numerically minimise $\mathcal{L}_T^{(m)}$). Let $\nabla = (\frac{\partial}{\partial a_1}, \dots, \frac{\partial}{\partial \boldsymbol{\theta}_q})$ and

13

$\nabla \mathcal{L}_T^{(m)} = (\nabla_{\boldsymbol{a}} \mathcal{L}_T^{(m)}, \nabla_{\boldsymbol{\theta}} \mathcal{L}_T^{(m)})$, where

$$\nabla_{\boldsymbol{\theta}} \mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta}) = \frac{1}{T_m} \Bigg\{ \Big(V_T(\nabla_{\boldsymbol{\theta}} \mathcal{F}^{-1}) + 2 \sum_{j=1}^{n} (a_{j,0} - a_j) D_T^{(j)}(\nabla_{\boldsymbol{\theta}} \mathcal{F}^{-1}) \Big)$$

$$+ \sum_{j_1, j_2 = 1}^{n} (a_{j_1,0} - a_{j_1})(a_{j_2,0} - a_{j_2}) H_T^{(j_1,j_2)}(\nabla_{\boldsymbol{\theta}} \mathcal{F}^{-1}) + \sum_{t=m/2}^{T-m/2} \int \frac{\nabla_{\boldsymbol{\theta}} \mathcal{F}_{t,m}(\boldsymbol{\theta}, \omega)}{\mathcal{F}_{t,m}(\boldsymbol{\theta}, \omega)} d\omega \Bigg\}$$

$$\nabla_{a_j} \mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta}) = \frac{-2}{T_m} \Big\{ D_T^{(j)}(\mathcal{F}^{-1}) + \sum_{j_1=1}^{n} (a_{j_1,0} - a_{j_1}) H_T^{(j,j_1)}(\mathcal{F}^{-1}) \Big\} \qquad (19)$$

and the second derivatives are

$$\nabla_{a_j} \nabla_{\boldsymbol{\theta}} \mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta}) = \frac{-2}{T_m} \Big\{ D_T^{(j)}(\nabla_{\boldsymbol{\theta}} \mathcal{F}^{-1}) + \sum_{j_1=1}^{n} (a_{j_1,0} - a_{j_1}) H_T^{(j,j_1)}(\nabla_{\boldsymbol{\theta}} \mathcal{F}^{-1}) \Big\}$$

$$\nabla_{a_{j_1}} \nabla_{a_{j_2}} \mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta}) = \frac{2}{T_m} H_T^{(j_1,j_2)}(\nabla_{\boldsymbol{\theta}} \mathcal{F}^{-1}) \qquad (20)$$

$$\nabla_{\boldsymbol{\theta}}^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta}) = \frac{1}{T_m} \Bigg[V_T(\nabla_{\boldsymbol{\theta}}^2 \mathcal{F}^{-1}) + 2 \sum_{j=1}^{n} (a_{j,0} - a_j) D_T^{(j)}(\nabla_{\boldsymbol{\theta}}^2 \mathcal{F}^{-1}) +$$

$$\sum_{j_1, j_2 = 1}^{n} (a_{j_1,0} - a_{j_1})(a_{j_2,0} - a_{j_2}) H_T^{(j_1,j_2)}(\nabla_{\boldsymbol{\theta}}^2 \mathcal{F}^{-1})$$

$$+ \frac{1}{T_m} \sum_{t=m/2}^{T-m/2} \Bigg\{ \int \Bigg[\frac{\nabla_{\boldsymbol{\theta}}^2 \mathcal{F}_{t,m}(\boldsymbol{\theta}, \omega)}{\mathcal{F}_{t,m}(\boldsymbol{\theta}, \omega)} - \frac{\nabla_{\boldsymbol{\theta}} \mathcal{F}_{t,m}(\boldsymbol{\theta}, \omega) \nabla_{\boldsymbol{\theta}} \mathcal{F}_{t,m}(\boldsymbol{\theta}, \omega)'}{\mathcal{F}_{t,m}(\boldsymbol{\theta}, \omega)^2} \Bigg] d\omega \Bigg\} \Bigg].$$

Since the Fourier coefficients in the quadratic forms of $\mathcal{L}_T^{(m)}$ and its derivatives are absolutely summable (under the stated assumptions), it can be shown that the covariance of the stochastic sequence $\{Z_{t.m}\}$, where $Z_{t,m} = \frac{1}{2\pi m} \sum_{k=1}^{m} \sum_{r=-k}^{m-k} X_t X_{t+r} g_{t+m/2-k}(r)\}_s$ is absolutely summable. This implies that the variance of $\mathcal{L}_T^{(m)}$ and its derivatives do not depend on m (so long as $m = o(T)$). We make this precise in the lemma below.

Lemma 5.1 *Suppose Assumptions 2.1, 5.1(i-iii) (and $\sup_j \mathbb{E}(\eta_{t,j}^4) < \infty$) and 5.2(i) are satisfied. Let $V_T(\mathcal{G})$ and $D_T^{(j)}(\mathcal{G})$ be defined as in (18), where $\sup_s \sum_r |g_s(r)| < \infty$ and $\sup_s \sum_r |\tilde{g}_s(r)| < \infty$. Then we have $\mathbb{E}(D_T^{(j)}(\mathcal{G})) = 0$,*

$$\mathbb{E}(V_T(\mathcal{G})) = \sum_{t=m/2}^{T-m/2} \int \mathcal{G}_t(\omega) \mathcal{F}_{t,m}(\boldsymbol{\theta}_0, \omega) d\omega, \qquad (21)$$

$$var(V_T(\mathcal{G})) \leq T(n+1) \sup_s \sum_r |g_s(r)| \Bigg\{ 2\Big(\sum_r \rho_2(r)\Big)^2 + \sum_{k_1,k_2,k_3} \rho_4(k_1, k_2, k_3) \Bigg\} \qquad (22)$$

and

$$var(D_T^{(j)}(\mathcal{G})) \leq T(n+1) \sup_s \Big(\sum_r |\tilde{g}_s(r)|\Big)\Big(\sum_r \rho_2(r)\Big), \qquad (23)$$

14

where

$$\rho_2(k) = \kappa_2^2 \sup_j \sum_i |\psi_{i,j}| \cdot |\psi_{i+k,j}|, \quad \rho_4(k_1, k_2, k_3) = \kappa_4 \sup_j \sum_i |\psi_{i,j}| \cdot |\psi_{i+k_1,j}| \cdot |\psi_{i+k_2,j}| \cdot |\psi_{i+k_3,j}|, (24)$$

$\kappa_2 = \sup_j var(\eta_{0,j})$ *and* $\kappa_4 = \sup_j cum(\eta_{0,j}, \eta_{0,j}, \eta_{0,j}, \eta_{0,j})$.

PROOF. See the Technical report. □

Applying the mean value theorem pointwise to $\nabla \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0)$ and using the uniform convergence $\sup_{\boldsymbol{a},\boldsymbol{\theta}} |\nabla^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta}) - \mathbb{E}(\nabla^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta}))| \xrightarrow{\mathcal{P}} 0$ we have

$$\begin{pmatrix} \hat{\boldsymbol{a}}_T - \boldsymbol{a}_0 \\ \hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0 \end{pmatrix} = \mathbb{E}(\nabla^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0))^{-1} \begin{pmatrix} \nabla_{\boldsymbol{a}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0) \\ \nabla_{\boldsymbol{\theta}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0) \end{pmatrix} + o_p(\frac{1}{\sqrt{T}}). \tag{25}$$

Thus by using the above, if Assumption 5.2(iii) holds and $m = o(T)$ then we have

$$(\hat{\boldsymbol{a}}_T - \boldsymbol{a}_0, \hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0) = O_p(\frac{1}{\sqrt{T}}).$$

In the following theorem we use the above expansion to show asymptotic normality of $(\hat{\boldsymbol{a}}_T, \hat{\boldsymbol{\theta}}_T)$. This requires us to evaluate $\mathbb{E}(\nabla^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0))$ and $var(\nabla \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0))$. The elements of $\mathbb{E}(\nabla^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0))$ are given by $\mathbb{E}(\nabla_{\boldsymbol{\theta}} \nabla_{\boldsymbol{a}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0) = 0$,

$$\mathbb{E}(\nabla_{\boldsymbol{a}}^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0))]_{j_1,j_2} = \frac{1}{Tm} \sum_{t=m/2}^{T-m/2} \int \mathcal{F}_{t,m}(\boldsymbol{\theta}_0, \omega)^{-1} J_{t,m}^{(j_1)}(\omega) J_{t,m}^{(j_2)}(-\omega) d\omega, \tag{26}$$

$$\mathbb{E}(\nabla_{\boldsymbol{\theta}}^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0)) = \frac{1}{Tm} \sum_{t=m/2}^{T-m/2} \int \frac{\nabla \mathcal{F}_{t,m}(\boldsymbol{\theta}_0, \omega) \nabla \mathcal{F}_{t,m}(\boldsymbol{\theta}_0, \omega)'}{(\mathcal{F}_{t,m}(\boldsymbol{\theta}_0, \omega))^2} d\omega,$$

and $var(\nabla \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0))$. In order to compare the limiting distributions of estimator 1 and 2, we rewrite $var(\nabla \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0))$ in terms of a block matrix

$$W_T = T var(\nabla \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0)) \tag{27}$$

$$= \begin{pmatrix} W_{1,T} & T cov(\nabla_{\boldsymbol{\beta}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0), \nabla_{\boldsymbol{\Sigma}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0)) \\ T cov(\nabla_{\boldsymbol{\beta}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0), \nabla_{\boldsymbol{\Sigma}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0)) & T var(\nabla_{\boldsymbol{\Sigma}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0)) \end{pmatrix}$$

with $\boldsymbol{\beta} = (\boldsymbol{a}, \boldsymbol{\vartheta})$ and

$$W_{1,T} = var \begin{pmatrix} \sqrt{T} \nabla_{\boldsymbol{a}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \boldsymbol{\sigma}_0) \\ \sqrt{T} \nabla_{\boldsymbol{\vartheta}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \boldsymbol{\sigma}_0) \end{pmatrix}. \tag{28}$$

Theorem 5.1 *Suppose Assumptions 2.1, 5.1 and 5.2 hold, then we have*

$$\sqrt{T} B_T^{-1/2} \begin{pmatrix} \hat{\boldsymbol{a}}_T - \boldsymbol{a}_0 \\ \hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0 \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}(0, I), \tag{29}$$

15

as $T_m/T \to 1$ and $T \to \infty$, where $B_T = V_T^{-1} W_T (V_T^{-1}$, W_T is defined in (27) and

$$V_T = \begin{pmatrix} \mathbb{E}(\nabla_{\boldsymbol{a}}^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0)) & 0 \\ 0 & \mathbb{E}(\nabla_{\boldsymbol{\theta}}^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0)) \end{pmatrix}.$$

PROOF. In the technical report. □

We observe that V_T is a block diagonal matrix, this is because straightforward calculations show $\mathbb{E}(\nabla_{\boldsymbol{a}} \nabla_{\boldsymbol{\theta}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0)) = 0$.

Remark 5.1 *Examining W_T we observe that the only parameters in W_T which we need to estimate (in addition to $(\boldsymbol{a}_0, \boldsymbol{\theta}_0)$) are the cumulants $cum(\eta_{t,j}, \eta_{t,j}, \eta_{t,j})$ and $cum(\eta_{t,j}, \eta_{t,j}, \eta_{t,j}, \eta_{t,j})$. We estimate the cumulants using the estimated moments. To estimate the moments, we group the observations $\{Y_t\}$ in $(n+1)$ blocks, each of length $M = T/(n+1)$, and evaluate the empirical third moment within each block. If the size of each block $M = T/(n+1)$ is large, we obtain the following approximate equations*

$$\frac{1}{M} \sum_{s=1}^{M} Y_{Mr+s}^3 \approx \frac{1}{M} \sum_{j=1}^{n+1} \mathbb{E}(\eta_{t,j}^3) \sum_{s=1}^{M} x_{Mr+s,j}^3 \sum_{i=0}^{\infty} \psi_{i,j}(\boldsymbol{\vartheta}_0)^3.$$

Obviously this equation is true for $r = 1, \ldots, (n+1)$. Therefore, we have $(n+1)$ linear simultaneous equations in the unknown $\{\mathbb{E}(\eta_{t,j}^3)\}$, which, if we replace $\boldsymbol{\vartheta}_0$ with $\hat{\boldsymbol{\vartheta}}_T$, we can solve for. Thus we have an estimator of $\mathbb{E}(\eta_{t,j}^3)$. Using a similar method, the empirical fourth moment, can be used to obtain an estimator of the fourth order cumulants.

5.3 Sampling properties of Estimator 2: $\tilde{\boldsymbol{\Sigma}}_T$, $\tilde{\boldsymbol{a}}_T$, $\tilde{\boldsymbol{\vartheta}}_T$

We now obtain the sampling properties of Estimator 2. We first consider the properties of the variance estimator $\tilde{\boldsymbol{\Sigma}}_T$.

Proposition 5.2 *Suppose Assumptions 2.1, 5.1 and 5.2 are satisfied, let $\mathcal{L}_T(\boldsymbol{a}, \boldsymbol{\Sigma})$ and $\tilde{\boldsymbol{\Sigma}}_T$ be defined as in (12). Then we have*

$$var(\nabla_{\boldsymbol{\Sigma}} \mathcal{L}_T(\boldsymbol{a}_0, \boldsymbol{\Sigma}_0))^{-1/2} \nabla_{\boldsymbol{\Sigma}} \mathcal{L}_T(\boldsymbol{a}_0, \boldsymbol{\Sigma}_0) \xrightarrow{\mathcal{D}} \mathcal{N}(0, I) \tag{30}$$

$$C_T^{-1/2} (\tilde{\boldsymbol{\Sigma}}_T - \boldsymbol{\Sigma}_0) \xrightarrow{\mathcal{D}} \mathcal{N}(0, I),$$

as $T \to \infty$, where $C_T = \mathbb{E}(\nabla_{\boldsymbol{\Sigma}}^2 \mathcal{L}_T(\boldsymbol{a}_0, \boldsymbol{\Sigma}_0))^{-1} var(\nabla_{\boldsymbol{\Sigma}} \mathcal{L}_T(\boldsymbol{a}_0, \boldsymbol{\Sigma}_0)) \mathbb{E}(\nabla_{\boldsymbol{\Sigma}}^2 \mathcal{L}_T(\boldsymbol{a}_0, \boldsymbol{\Sigma}_0))^{-1}.$

We now consider the properties of $(\tilde{\boldsymbol{a}}_T, \tilde{\boldsymbol{\vartheta}}_T)$, which are obtained in Step 2 of Estimator 2.

Theorem 5.2 *Suppose Assumptions 2.1, 5.1 and 5.2 hold, then we have*

$$\sqrt{T}(\tilde{V}_T^{(m)})^{1/2}(\tilde{W}_T^{(m)})^{-1/2}(\tilde{V}_T^{(m)})^{1/2}\begin{pmatrix} \tilde{\boldsymbol{a}}_T - \boldsymbol{a}_0 \\ \tilde{\boldsymbol{\vartheta}}_T - \boldsymbol{\vartheta}_0 \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}(0, I), \tag{31}$$

for $m = o(T)$ as $T \to \infty$, $C_T = \tilde{V}_T^{-1}\tilde{W}_T\tilde{V}_T^{-1}$, where

$$\tilde{W}_T = W_{T,1} + \begin{pmatrix} 0 & \Xi_1 \\ \Xi_1' & \Xi_2 \end{pmatrix}, \quad \tilde{V}_T^{(m)} = \begin{pmatrix} \mathbb{E}(\nabla_{\boldsymbol{a}}^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \boldsymbol{\Sigma}_0)) & 0 \\ 0 & \mathbb{E}(\nabla_{\boldsymbol{\vartheta}}^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \boldsymbol{\Sigma}_0)) \end{pmatrix}$$

with $W_{T,1}$ defined as in (28) and

$$\Xi_1 = cov\left(\sqrt{T}\nabla_{\boldsymbol{a}}\mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \boldsymbol{\Sigma}_0), \sqrt{T}\nabla_{\boldsymbol{\Sigma}}\mathcal{L}_T(\boldsymbol{a}_0, \boldsymbol{\Sigma}_0)\right)\mathcal{Q}_T'$$

$$\Xi_2 = 2cov\left(\sqrt{T}\nabla_{\boldsymbol{\vartheta}}\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\vartheta}_0, \boldsymbol{\Sigma}_0), \sqrt{T}\nabla_{\boldsymbol{\Sigma}}\mathcal{L}_T(\boldsymbol{a}_0, \boldsymbol{\Sigma}_0)\right)\mathcal{Q}_T' + \mathcal{Q}_T var(\sqrt{T}\nabla_{\boldsymbol{\Sigma}}\mathcal{L}_T(\boldsymbol{a}_0, \boldsymbol{\Sigma}_0))\mathcal{Q}_T',$$

Q_T is a $q \times (n+1)$-dimensional matrix defined by

$$\mathcal{Q}_T = \left(\frac{1}{T_m}\int\sum_{t=m/2}^{T-m/2}(\nabla_{\boldsymbol{\vartheta}}\mathcal{F}_{t,m}(\boldsymbol{\vartheta}_0, \boldsymbol{\Sigma}_0, \omega)^{-1}) \otimes \underline{H}_{(t,m)}(\omega)d\omega\right)\mathbb{E}(\nabla_{\boldsymbol{\Sigma}}^2\mathcal{L}_T(\boldsymbol{a}_0, \boldsymbol{\Sigma}_0))^{-1}$$

$$h_j^{(t,m)}(\omega) = \int_{-\pi}^{\pi} I_{t,m}^{(j)}(\lambda)f_j(\boldsymbol{\vartheta}_0, \omega - \lambda)d\lambda, \tag{32}$$

$\underline{H}_{(t,m)}(\omega) = (h_1^{(t,m)}(\omega), \ldots, h_{n+1}^{(t,m)}(\omega))$ and noting that $\otimes$ denotes the tensor product.

Remark 5.2 *It is unclear which of the two estimators have the smallest variance. However, comparing the variances of the two estimators in (29) and (31), we observe that they are similar. In particular $\tilde{V}_T$ is a submatrix of V_T. The terms Ξ_1 and Ξ_2 in $\tilde{W}_T$, are due to the estimation of $\boldsymbol{\Sigma}_0$ in the first stage of the scheme.*

5.4 The Gaussian likelihood and asymptotic efficiency of Estimator 1

In this section we compare the asymptotic properties of the frequency domain estimator $(\hat{\boldsymbol{a}}_T, \hat{\boldsymbol{\theta}}_T)$ with the Gaussian maximum likelihood estimator (GMLE). We recall the GMLE is constructed as if the stochastic coefficients $\{\alpha_{t,j}\}$ and errors $\{\varepsilon_t\}$ were Gaussian. However, unlike the frequency domain estimators, in general there does not exist an explicit expression for the asymptotic variance of the Gaussian maximum likelihood. Instead we will consider a subclass of SCR models, where the regressors vary slowly over time and do the comparison over this subclass. We will show that for this subclass an asymptotic expression for the asymptotic distributional variance of the GMLE

can be derived. We will assume that the regressors are such that there exists a 'smooth' function, $x_j(\cdot)$, such that $x_{t,j} = x_j(\frac{t}{N})$ and $Y_t := Y_{t,N}$ satisfies

$$Y_{t,N} = \sum_{j=1}^{n}(a_{j,0} + \alpha_{t,j})x_j(\frac{t}{N}) + \varepsilon_t \quad t = 1, \ldots, T. \tag{33}$$

In the following lemma we obtain the asymptotic distribution of the GMLE under the asymptotic framework that both T and $N \to \infty$.

Lemma 5.2 *Let us suppose that $\{Y_{t,N}\}$ satisfies (33), where the $\{\alpha_{t,j}\}$ and $\{\varepsilon_t\}$ are Gaussian and satisfy Assumption 2.1. Let*

$$\mathcal{F}(v, \boldsymbol{\theta}_0, \omega) = \sum_{j=1}^{n} x_j(v)^2 \sigma_{j,0}^2 f_j(\boldsymbol{\vartheta}_0, \omega) + \sigma_{n+1,0}^2 f_{n+1}(\boldsymbol{\vartheta}_0, \omega). \tag{34}$$

We assume that there does not exist another $\boldsymbol{\theta} \in \Theta_1 \otimes \Theta_2$ such that $\mathcal{F}(v, \boldsymbol{\theta}_0, \omega) = \mathcal{F}(v, \boldsymbol{\theta}_0, \omega)$ for all $v \in [0, T/N]$ and the matrix $\frac{N}{T} \int_0^{T/N} \mathbf{x}(v)\mathbf{x}(v)'dv$, (with $\mathbf{x}(v)' = (x_1(v), \ldots, x_n(v))$) has eigenvalues which are bounded from above and away from zero. Suppose $(\boldsymbol{a}_{mle}, \boldsymbol{\theta}_{mle})$ is the Gaussian maximum likelihood estimator of the parameters $(\boldsymbol{a}_0, \boldsymbol{\theta}_0)$. Then we have

$$\sqrt{T} \begin{pmatrix} \boldsymbol{a}_{mle} - \boldsymbol{a}_0 \\ \boldsymbol{\theta}_{mle} - \boldsymbol{\theta}_0 \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \begin{pmatrix} \Delta_1^{-1} & 0 \\ 0 & \Delta_2^{-1} \end{pmatrix}\right)$$

with $N \to \infty$ as $T \to \infty$, where

$$(\Delta_1)_{j_1,j_2} = \frac{N}{T} \int_0^{T/N} x_{j_1}(v)x_{j_2}(v)\mathcal{F}(v, \boldsymbol{\theta}_0, 0)^{-1}dv,$$

$$\Delta_2 = 2\frac{N}{T} \int_0^{T/N} \int_0^{2\pi} \frac{\nabla_{\boldsymbol{\theta}}\mathcal{F}(v, \boldsymbol{\theta}_0, \omega)(\nabla_{\boldsymbol{\theta}}\mathcal{F}(v, \boldsymbol{\theta}_0, -\omega))'}{|\mathcal{F}(v, \boldsymbol{\theta}_0, \omega)|^2}d\omega dv,$$

$$a(v, k) = \int \frac{1}{\mathcal{F}(v, \boldsymbol{\theta}_0, \omega)} \exp(ik\omega)d\omega \quad \underline{b}(v, k) = \int \nabla_{\boldsymbol{\theta}}\mathcal{F}(v, \boldsymbol{\theta}_0, \omega)^{-1} \exp(ik\omega)d\omega$$

$a(v) = \{a(v, k)\}$ and $\bar{\underline{b}}(v) = \{\underline{b}(v, -k)\}$.

In practice, for any given set of regressors $\{x_{t,j}\}$, N will not be known, but a lower bound for N can be obtained from $\{x_{t,j}\}$. To ensure the magnitude of the regressors do not influence N, we will assume that the regressors satisfy $\frac{1}{T}\sum_{t=1}^{T} x_{t,j}^2 = 1$ (for all j). To measure the smoothness of the regressors define

$$\hat{N} = \frac{1}{\sup_{t,j} |x_{t,j} - x_{t-1,j}|}. \tag{35}$$

Clearly if $\hat{N}$ is large, this indicates that the regressors are smooth.

We now compare the asymptotic variance of the GMLE and Estimator 1.

18

Proposition 5.3 *Suppose Assumptions 2.1, 5.1 and 5.2 hold and*

$$\sup_j \int \Big|\frac{d^2 f_j(\boldsymbol{\vartheta}_0, \omega)}{d\omega^2}\Big|^2 d\omega < \infty \quad \text{and} \sup_j \int \Big|\frac{d^2 \nabla_{\boldsymbol{\vartheta}} f_j(\boldsymbol{\vartheta}_0, \omega)}{d\omega^2}\Big|^2 d\omega < \infty. \tag{36}$$

Let $V_T^{(m)}$, $W_T^{(m)}$, $\Delta_{T,N,1}$, $\Delta_{T,N,2}$ and $\hat{N}$ be defined as in (26), (27), Lemma 5.2 and (35), respectively. Then we have

$$\Big| W_T^{(m)} - \Big(\begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} + \begin{pmatrix} 0 & \Gamma_{1,2} \\ \Gamma_{1,2}' & \Gamma_2 \end{pmatrix} \Big) \Big| \; \le \; K \Big\{ \frac{1}{\hat{N}} + \frac{1}{m} + \frac{1}{T_m} + \frac{m}{\hat{N}} \Big\} \tag{37}$$

$$\Big| V_T^{(m)} - \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix} \Big| \; \le \; K \frac{m}{\hat{N}} \tag{38}$$

where K is a finite constant,

$$\Gamma_2 \;=\; \frac{N}{T} \int_0^{T/N} \int_0^{2\pi} \int_0^{2\pi} \frac{\nabla_{\boldsymbol{\theta}} \mathcal{F}(v, \boldsymbol{\theta}_0, \omega_1) \nabla_{\boldsymbol{\theta}} \mathcal{F}(v, \boldsymbol{\theta}_0, \omega_1)'}{\mathcal{F}(v, \boldsymbol{\theta}_0, \omega_1)^2 \mathcal{F}(v, \boldsymbol{\theta}_0, \omega_2)^2} \mathcal{F}_4(v, \boldsymbol{\vartheta}_0, \omega_1, \omega_2, -\omega_1) d\omega_1 d\omega_2 dv,$$

$$\Gamma_{1,2} \;=\; \frac{N}{T} \int_0^{T/N} \mathbf{x}(v) \int_0^{2\pi} \int_0^{2\pi} \frac{\nabla_{\boldsymbol{\theta}} \mathcal{F}(v, \boldsymbol{\theta}_0, \omega_2)'}{\mathcal{F}(v, \boldsymbol{\theta}_0, \omega_1) \mathcal{F}(v, \boldsymbol{\theta}_0, \omega_2)^2} \mathcal{F}_3(v, \boldsymbol{\vartheta}_0, \omega_1, \omega_2) \exp(ir\omega_2) d\omega_1 d\omega_2 dv,$$

$$\tag{39}$$

$\mathbf{x}(v)' = (x_1(v), \ldots, x_n(v))$, with $\mathcal{F}(v, \boldsymbol{\theta}, \omega)$ defined as in (34),

$$\mathcal{F}_3(v, \boldsymbol{\vartheta}, \omega_1, \omega_2) \;=\; \sum_{j=1}^{n+1} \kappa_{j,3} x_j(v)^3 A_j(\boldsymbol{\vartheta}, \omega_1) A_j(\boldsymbol{\vartheta}, \omega_2) A_j(\boldsymbol{\vartheta}, -\omega_1 - \omega_2)$$

$$\mathcal{F}_4(v, \boldsymbol{\vartheta}, \omega_1, \omega_2, \omega_3) \;=\; \sum_{j=1}^{n+1} \kappa_{j,4} x_j(v)^4 A_j(\boldsymbol{\vartheta}, \omega_1) A_j(\boldsymbol{\vartheta}, \omega_2) A_j(\boldsymbol{\vartheta}, \omega_3) A_j(\boldsymbol{\vartheta}, -\omega_1 - \omega_2 - \omega_3),$$

$\kappa_{j,3} = cum(\eta_{0,j}, \eta_{0,j}, \eta_{0,j})$ and $\kappa_{j,4} = cum(\eta_{0,j}, \eta_{0,j}, \eta_{0,j}, \eta_{0,j})$.

Remark 5.3 (Selecting m) *Let us consider the case that $\{\alpha_{t,j}\}$ and $\{\varepsilon_t\}$ are Gaussian, in this case $\Gamma_{1,2} = 0$ and $\Gamma_2 = 0$. Comparing the asymptotic variances of the GMLE and $(\hat{\mathbf{a}}_T, \hat{\boldsymbol{\theta}}_T)$ we see that if we let $\hat{N} \to \infty$, $m \to \infty$ and $m/\hat{N} \to 0$ (noting that we have replaced $\hat{N}$ with N) as $T \to \infty$, then the GMLE $((\mathbf{a}_{mle}, \boldsymbol{\theta}_{mle}))$ and $(\hat{\mathbf{a}}_T, \hat{\boldsymbol{\theta}}_T)$ both have the same asymptotic distribution. Hence within this framework, the relative efficiency of the frequency domain estimator compared with the GMLE is one.*

Furthermore, in the case that $\{\alpha_{t,j}\}$ and $\{\varepsilon_t\}$ are Gaussian, (37) suggests a method for selecting m. Since in this case the GMLE is efficient, by using (37) we have

$$\big| (V_T^{(m)})^{-1} W_T^{(m)} (V_T^{(m)})^{-1} - diag(\Delta_1^{-1}, \Delta_2^{-1}) \big| = O_p\Big(\frac{1}{\hat{N}} + \frac{1}{m} + \frac{1}{T_m} + \frac{m}{\hat{N}} \Big).$$

Hence the above difference is minimised when $m = \hat{N}^{1/2}$.

5.4.1 An expression for $W_T^{(m)}$

An expression for $V_T^{(m)}$ in terms of $\mathcal{F}_{t,m}$ is given in (26) we now obtain a similar expression for $W_T^{(m)}$. The expression for $W_T^{(m)}$ is quite cumbersome, to make it more interpretable we state it in terms of operators. Let ℓ_2 denote the space of all square summable (vector) sequences. We define the general operator $\Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(\mathbf{j})}(\cdot)$ which acts on the d_1 and d_2-dimensional column vector sequences $\underline{g} = \{\underline{g}(k)\} \in \ell_2$ and $\underline{h} = \{\underline{h}(k)\} \in \ell_2$ (to reduce notation we do not specify the dimensions of the vectors in ℓ_2)

$$\Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(\mathbf{j})}(\underline{g},\underline{h})(s,u) = \sum_{r_1=-k_1}^{m-k_1} \sum_{r_2=-k_2}^{m-k_2} \underline{g}(r_1)\underline{h}(r_2)' \mathcal{K}_{\mathbf{x}_s,\mathbf{x}_{s+r_1},\mathbf{x}_u,\mathbf{x}_{u+r_2}}^{(\mathbf{j})}(s-u,r_1,r_2), \tag{40}$$

where $\mathbf{x}_s = (x_{s,1},\ldots,x_{s,n},1)$ are the regressors observed at time s and the kernel $\mathcal{K}^{(\mathbf{j})}$ is defined as either

$$\mathcal{K}_{\mathbf{x}_{s_1},\mathbf{x}_{s_2},\mathbf{x}_{s_3},\mathbf{x}_{s_4}}^{(0)}(k,r_1,r_2) = \sum_{j=1}^{n+1} x_{s_1,j}x_{s_2,j}x_{s_3,j}x_{s_4,j}\mathrm{cov}(\alpha_{0,j}\alpha_{r_1,j},\alpha_{k,j}\alpha_{k+r_2,j})$$

$$\mathcal{K}_{\mathbf{x}_{s_1},\mathbf{x}_{s_2},\mathbf{x}_{s_3},\mathbf{x}_{s_4}}^{(j_1)}(k,r_1,r_2) = \sum_{j=1}^{n+1} x_{s_1,j}x_{s_2,j}x_{s_3,j}x_{s_4,j_1}\mathrm{cov}(\alpha_{0,j}\alpha_{r_1,j},\alpha_{k,j}) \quad 1 \le j_1 \le n$$

$$\text{or } \mathcal{K}_{\mathbf{x}_{s_1},\mathbf{x}_{s_2},\mathbf{x}_{s_3},\mathbf{x}_{s_4}}^{(j_1,j_2)}(k,r_1,r_2) = \sum_{j=1}^{n+1} x_{s_1,j}x_{s_2,j_1}x_{s_3,j}x_{s_4,j_2}\mathrm{cov}(\alpha_{0,j},\alpha_{k,j}) \quad 1 \le j_1, j_2 \le n \tag{41}$$

where for notational convenience we set $\alpha_{t,n+1} := \varepsilon_t$ and $x_{t,n+1} := 1$. We will show that $\mathrm{var}(\nabla \mathcal{L}_T^{(m)}(\mathbf{a}_0,\boldsymbol{\theta}_0))$ can be written in terms of $\Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(\mathbf{j})}$. Suppose $\mathcal{G}_s(\cdot)$ and $\mathcal{H}_u(\cdot)$ are square integrable vector functions, then it can be shown

$$\mathrm{var}\left(\begin{array}{c} \int_0^{2\pi} \mathcal{H}_u(\omega)X_u \sum_{r_2=-k_2}^{m-k_2} x_{u+r_1,j_1}\exp(ir_2\omega)d\omega \\ \int_0^{2\pi} \mathcal{G}_s(\omega)X_s \sum_{r_1=-k_1}^{m-k_1} X_{s+r_1}\exp(ir_1\omega)d\omega \end{array} \right)$$

$$= \left(\begin{array}{cc} \Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_2,k_2),(j_1,j_1)}(\underline{h}_u,\bar{\underline{h}}_u)(u,u) & \Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(j_1)}(\bar{\underline{g}}_s,\underline{h}_u)(s,u) \\ \Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(j_1)}(\underline{g}_s,\bar{\underline{h}}_u)(s,u) & \Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_1),(0)}(\underline{g}_s,\bar{\underline{g}}_s)(s,s) \end{array} \right) \tag{42}$$

where $\underline{g}_s = \{\underline{g}_s(r) = \int \mathcal{G}_s(\omega)\exp(ir\omega)d\omega\}_r$, $\underline{h}_s = \{\underline{h}_s(r) = \int \mathcal{H}_s(\omega)\exp(r\omega)d\omega\}_r$, $\bar{\underline{g}}_s = \{\underline{g}_s(-r)\}_r$ and $\bar{\underline{h}}_s = \{h_s(-r)\}_r$.

Using the above notation we can write

$$W_T^{(m)} = T\mathrm{var}(\nabla \mathcal{L}_T^{(m)}(\mathbf{a}_0,\boldsymbol{\theta}_0)) = \frac{1}{m^2 T_m} \sum_{k_1,k_2=1}^{m} \sum_{s=k_1}^{T-m+k_1} \sum_{u=k_2}^{T-m+k_2} \tag{43}$$

$$\left(\begin{array}{cc} \Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(\mathbf{j},\mathbf{j})}(a_{s+m/2-k_1},a_{u+m/2-k_2})(s,u) & \Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(\mathbf{j})}(\underline{b}_{s+m/2-k_1},a_{u+m/2-k_2})(s,u) \\ \Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(\mathbf{j})}(\bar{\underline{b}}_{s+m/2-k_1},a_{u+m/2-k_2})(s,u) & \Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(0)}(\underline{b}_{s+m/2-k_1},\bar{\underline{b}}_{u+m/2-k_2})(s,u) \end{array} \right),$$

where $\Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(\mathbf{j},\mathbf{j})} = \{\Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(j_1,j_2)}\}_{j_1,j_2=1,\ldots,n}$ and $\Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(\mathbf{j})} = \{\Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(j)}\}_{j=1,\ldots,n}$, $\underline{b}_s = \{\underline{b}_s(k)\}$, $a_s = \{a_s(k)\}$ with

$$a_s(k) = \int \mathcal{F}_{s,m}(\boldsymbol{\theta}_0,\omega)^{-1}\exp(ik\omega)d\omega, \quad \underline{b}_s(k) = \int \nabla_{\boldsymbol{\theta}}\mathcal{F}_{s,m}(\boldsymbol{\theta}_0,\omega)^{-1}\exp(ik\omega)d\omega, \tag{44}$$

(noting that $\mathcal{F}_{s,m}(\boldsymbol{\theta}_0,\omega)^{-1}$ is a symmetric function hence $a_s = \bar{a}_s$). We will use (91) when we compare Estimator 1 with the Gaussian maximum likelihood estimator.

6 Real data analysis

We now consider two real data examples.

Example 1: Application to financial time series: Modelling of T-bills and Inflation rates in the US

There are many possible applications of stochastic coefficient regression models in econometrics. One such application is modelling the influence of the nominal interest rate of three month (short term) Treasurey bills (T-bills) on monthly inflation. Fama (1977) argues that the relationship between the T-bills and inflation rate determines whether the market for short term Treasurey bills is efficient or not. In this section we will consider three month T-bills and monthly inflation data observed monthly between January 1959 to December 2008, the data can be obtained from the US Federal reserve,

`http://www.federalreserve.gov/releases/h15/data.htm#fn26` and

`http://inflationdata.com/inflation/Inflation_Rate/HistoricalInflation.aspx` respectively. A plot of the time series of both sets of observations is given in Figure 1. The estimated correlation coefficient between the three month T-bills and monthly inflation is 0.72. Let Y_t and x_t denote monthly inflation and T-bills interest rate at time t respectively. Fama (1977) and Newbold and Bos (1985) consider the nominal interest rate of three month T-bills and inflation rate data observed every three months between 1953-1980. Fama (1977) fitted the linear regression model $Y_t = a_1 x_t + \varepsilon_t$ ($\{\varepsilon_t\}$ are iid) to the data, and showed that there wasn't a significant departure of a_1 from one, he used this to argue that the T-bills market was efficient. However, Newbold and Bos (1985) argue that the relationship between T-bills and inflation is more complex and suggest that the SCR may be a more appropriate model, where the coefficient of x_t is stochastic and follows an AR(1) model. In other words

$$Y_t = a_0 + (a_1 + \alpha_{t,1})x_t + \varepsilon_t, \quad \alpha_{t,1} = \vartheta_1\alpha_{t-1,1} + \eta_t \tag{45}$$
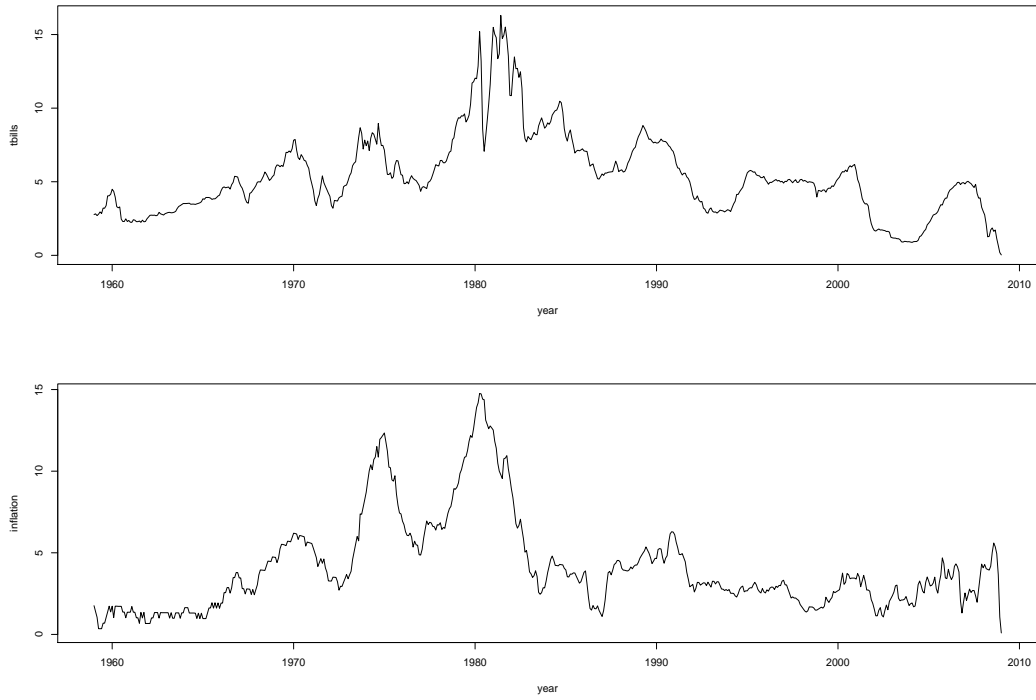
Figure 1: The top plot is 3-month T-bill nominal interest rate taken monthly and lower plot is the monthly inflation rate

where $\{\varepsilon_t\}$ and $\{\eta_t\}$ are iid random variables with $\mathbb{E}(\varepsilon_t) = 0$, $\mathbb{E}(\eta_t) = 0$, $\text{var}(\varepsilon_t) = \sigma_\varepsilon^2 < \infty$ and $\text{var}(\eta_t) = \sigma_\eta^2 < \infty$. Using the GMLE they obtain the parameter estimates $a_0 = -0.97$, $a_1 = 1.09$, $\vartheta_1 = 0.89$, $\sigma_\varepsilon^2 = 1.41$ and $\sigma_\eta^2 = 0.013$. We now fit the same model to the T-bills data observed monthly from January 1959 to December 2008 (600 observations), and use the two-step Estimator 2 to estimate the parameters a_0, a_1, ϑ_1, $\sigma_\alpha^2 = \text{var}(\alpha_{t,1})$ and $\sigma_\varepsilon^2 = \text{var}(\varepsilon_t)$ using Estimator 2. The variances σ_α^2 and σ_ε^2 are estimated in the first step of Estimator 2, the estimates with their standard errors are given Table 1. Note that comparing the estimates with their standard errors we observe that both parameters σ_ε^2 and σ_α^2 appear significant. In the second stage of the scheme we estimate a_0, a_1 and ϑ_1 (we note that because the intercept a_0 appears to be insignificant we also do the estimation excluding the intercept), these estimates are also summarised in Table 1. The estimates for different m are quite close. The model found to be most suitable for the above data when $m = 200$ is

$$Y_t \;\; = \;\; (0.73 + \alpha_{t,1})x_t + \varepsilon_t, \quad \alpha_{t,1} = 0.960\alpha_{t-1,1} + \eta_t,$$

where $\sigma_\varepsilon^2 = 1.083^2$ and $\sigma_\alpha^2 = 0.285^2$ (hence $\sigma_\eta^2 = 0.079^2$). We observe that the AR(1) parameter estimate of the stochastic coefficient $\{\alpha_{t,1}\}$ is 0.96. This value is close to one, suggesting that the stochastic coefficients $\{\alpha_{t,1}\}$ could come from a unit root process.

To assess the validity of this model, we obtain one step ahead best linear predictors of Y_t given $\{Y_s\}_{s=1}^{t-1}$ and the current T-bills rate x_t, every month in the year 2008. In order to do the prediction we re-estimate the parameters α_1, θ, σ_ε^2 and σ_α^2 using the observations from 1959-2007. We use the two-step Estimator 2 with $m = 200$ to obtain

$$Y_t \;\; = \;\; (0.77 + \alpha_{t,1})x_t + \varepsilon_t, \quad \alpha_{t,1} = 0.965\alpha_{t-1,1} + \eta_t, \tag{46}$$

with $\sigma_\varepsilon^2 = 0.79^2$ and $\sigma_\alpha^2 = 0.30^2$ (hence $\sigma_\eta^2 = 0.18^2$). We also fit the linear regression model $Y_t = a_1 x_t + \varepsilon_t$ to the data and use OLS to obtain the model $Y_t = 0.088 + 0.75x_t + \varepsilon_t$. The predictor using the usual multiple linear regression model and and one-step ahead predictor using the SCR model are given in Figure 2. To do the one-step ahead prediction we use the Kalman filter (using the R package ss1.R, see Shumway and Stoffer (2006), Chapter 6 for the details). The mean squared prediction errors over the 12 months using the multiple regression and the SCR model are 8.99 and 0.89 respectively. We observe from the plots in Figure 2 that the multiple regression model always underestimates the true value and the mean square error is substantially larger than the SCR model.
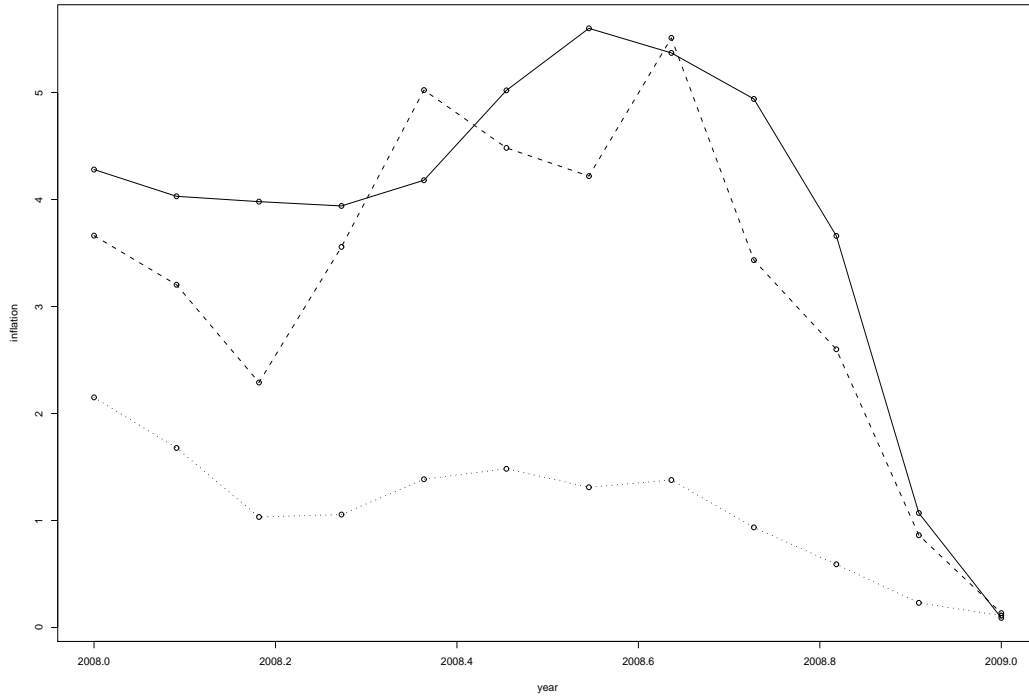
Figure 2: We compare the true inflation rates with their predictions. The continuous thick line $-$ is the true inflation rate. The broad dashed line $--$ is SCR one step ahead predictor given in (46). The fine dashed line $\cdots$ is the linear regression predictor $\hat{Y}_t = 0.088 + 0.750 x_t$.

	a_0	a_1	ϑ_1	σ_α	σ_ε
OLS	0.088	0.750			
(s.e.)	(0.18)	(0.029)			
Stage 1	0.088	0.74		0.285	1.083
(s.e.)				(0.011)	(0.059)
$m = 10$ (with intercept)	0.618	0.625	0.981		
(s.e.)	(0.325)	(0.069)	(0.042)		
$m = 10$ (without intercept)		0.741	(0.971)		
(s.e.)		(0.0325)	(0.05)		
$m = 50$ (with intercept)	0.309	0.687	0.969		
(s.e.)	(0.35)	(0.069)	(0.026)		
$m = 50$ (without intercept)		0.743	0.957		
(s.e.)		(0.032)	(0.038)		
$m = 200$ (with intercept)	0.223	0.7327	0.96088		
(s.e.)	(0.44)	(0.022)	(0.024)		
$m = 200$(without intercept)		0.765	0.951		
(s.e.)		(0.029)	(0.030)		
$m = 400$ (with intercept)	0.367	0.725	0.963		
(s.e.)	(0.48)	(0.070)	(0.023)		
$m = 400$(without intercept)		0.773	0.957		
(s.e.)		(0.029)	(0.026)		

Table 1: We fit the model $Y_t = a_0 + (a_1 + \alpha_{t,1})x_t + \varepsilon_t$, where $\alpha_{t,1} = \vartheta_1 \alpha_{t-1,1} + \eta_t$, with and without the intercept a_0. The estimates using least squares and the frequency domain estimator for different m are given. The values in the brackets are the corresponding standard errors.

Example 2: Application to environmental time series: Modelling of visibility and air pollution

It is known that air visibility quality depends on the amount of particulate matter (particulate matter negatively effects visibility). Furthermore, air pollution is known to influence the amount of particulate matter (see Hand et al. (2008)). To model the influence of air pollution on particulate matter Hand et al. (2008) (see equation (6)) fit a linear regression model. However, Burnett and Guthrie (1970) argue that the influence of air pollution on air visibility may vary each day, depending on meteorological conditions, and suggest that a SCR model may be more appropriate than a multiple linear regression model. In this section we investigate this possibility. We consider the influence of man made emissions on Particulate Matter (PM2.5-10) in Shenandoah National Park, Virginia, USA. The data we consider is Ammonium Nitrate Extinction (ammNO3f), Ammonium Sulfate (ammSO4f), Carbon Elemental Total (ECF) and Particulate Matter (PM2.5-10) (ammNO3f, ammSO4f and ECF are measured in ug/m3) which has been collected every three days between 2000-2005 (600 observations). We obtain the data from the VIEWS website `http://vista.cira.colostate.edu/views/Web/Data/DataWizard.aspx`. We mention that the influence of man made emissions on air visibility is of particular importance to the US national parks service (NPS), who collected and compiled this data. An explanation of the data and how air pollution influences visibility (light scattering) can be found in Hand et al. (2008).

The plots of both the air pollution and PM2.5-10 data is given in Figure 3 and 4 respectively. There is a clear seasonal component in all the data sets as seen from their plots. Therefore to prevent spurious correlation between the PM2.5-10 and air pollution we detrended and deseasonalised the PM2.5-10 and emissions data. To identify the dominating harmomics we used the maximum periodogram methods suggested in Quinn and Fernandez (1991) and Kavalieris and Hannan (1994). To the detrended and deseasonalise PM2.5-10 and air pollution data we fitted the following model

$$Y_t = (a_1 + \alpha_{t,1})x_{t,1} + (a_2 + \alpha_{t,2})x_{t,2} + (a_3 + \alpha_{t,3})x_{t,3} + \varepsilon_t,$$

where $\{x_{t,1}\}$, $\{x_{t,2}\}$, $\{x_{t,3}\}$ and $\{Y_t\}$ are the detrended and deseasonalised ammNO3f, ammSO4f, ECF and Particulate Matter (PM2.5-10), and $\{\alpha_{t,j}\}$ and ε_t satisfy

$$\alpha_{t,j} = \vartheta_j \alpha_{t-1,j} + \eta_{t,j}, \quad \text{for and } j = 1,2,3, \qquad \varepsilon_t = \vartheta_4 \varepsilon_{t-1} + \eta_{t,4},$$

$\{\eta_{t,i}\}$ iid random variables. Let $\sigma_\varepsilon^2 = \text{var}(\varepsilon_t)$, $\sigma_{\alpha,1}^2 = \text{var}(\alpha_{t,1})$, $\sigma_{\alpha,2}^2 = \text{var}(\alpha_{t,2})$ and $\sigma_{\alpha,3}^2 = \text{var}(\alpha_{t,3})$.

We used Estimator 2 to estimate the parameters $a_1, a_2, a_3, \alpha_{t,1}, \alpha_{t,2}, \alpha_{t,3}, \alpha_{t,4}, \sigma_\varepsilon^2, \sigma_{\alpha,1}^2, \sigma_{\alpha,2}^2$ and $\sigma_{\alpha,3}^2$. At the start off the minimisations of the objective functions $\mathcal{L}_T$ and $\mathcal{L}_T^{(m)}$, as initial values we
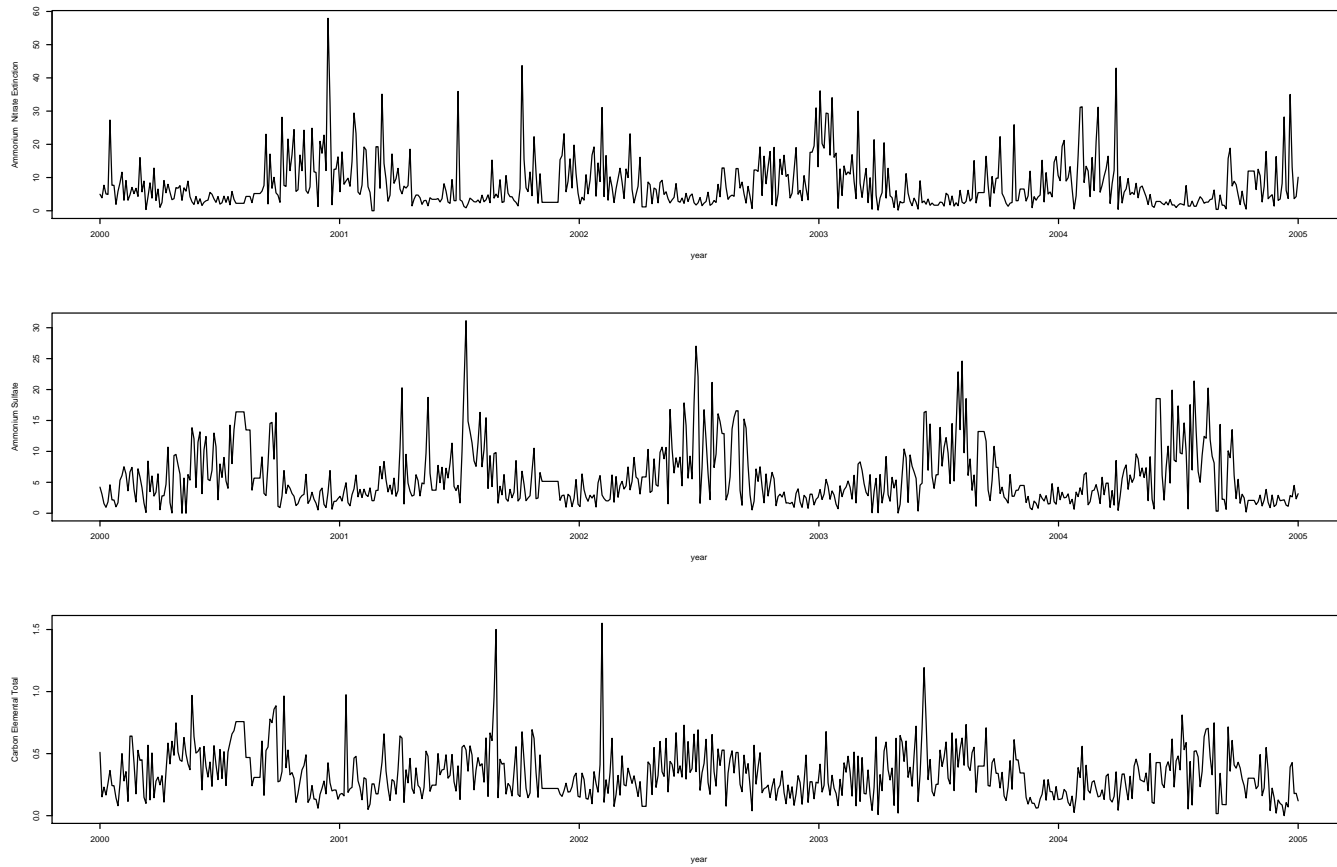
26

Figure 3: The top plot is 3-day Ammonium Nitrate Extinction (Fine), middle plot is 3-day Ammonium Sulfate (Fine) and lower plot is three day Carbon Elemental Total (Fine)
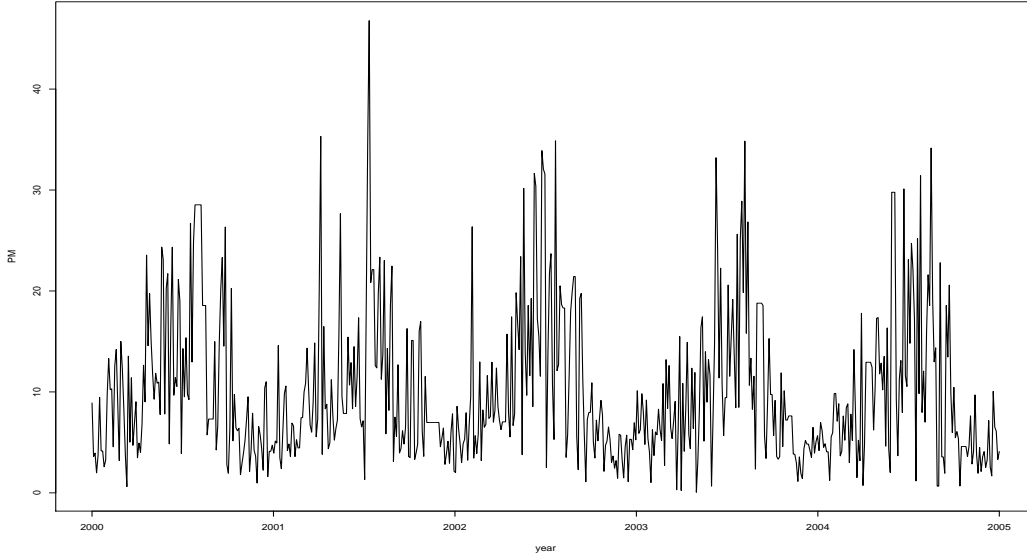
27

Figure 4: The plot is 3-day Particulate Matter (PM2.5 - PM10)

gave the least squares estimates of $\boldsymbol{a}_0$ for the mean regression coefficients and 0.1 for all the other unknown parameters. In the first stage of the scheme we estimated a_1, a_2, a_3 and $\sigma_\varepsilon^2, \sigma_{\alpha,1}^2, \sigma_{\alpha,2}^2$ and $\sigma_{\alpha,3}^2$. We also fitted parsimonious models where some of the coefficients were kept fixed rather stochastic. The results are summarised in Table 2 (step 1). We observe that the estimate of $\sigma_{\alpha,1}$ is extremely small and insignificant. We observe that the minimum value of the objective function $\mathcal{L}_T$ is about the same when the coefficient of ammNO3f $\{x_{t,1}\}$ is fixed and random (it is 2.012). This suggests that the coefficient of ammNO3f $\{x_{t,1}\}$ is deterministic. This may indicate that the relative contribution of NO3f ($\{x_{t,1}\}$) to the response is constant throughout the period of time and is not influenced by any other extraneous factors. We re-did the minimisation systematically removing $\sigma_{\alpha,2}$ and $\sigma_{\alpha,3}$, but the minimum value of $\mathcal{L}_T$, changed quite substantially (compare the minimum of the objective functions 2.012 with 4.09, 3.99 and 4.15). Hence the most appropriate model appears to be

$$Y_t = a_1 x_{t,1} + (a_2 + \alpha_{t,2})x_{t,2} + (a_3 + \alpha_{t,3})x_{t,3} + \varepsilon_t,$$

where $\{\alpha_{t,1}\}$ and $\{\alpha_{t,2}\}$ are stochastic coefficients. It is of interest to investigate whether the coefficients of ammSO4f and ECF are purely random or correlated, and we investigate this in the second stage of the frequency domain scheme, where we modelled $\{\alpha_{t,2}\}$ and $\{\alpha_{t,3}\}$ both as iid random variables and as the AR(1) model $\alpha_{t,j} = \vartheta_j \alpha_{t-1,j} + \eta_{t,j}$, for $j = 2, 3$. The estimates for various different models and different values of m are given in Table 2. If we compare the minimum

28

of the objective function where $\{\alpha_{t,2}\}$ and $\{\alpha_{t,3}\}$ are modelled as both iid and satisfying an AR(1) model, we see that there is very little difference between them. Moreover the standard errors for the estimates of ϑ_2 and ϑ_3, are large. Altogether, this suggests that ϑ_2 and ϑ_3 are not significant and $\{\alpha_{t,2}\}$ and $\{\alpha_{t,3}\}$ are uncorrelated over time. Hence it seems plausible that the coefficients of ammSO4f and ECF are random, but independent. To check the possibility that the errors $\{\varepsilon_t\}$ are correlated, we fitted an AR(1) model to the errors. However we observe from Table 2, that the AR(1) parameter does not appear to be significant. Moreover, comparing the minimum of the objective function $\mathcal{L}_{600}^{(m)}$ (for different values of m) fitting iid $\{\varepsilon_t\}$ and an AR(1) to $\{\varepsilon_t\}$ gives almost the same value. This suggests that the errors are independent. In summary, our analysis suggests that the influence of ammNO3f on PM2.5-10 is fixed over time, whereas the influence of ammSO4f and ECF varies purely randomly over time. Using the estimator obtained when $m = 200$ this suggests the model

$$Y_t = 0.255x_{t,1} + (4.58 + \alpha_{t,2})x_{t,2} + (1.79 + \alpha_{t,3})x_{t,3} + \varepsilon_t,$$

where $\{\alpha_{t,2}\}$ and $\{\alpha_{t,3}\}$ are iid random variables, with $\sigma_{\alpha,2} = 1.157$, $\sigma_{\alpha,3} = 0.84$, $\sigma_\varepsilon = 1.296$. Based on our analysis it would appear that the coefficients of pollutants are random, but there is no linear dependence between the current coefficient and the previous coefficient. On possible explanation for the lack of dependence is that the data is taken every three days and not daily. This could mean that the meterological conditions from three days ago has little influence on today's particulate matter. On the other hand if we were to analyse the daily pollutants and daily PM2.5-10 the conclusions could have been different. But this daily data is not available. It is likely that since the data is aggregated (smoothed) over a three day period any possible dependence in the data was removed.

Acknowledgements

A Appendix

A.1 Proofs for Section 2.2.2 and 4

We first prove Proposition 2.2.

	a_1	a_2	a_3	ϑ_2	ϑ_3	ϑ_4	$\sqrt{\text{var}(\alpha_{t,1})}$	$\sqrt{\text{var}(\alpha_{t,2})}$	$\sqrt{\text{var}(\alpha_{t,3})}$	$\sqrt{\text{var}(\varepsilon_t)}$	minL
OLS	0.29	4.57	1.76								
	(0.078)	(0.088)	(0.0908)								
Stage 1	0.38	4.53	1.58				7.10^{-7}	1.25	0.84	1.29	2.102
(s.e.)	(0.048)	(0.079)	(0.076)				(0.07)	(0.097)	(0.098)	(0.046)	
Stage 1	0.56	3.28	-3.09				0.75		3.315	4.496	4.09
(s.e.)	(0.170)	(0.181)	(0.29)				(0.042)		(0.159)	(0.049)	
Stage 1	0.387	4.53	1.58					1.157	0.84	1.296	2.102
(s.e.)	(0.048)	(0.080)	(0.076)					(0.009)	(0.009)	(0.002)	
Stage 1	0.287	3.52	-3.11						3.00	3.83	3.99
(s.e.)	(0.139)	(0.171)	(0.27)						(0.164)	(0.07)	
Stage 1	0.521	5.09	-2.93							4.42	4.15
(s.e.)	(0.131)	(0.145)	(0.65)							(0.011)	
m=10	0.393	4.47	1.630	0.883	-0.144						2.066
(s.e.)	(0.048)	(0.079)	(0.076)	(0.095)	(0.351)						
m=10	0.39	4.47	1.63		-0.13						2.066
(s.e.)	(0.05)	(0.077)	(0.071)		(0.34)						
m=10	0.388	4.467	1.661			0.94					2.084
(s.e.)	(0.039)	(0.061)	(0.058)			(0.029)					
m=50	0.327	4.55	1.75	0.617	-0.235						2.199
(s.e.)	(0.056)	(0.070)	(0.071)	(0.30)	(0.659)						
m=50	0.324	4.54	1.746		-0.32						2.22
(s.e.)	(0.055)	(0.073)	(0.070)		(0.54)						
m=50	0.308	4.55	1.764			0.244					2.21
(s.e.)	(0.049)	(0.062)	(0.062)			(0.127)					
m=200	0.261	4.595	1.770	0.538	0.9722						2.111
(s.e.)	(0.06)	(0.067)	(0.070)	(0.322)	(0.042)						
m=200	0.261	4.60	1.77		-0.087						2.124
(s.e.)	(0.06)	(0.068)	(0.068)		(1.2)						
m=200	0.255	4.58	1.797			0.458					2.1339
(s.e.)	(0.051)	(0.058)	(0.058)			(0.145)					
m=400	0.2531	4.597	1.793	0.979	0.932						2.116
(s.e.)	(0.061)	(0.068)	(0.070)	(0.032)	(0.143)						
m=400	0.25	4.59	1.79		0.92						2.128
(s.e.)	(0.06)	(0.068)	(0.068)		(0.167)						
m=400	0.250	4.580	1.807			0.478					2.139
(s.e.)	(0.051)	(0.0583)	(0.059)			(0.148)					

Table 2: In stage 1 we fitted the model $Y_t = (a_1 + \alpha_{t,1})x_{t,1} + (a_2 + \alpha_{t,2})x_{t,2} + (a_3 + \alpha_{t,3})x_{t,3} + \varepsilon_t$, and various subsets (here we did not model any dependence in the stochastic coefficients). In the second stage (for $m = 10, 50, 200$ and 400) we fitted AR(1) models to $\{\alpha_{t,2}\}$, $\{\alpha_{t,3}\}$ and $\{\varepsilon_t\}$, that is $\alpha_{t,2} = \vartheta_2\alpha_{t-1,2} + \eta_{t,2}$, $\alpha_{t,3} = \vartheta_3\alpha_{t-1,3} + \eta_{t,3}$ and $\varepsilon_t = \vartheta_4\varepsilon_{t-1} + \eta_{t,4}$. The value of the frequency domain likelihood at the minimal value is also given in the column minL. The standard errors of the estimates are given below each estimate in brackets.

PROOF of Proposition 2.2 We first observe that because $\tilde{X}_{t,N}$ is locally stationary there exists a function $A(\cdot)$, such that

$$\tilde{X}_{t,N} = \int A(\frac{t}{N}, \omega) \exp(it\omega) dZ(\omega) + O_p(N^{-1}). \tag{47}$$

Now since $A(\cdot) \in L_2([0,1] \otimes [-\pi, \pi])$, for any δ there exists a n_δ such that

$$\left| A(v, \omega) - \sum_{j=1}^{n_\delta} A_j(\omega)x_j(u) \right| \leq K\delta,$$

where $A_j(\omega) = \int A(v,\omega)x_j(v)dv$, and K is a finite constant independent of δ. Substituting this into (48) gives

$$\tilde{X}_{t,N} = \sum_{j=1}^{n_\delta}\left\{\int A_j(\omega)\exp(it\omega)dZ(\omega)\right\}x_j(\frac{t}{N}) + O_p(\delta + N^{-1}).\tag{48}$$

Let $\alpha_{t,j} = \int A_j(\omega)\exp(it\omega)dZ(\omega)$, since $Z(\omega)$ is a orthogonal increment process with $\mathbb{E}|dZ(\omega)|^2 = d\omega$ it is clear that $\{\boldsymbol{\alpha}_t = (\alpha_{t,1},\ldots,\alpha_{t,n})\}$ is a second order stationary vector time series. Thus for every δ there exists a stochastic coefficient regression expansion, thus we have the desired result. $\square$

We now prove Proposition 4.1.

PROOF of Proposition 4.1 We first prove the result under the null. We write the residuals as $\hat{\varepsilon}_t = \varepsilon_t + q(\hat{\underline{a}} - \underline{a}, x_t)$, where $q(\hat{\underline{a}} - \underline{a}, x_t) = (\hat{a}_0 - a_0) + (\hat{a}_1 - a_1)x_t$ and $\hat{a}_0, \hat{a}_1$ are the OLS estimators of (a_0, a_1). Therefore $\mathcal{S}_1 = \tilde{\mathcal{S}}_1 + I_1$, where

$$\tilde{\mathcal{S}}_1 = \frac{1}{T}\sum_{t=1}^{T}x_t^2\epsilon_t^2 - \left(\frac{1}{T}\sum_{t=1}^{T}x_t^2\right)\left(\frac{1}{T}\sum_{t=1}^{T}\epsilon_t^2\right)$$

and

$$I_1 = \frac{1}{T}\sum_{t=1}^{T}x_t^2\left(q(\hat{\underline{a}} - \underline{a}, x_t)\varepsilon_t + q(\hat{\underline{a}} - \underline{a}, x_t)^2\right) + \left(\frac{1}{T}\sum_{t=1}^{T}x_t^2\right)\frac{1}{T}\sum_{t=1}^{T}\left(q(\hat{\underline{a}} - \underline{a}, x_t)\varepsilon_t + q(\hat{\underline{a}} - \underline{a}, x_t)^2\right).$$

Under the stated conditions it is straightforward to show that $I_1 = O_p(T^{-1})$, therefore $\mathcal{S}_1 = \tilde{\mathcal{S}}_1 + O_p(T^{-1})$. By using the central limit theorem for independent random variables it can be seen that $\sqrt{T}\Gamma_{1,T}^{-1/2}\tilde{\mathcal{S}}_1 \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$, thus $\sqrt{T}\Gamma_{1,T}^{-1/2}\mathcal{S}_1 \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$ which gives us the result under the null. A similar method can be used to prove the result under the alternative. $\square$

A.2 Proofs for Sections 5.2

PROOF of Lemma 5.1. The proof of (21) is straightforward, hence we omit the details. To prove (22) we first note that $|\text{cov}(X_t, X_s)| \leq (n+1)\sup_{t,j}|x_{t,j}|^2\rho_2(t-s)$ and $|\text{cum}(X_{t_1}, X_{t_2}, X_{t_3}, X_{t_4})| \leq (n+1)\sup_{t,j}|x_{t,j}|^4\rho_4(t_2 - t_1, t_3 - t_1, t_4 - t_1)$. Expanding $\text{var}(V_T(\mathcal{G}))$ gives

$$\text{var}(V_T(\mathcal{G})) = \frac{1}{(2\pi m)^2}\sum_{k_1,k_2=1}^{m}\sum_{s=k_1}^{T-m/2+k_1}\sum_{u=k_2}^{T-m/2+k_2}\sum_{r_1=-k_1}^{m-k_1}\sum_{r_2=-k_2}^{m-k_2}\text{cov}(X_sX_{s+r_1}, X_u, X_{u+r_2})g_s(r_1)g_u(r_2).\tag{49}$$

From the definition of $\rho_2(\cdot)$ and $\rho_4(\cdot)$ it is clear that

$$|\text{cov}(X_sX_{s+r_1}, X_u, X_{u+r_2})|$$
$$\leq \rho_2(s-u)\rho_2(s+r_1-u-r_2) + \rho_2(s+r_1-u)\rho_2(s-u-r_2) + \rho_4(r_1, u-s-r_1, u-s+r_1-r_2).$$

31

Substituting the above into (49) gives us (23). Using the same methods we have (23). □

We use the following lemma to prove consistency of the estimator and obtain the rate of convergence.

Lemma A.1 *Suppose Assumptions 2.1, 5.1(i-v) and 5.2(i) are satisfied. Then for all $\boldsymbol{a} \in \Omega$ and $\boldsymbol{\theta} \in \Theta$ we have*

$$
var\left(\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta})\right) = O(\frac{1}{T_m}), \quad var\left(\nabla\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta})\right) = O(\frac{1}{T_m}), \quad var\left(\nabla^2\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta})\right) = O(\frac{1}{T_m}). \tag{50}
$$

PROOF. To prove (50), we use Lemma 5.1, which means we have to verify the condition $\sum_r |\int \nabla_{\boldsymbol{\theta}}^k \mathcal{F}_{t,m}^{-1}(\boldsymbol{\theta}, \omega) \exp(ir\omega) d\omega| < \infty$, for $k = 0, 1, 2$. This follows from (112) (in Lemma A.8), hence we obtain the result. □

To show convergence in probability of the estimator we need to show equicontinuity in probability of the sequence of random functions $\{\mathcal{L}_T^{(m)}\}_T$. We recall that a sequence of random functions $\{\mathcal{H}_T(\boldsymbol{\theta})\}$ is equicontinuous in probability if for every $\epsilon > 0$ and $\eta > 0$ there exists a $\delta > 0$ such that $\lim_{T\to\infty} \sup P(\sup_{\|\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2\|_1 \leq \delta} |\mathcal{H}_T(\boldsymbol{\theta}_1) - \mathcal{H}_T(\boldsymbol{\theta}_2)| > \eta) < \varepsilon$. In our case $\mathcal{H}_T(\boldsymbol{\theta}_1) := \mathcal{L}_T^{(m)}$ and pointwise it converges to its expectation

$$
\mathbb{E}(\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta})) \tag{51}
$$
$$
= \frac{1}{T_m} \sum_{t=m/2}^{T-m/2} \int \left\{ \frac{\mathcal{F}_{t,m}(\boldsymbol{\theta}_0, \omega)}{\mathcal{F}_{t,m}(\boldsymbol{\theta}, \omega)} + \sum_{j_1,j_2=1}^{n} (a_{j_1,0} - a_{j_1})(a_{j_2,0} - a_{j_2}) \frac{J_{t,m}^{(j_1)}(\omega) J_{t,m}^{(j_2)}(-\omega)}{\mathcal{F}_{t,m}(\boldsymbol{\theta}, \omega)} + \log \mathcal{F}_{t,m}(\boldsymbol{\theta}, \omega) \right\} d\omega,
$$

Lemma A.2 *Suppose Assumptions 2.1, 5.1(i-v) and 5.2(i) are satisfied, then*

(i) $\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta})$ *is equicontinuous is probability and* $|\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta}) - \mathbb{E}(\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta}))| \overset{\mathcal{P}}{\to} 0$

(ii) $\nabla^2\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta})$ *is equicontinuous in probability and* $|\nabla^2\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta}) - \mathbb{E}(\nabla^2\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta}))| \overset{\mathcal{P}}{\to} 0$,

as $T \to \infty$.

PROOF. We first prove (i). By the mean value theorem

$$
|\mathcal{L}_T^{(m)}(\boldsymbol{a}_1, \boldsymbol{\theta}^{(1)}) - \mathcal{L}_T^{(m)}(\boldsymbol{a}_2, \boldsymbol{\theta}^{(2)})| \leq \sup_{\boldsymbol{a} \in \Omega, \boldsymbol{\theta} \in \Theta_1 \otimes \Theta_2} |\nabla\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta})| \cdot |(\boldsymbol{a}_1 - \boldsymbol{a}_2, \boldsymbol{\theta}^{(1)} - \boldsymbol{\theta}^{(2)})|, \tag{52}
$$

where $\nabla\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta}) = (\nabla_{\boldsymbol{a}}\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta}), \nabla_{\boldsymbol{\theta}}\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta}))$. From above it is clear if we can show $\nabla\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta})$ is bounded in probability, then equicontinuity of $\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta})$ immediately follows. Therefore we now

bound $\sup_{\boldsymbol{a}\in\Omega\boldsymbol{\theta}\in\Theta_1\otimes\Theta_2}|\nabla\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta})|$, with a random variable which is bounded in probability. Inspecting (19) and under Assumption 5.1(i) we note that

$$\sup_{\boldsymbol{a},\boldsymbol{\theta}}|\nabla_{a_j}\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta})| \leq \frac{K}{T_m}\Big\{\sup_{t,j}|x_{t,j}|\cdot m\cdot S_T + \sum_{j_2=1}^{n}\sup_{\boldsymbol{\theta}}|H_T^{(j,j_2)}(\mathcal{F}_{\boldsymbol{\theta}}^{-1})|\Big\} \tag{53}$$

$$\sup_{\boldsymbol{a},\boldsymbol{\theta}}|\nabla_{\boldsymbol{\theta}}\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta})| \leq \frac{K}{T_m}\Big\{\sup_{\boldsymbol{\theta}}V_T(\nabla_{\boldsymbol{\theta}}\mathcal{F}_{\boldsymbol{\theta}}^{-1}) + \sup_{t,j}|x_{t,j}|\cdot m\cdot S_T$$

$$+ \sum_{j_1,j_2}\sup_{\boldsymbol{\theta},\boldsymbol{a}}H_T^{(j_1,j_2)}(\nabla_{\boldsymbol{\theta}}\mathcal{F}_{\boldsymbol{\theta}}^{-1}) + \sum_{t=m/2}^{T-m/2}\sup_{\boldsymbol{\theta}}\int\frac{|\nabla_{\boldsymbol{\theta}}\mathcal{F}_{t,m}(\boldsymbol{\theta},\omega)|}{\mathcal{F}_{t,m}(\boldsymbol{\theta},\omega)}d\omega\Big\}, \tag{54}$$

noting that to obtain the above we bound the $D_T^{(j)}(\mathcal{F}_{\boldsymbol{\theta}}^{-1})$ in $\nabla_{a_j}\mathcal{L}^{(m)}$ (see (19)) with $|D_T^{(j)}(\mathcal{F}_{\boldsymbol{\theta}}^{-1})| \leq \sup_{t,\omega}|\mathcal{F}_{t,m}(\boldsymbol{\theta}_0,\omega)^{-1}|\sup_{t,j}|x_{t,j}|\cdot S_T$, where $S_T = \sum_{s=1}^{T}|X_s|$. With the exception of $T_m^{-1}\sup_{\boldsymbol{\theta}}V_T(\nabla_{\boldsymbol{\theta}}\mathcal{F}_{\boldsymbol{\theta}}^{-1})$ and $T_m^{-1}S_T$, all the terms on the right hand side of $\sup_{\beta,\boldsymbol{\theta}}|\nabla_{\boldsymbol{\theta}}\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta})|$ and $\sup_{\boldsymbol{a},\boldsymbol{\theta}}|\nabla_{a_j}\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta})|$, above, are deterministic and, under Assumptions 5.1(i,ii,iii) (with $k = 1$) and 5.2(i), are bounded. It is straightforward to show that $\sup_T\mathbb{E}(T_m^{-1}S_T) < \infty$ and $\text{var}(T_m^{-1}S_T) = O(T^{-1})$, therefore $T_m^{-1}S_T$ is bounded in probability. By using Lemma 5.1 we have that $T_m^{-1}\mathbb{E}(\sup_{\boldsymbol{\theta}}V_T(\nabla_{\boldsymbol{\theta}}\mathcal{F}_{\boldsymbol{\theta}}^{-1})) < \infty$ and $\text{var}\big[\frac{1}{T_m}V_T(\nabla_{\boldsymbol{\theta}}\mathcal{F}_{\boldsymbol{\theta}}^{-1})\big] = O(\frac{1}{T_m})$, hence $\frac{1}{T_m}V_T(\nabla_{\boldsymbol{\theta}}\mathcal{F}_{\boldsymbol{\theta}}^{-1})$ is bounded in probability. Altogether this gives that $\sup_{\boldsymbol{a},\boldsymbol{\theta}}|\nabla_{\boldsymbol{\theta}}\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta})|$ and $\sup_{\boldsymbol{a},\boldsymbol{\theta}}|\nabla_{a_j}\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta})|$ are both bounded in probability. Finally, since $\nabla\mathcal{L}_T^{(m)}$ is bounded in probability, equicontinuity of $\mathcal{L}_T^{(m)}(\cdot)$ immediately follows from (52).

The proof of (ii) is identical to the proof of (i), the main difference is that it is based on the boundedness of the third derivative of f_{j_1,j_2}, rather than the first derivative of f_{j_1,j_2}. We omit the details. □

We use equicontinuity now to prove consistency of $(\hat{\boldsymbol{a}}_T,\hat{\boldsymbol{\theta}}_T)$.

PROOF of Proposition 5.1. The proof is split into two stages, we first show that for a large enough T, $(\boldsymbol{a}_0,\boldsymbol{\theta}_0)$ is the unique minimum of the limiting deterministic function function $\mathbb{E}(\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta}))$ (defined in (51)) and second we show that $\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta})$ converges uniformly in probability, together they prove the result. Let us consider the difference

$$\mathbb{E}(\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta})) - \mathbb{E}(\mathcal{L}_T^{(m)}(\boldsymbol{a}_0,\boldsymbol{\theta}_0)) = I + II$$

where

$$I = \frac{1}{T_m}\sum_{t=m/2}^{T-m/2}\int\Big\{\frac{\mathcal{F}_{t,m}(\boldsymbol{\theta}_0,\omega)}{\mathcal{F}_{t,m}(\boldsymbol{\theta},\omega)} - \log\frac{\mathcal{F}_{t,m}(\boldsymbol{\theta}_0,\omega)}{\mathcal{F}_{t,m}(\boldsymbol{\theta},\omega)}\Big\}d\omega - 1$$

$$II = \frac{1}{T_m}\sum_{t=m/2}^{T-m/2}\sum_{j_1,j_2=1}^{n}(a_{j_1,0}-a_{j_1})(a_{j_2,0}-a_{j_2})\int J_{t,m}^{(j_1)}(\omega)J_{t,m}^{(j_2)}(-\omega)d\omega.$$

We now show that I and II are positive. We first show that I is positive. Let $x_{t,\omega} = \frac{\mathcal{F}_{t,m}(\boldsymbol{\theta}_0,\omega)}{\mathcal{F}_{t,m}(\boldsymbol{\theta},\omega)}$, since $x_{t,\omega}$ is positive, it is obvious that $x_{t,\omega} - \log x_{t,\omega} - 1$ is a positive function. Moreover $x_{t,\omega} - \log x_{t,\omega} - 1$ has a unique minimum when $x_{t,\omega} = 1$. Since $I = \frac{1}{T_m}\sum_{t=m/2}^{T-m/2}\int(x_{t,\omega} - \log x_{t,\omega} - 1)d\omega$, I is positive and has minimum when $\mathcal{F}_{t,m}(\boldsymbol{\theta}_0,\omega) = \mathcal{F}_{t,m}(\boldsymbol{\theta}_0,\omega)$. To show that II is positive, we recall from the definition of $\mathcal{J}_{T,m}(\omega)'$ given in Assumption 5.2(ii), that $II = (\boldsymbol{a} - \boldsymbol{a}_0)'\int\mathcal{J}_{T,m}(\boldsymbol{\theta},\omega)\mathcal{J}_{T,m}(\boldsymbol{\theta},\omega)'d\omega(\boldsymbol{a} - \boldsymbol{a}_0)$, where $(\boldsymbol{a} - \boldsymbol{a}_0) = (a_1 - a_{1,0}, \ldots, a_n - a_{n,0})'$, which is clearly positive. Altogether this gives $\mathbb{E}(\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta})) - \mathbb{E}(\mathcal{L}_T^{(m)}(\boldsymbol{a}_0,\boldsymbol{\theta}_0)) = I + II \geq 0$, and for all values of T, $\mathbb{E}(\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta}))$ has a minimum at $\boldsymbol{a} = \boldsymbol{a}_0$ and $\boldsymbol{\theta} = \boldsymbol{\theta}_0$. To show that $(\boldsymbol{a}_0,\boldsymbol{\theta}_0)$ is the unique minimum of $\mathbb{E}(\mathcal{L}_T^{(m)}(\cdot))$, suppose by contradiction, there exists another $(\boldsymbol{a}^*,\boldsymbol{\theta}^*)$ such that $\mathbb{E}(\mathcal{L}_T^{(m)}(\boldsymbol{a}^*,\boldsymbol{\theta}^*)) = \mathbb{E}(\mathcal{L}_T^{(m)}(\boldsymbol{a}_0,\boldsymbol{\theta}_0))$. Then by using the arguments above, for all values of t and ω we must have $\mathcal{F}_{t,m}(\boldsymbol{\theta}^*,\omega) = \mathcal{F}_{t,m}(\boldsymbol{\theta}_0,\omega)$ and $(\boldsymbol{a}^* - \boldsymbol{a}_0)'\int\mathcal{J}_{T,m}(\boldsymbol{\theta},\omega)\mathcal{J}_{T,m}(\boldsymbol{\theta},\omega)'d\omega(\boldsymbol{a}^* - \boldsymbol{a}_0) = 0$. However this contradicts Assumption 5.2(ii), therefore we can conclude that for a large T, $(\boldsymbol{a}_0,\boldsymbol{\theta}_0)$ is the unique minimum of $\mathbb{E}(\mathcal{L}_T^{(m)}(\cdot))$.

We now show that $\hat{\boldsymbol{\theta}}_T \overset{\mathcal{P}}{\to} \boldsymbol{\theta}_0$ and $\hat{\boldsymbol{a}}_T \overset{\mathcal{P}}{\to} \boldsymbol{a}_0$. To prove this we need to show uniform convergence of $\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta})$. Uniform convergence follows from the probabilistic version of the Arzela-Ascoli lemma which means we have to verify (a) pointwise convergence of $\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta})$ (b) equicontinuity in probability of $\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta})$ and (c) compactness of the parameter space. It is clear from Lemma A.1 that for every $\boldsymbol{\theta} \in \Theta_1 \otimes \Theta_2$ and $\boldsymbol{a} \in \Omega$ we have $|\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta}) - \mathbb{E}(\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta}))| \overset{\mathcal{P}}{\to} 0$ as $T \to \infty$ ((a) holds), by Lemma A.2 we have that equicontinuity of $\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta})$ ((b) holds) and under Assumption 5.1(iii) Θ and Ω are compact sets ((c) holds). Therefore $\sup_{\boldsymbol{a},\boldsymbol{\theta}}|\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta}) - \mathbb{E}(\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta}))| \overset{\mathcal{P}}{\to} 0$. Finally, since $\mathcal{L}_T^{(m)}(\boldsymbol{a}_0,\boldsymbol{\theta}_0) \geq \mathcal{L}_T^{(m)}(\hat{\boldsymbol{a}}_T,\hat{\boldsymbol{\theta}}_T) \overset{\mathcal{P}}{\to} \mathbb{E}(\mathcal{L}_T^{(m)}(\hat{\boldsymbol{a}}_T,\hat{\boldsymbol{\theta}}_T)) \geq \mathbb{E}(\mathcal{L}_T^{(m)}(\boldsymbol{a}_0,\boldsymbol{\theta}_0))$, we have that $|\mathcal{L}_T^{(m)}(\hat{\boldsymbol{a}}_T,\hat{\boldsymbol{\theta}}_T) - \mathbb{E}(\mathcal{L}_T^{(m)}(\boldsymbol{a}_0,\boldsymbol{\theta}_0))| \leq \max\{|\mathcal{L}_T^{(m)}(\hat{\boldsymbol{a}}_T,\hat{\boldsymbol{\theta}}_T) - \mathbb{E}(\mathcal{L}_T^{(m)}(\hat{\boldsymbol{a}}_T,\hat{\boldsymbol{\theta}}_T))|, |\mathcal{L}_T^{(m)}(\boldsymbol{a}_0,\boldsymbol{\theta}_0) - \mathbb{E}(\mathcal{L}_T^{(m)}(\boldsymbol{a}_0,\boldsymbol{\theta}_0))|\} \leq \sup_{\boldsymbol{a},\boldsymbol{\theta}}|\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta}) - \mathbb{E}(\mathcal{L}_T^{(m)}(\boldsymbol{a},\boldsymbol{\theta}))| \overset{\mathcal{P}}{\to} 0$. Because $(\boldsymbol{a}_0,\boldsymbol{\theta}_0)$ is the unique minimum of $\mathbb{E}(\mathcal{L}_T^{(m)}(\cdot))$, it follows that $\hat{\boldsymbol{a}}_T \overset{\mathcal{P}}{\to} \boldsymbol{a}_0$ and $\hat{\boldsymbol{\theta}}_T \overset{\mathcal{P}}{\to} \boldsymbol{a}_0$. $\qquad\square$

By approximating $\nabla\mathcal{L}_T^{(m)}(\boldsymbol{a}_0,\boldsymbol{\theta}_0)$ with the sum of martingale differences, we will use the martingale central limit theorem to prove Theorem 5.1. Using (19) we have

$$\nabla_{\boldsymbol{\theta}}\mathcal{L}_T^{(m)}(\boldsymbol{a}_0,\boldsymbol{\theta}_0) = -\frac{1}{T_m}\left\{V_T(\nabla\mathcal{F}_{\boldsymbol{\theta}_0}^{-1}) - \mathbb{E}[V_T(\nabla\mathcal{F}_{\boldsymbol{\theta}_0}^{-1})]\right\} = -\frac{1}{T_m}\left[\mathcal{Z}_T^{(m)} + U_T^{(m)}\right],$$

$$\text{and } \nabla_{\boldsymbol{a}}\mathcal{L}_T^{(m)}(\boldsymbol{a}_0,\boldsymbol{\theta}_0)]_j = \frac{1}{T_m}D_T^{(j)}(\mathcal{F}_{\boldsymbol{\theta}_0}^{-1}) = \frac{1}{T_m}\{\mathcal{B}_T^{(m,j)} + C_T^{(m,j)}\}$$

where

$$\mathcal{Z}_T^{(m)} = \frac{1}{2\pi m} \sum_{k=1}^{m} \sum_{j_1,j_2=1}^{n+1} \sum_{s=k}^{T-m+k} \sum_{r=-k}^{m-k} x_{s,j_1} x_{s+r,j_2} \underline{b}_{s+m/2-k,m}(r) \tag{55}$$

$$\times \sum_{i_1,i_2=0}^{s-1,s+r-1} \psi_{i_1,j_1} \psi_{i_2,j_2} \big(\eta_{s-i_1,j_1} \eta_{s+r-i_2,j_2} - \mathbb{E}(\eta_{s-i_1,j_1} \eta_{s+r-i_2,j_2}) \big)$$

$$U_T^{(m)} = \frac{1}{2\pi m} \sum_{k=1}^{m} \sum_{j_1,j_2=1}^{n+1} \sum_{s=k}^{T-m+k} \sum_{r=-k}^{m-k} x_{s,j_1} x_{s+r,j_2} \underline{b}_{s+m/2-k,m}(r) \tag{56}$$

$$\times \sum_{i_1 \geq s \text{ or } i_2 \geq s+r} \psi_{i_1,j_1} \psi_{i_2,j_2} \big(\eta_{s-i_1,j_1} \eta_{s+r-i_2,j_2} - \mathbb{E}(\eta_{s-i_1,j_1} \eta_{s+r-i_2,j_2}) \big)$$

$$\tag{57}$$

$$\mathcal{B}_T^{(m,j)} = \frac{1}{2\pi m} \sum_{j_2=1}^{n} \sum_{k=1}^{m} \sum_{s=k}^{T-m+k} \sum_{r=-k}^{m-k} a_{s+m/2-k,m}(r) x_{s,j_2} x_{s+r,j} \sum_{i=1}^{s_1} \psi_{i,j_2} \eta_{s_1-i,j_2}$$

$$C_T^{(m,j)} = \frac{1}{2\pi m} \sum_{j_2=1}^{n} \sum_{k=1}^{m} \sum_{s=k}^{T-m+k} \sum_{r=-k}^{m-k} a_{s+m/2-k,m}(r) x_{s_1,j_2} x_{s_2,j} \sum_{i=s+1}^{\infty} \psi_{i,j_2} \eta_{s-i,j_2}, \tag{58}$$

with $\{\underline{b}_s(k)\}$, $\{a_s(k)\}$ defined in (92) and setting $x_{t,n+1} = 1$.

In the following results we show that $\sqrt{T} \nabla_{\boldsymbol{\theta}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0) = \frac{-1}{\sqrt{T}} Z_T^{(m)} + o_p(1)$ and $\sqrt{T} \nabla_{\boldsymbol{a}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0)\rfloor_j = \frac{1}{\sqrt{T}} B_T^{(m,j)} + o_p(1)$.

Proposition A.1 *Suppose Assumptions 2.1, 5.1 and 5.2(i) are satisfied and $U_T^{(m)}$ and $W_T^{(m)}$ satisfy (55). Then we have*

$$[\mathbb{E}(|U_T^{(m)}|^2)]^{1/2} \leq K \tag{59}$$

where K is independent of T and $\mathcal{Z}_T^{(m)} = \sum_{d=1}^{T} Z_d^{(m)}$, $\{Z_d^{(m)}\}_d$ are martingale difference vectors with

$$Z_d^{(m)} = \sum_{j_1,j_2=1}^{n+1} \Big(\{\eta_{d,j_1} \eta_{d,j_2} - \mathbb{E}[\eta_{d,j_1} \eta_{d,j_2}]\} A_{d,d}^{(j_1,j_2)} + 2\eta_{d,j_1} \sum_{d_1<d} \eta_{d_1,j_2} A_{d,d_1}^{(j_1,j_2)} \Big) \tag{60}$$

$\psi_{-i,j_i} = 0$ for $i \geq 1$, and

$$A_{d,d_1}^{(j_1,j_2)} = \frac{1}{2\pi m} \sum_{k=1}^{m} \sum_{s=k}^{T-m+k} \sum_{r=-k}^{m-k} x_{s,j_1} x_{s+r,j_2} \psi_{s+d,j_1} \psi_{s+r+d_1,j_2} \underline{b}_{s+m/2-k,m}(r). \tag{61}$$

For all d, d_2, j_1 and j_2 we have

$$\sup_d \sum_{d_2=1}^{d} |A_{d,d_2}^{(j_1,j_2)}|^2 < \infty \quad and \quad \sup_d |A_{d,d}^{(j_1,j_2)}| < \infty. \tag{62}$$

35

PROOF. We first prove (59). To make the notation in the proof less cumbersome, we prove the result for $X_t = x_{t,1}\alpha_{t,1}$, where $\alpha_{t,1}$ satisfies Assumption 2.1. The proof for the general case is the same. We observe $U_T^{(m)}$ contains all the terms in $\nabla_{\boldsymbol{\theta}}\mathcal{L}^{(m)}$ which have $\{\eta_t\}_{t\leq 0}$. We partition $U_T^{(m)} = R_{T,1} + R_{T,2}$, where

$$R_{T,1} = \frac{2}{2\pi m}\sum_{k=1}^{m}\sum_{s=k}^{T-m+k}\sum_{r=-k}^{m-k}x_{s,1}x_{s+r,1}\underline{b}_{s+m/2-k,m}(r)\sum_{i_1\geq s}^{s+r-1}\sum_{i_2=0}\psi_{i_1,1}\psi_{i_2,1}\big(\eta_{s-i_1,1}\eta_{s+r-i_2,1} - \mathbb{E}(\eta_{s-i_1,1}\eta_{s+r-i_2,1})\big),$$

$$R_{T,2} = \frac{1}{2\pi m}\sum_{k=1}^{m}\sum_{s=k}^{T-m+k}\sum_{r=-k}^{m-k}x_{s,1}x_{s+r,1}\underline{b}_{s+m/2-k,m}(r)\sum_{i_1\geq s}\sum_{i_2\geq s+r}\psi_{i_1,1}\psi_{i_2,1}\big(\eta_{s-i_1,1}\eta_{s+r-i_2,1} - \mathbb{E}(\eta_{s-i_1,1}\eta_{s+r-i_2,1})\big).$$

We bound each term. Studying $R_{T,1}$ we see that due to the ranges in the summands, $\eta_{s-i_1,1}$ and $\eta_{s+r-i_2,1}$ are independent, therefore the second moment of $(\mathbb{E}|R_{T,1}|^2)^{1/2}$ is bounded by

$$(\mathbb{E}|R_{T,1}|^2)^{1/2}$$

$$\leq \frac{2}{m}\mathbb{E}(\eta_{0,1}^2)^{1/2}\sup_t|x_{t,1}|^2\sum_{k=1}^{m}\sup_s\sum_{r=-k}^{m-k}|\underline{b}_{s,m}(r)|\sum_{s=k}^{T-m+k}\sum_{i_1\geq s}|\psi_{i_1,1}|\left\{\mathbb{E}\left(\sum_{i_2=1}^{s+r}\psi_{i_2,1}\eta_{s-i_2,1}\right)^2\right\}^{1/2}$$

$$\leq 2\mathbb{E}(\eta_{0,1}^2)\sup_t|x_{t,1}|^2\sup_s\sum_r|\underline{b}_{s,m}(r)|\{\sum_{s=1}^{\infty}|s\psi_{s,1}|\}\{\sum_{i=0}^{\infty}|\psi_{i,1}|^2\}^{1/2} < \infty,$$

where to obtain the last inequality in the above we use that $\sum_{s=k}^{T-m+k}\sum_{i_1\geq s}|\psi_{i_1,1}| \leq \sum_{i=1}^{\infty}|i\cdot\psi_{i,1}|$. To bound $(\mathbb{E}|R_{T,2}|^2)^{1/2}$, it is straightforward to show

$$(\mathbb{E}|R_{T,2}|^2)^{1/2} \leq \sup_t|x_{t,1}|^2\max(\mathbb{E}|\eta_0^4|^{1/2}, \mathbb{E}|\eta_0^2|)\{\sup_s\sum_r|\underline{b}_{s,m}(r)|\sum_{s=0}^{\infty}|s\psi_{s,1}|\}^2 < \infty$$

Using the above bounds we have $(\mathbb{E}|U_T^{(m)}|^2)^{1/2} < \infty$ thus giving (59). To prove (60), we change indices and let $d_1 = s - i_1$ and $d_2 = s + r - i_2$, substituting this into $\mathcal{Z}_T^{(m)}$ gives

$$\mathcal{Z}_T^{(m)} = \frac{1}{2\pi m}\sum_{k=1}^{m}\sum_{j_1,j_2=1}^{n}\sum_{s=k}^{T-m+k}\sum_{r=-k}^{m-k}\underline{b}_{s+m/2-k,m}(r)\sum_{d=0}^{s}\sum_{d_1=0}^{s+r}\psi_{s+d,j_1}\psi_{s+r+d_1,j_2}\big(\eta_{d,j_1}\eta_{d_1,j_2} - \mathbb{E}(\eta_{d,j_1}\eta_{d_1,j_2})\big).$$

Using the above and partitioning into the cases $d_1 < d$ and $d_1 = d$ we obtain the martingale difference decomposition in (60).

Finally from the definition of $A_{d,d_2}^{(j_1,j_2)}$ we have

$$\sum_{d_2=1}^{d}|A_{d,d_2}^{(j_1,j_2)}|^2 \leq \frac{1}{m^2}\sum_{k_1,k_2=1}^{m}\sum_{d_2=1}^{d}\sum_{s=k_1}^{T-m+k_1}\sum_{u=k_2}^{T-m+k_2}\sum_{r_1=-k_1}^{m-k_1}\sum_{r_2=-k_2}^{m-k_2}|x_{s,j_1}x_{s+r_1,j_2}x_{u,j_1}x_{u+r_2,j_2}|$$

$$|\underline{b}_{s+m/2-k_1,m}(r_1)|\cdot|\underline{b}_{s+m/2-k_2,m}(r_2)|\psi_{s+d,j_1}\psi_{s+r+d,j_2}\psi_{u+d,j_1}\psi_{u+r_2+d,j_2}$$

$$\leq \sup_{t,j}|x_{t,j}|^4(\sup_s\sum_r|\underline{b}_{s,m}(r)|)^2\sup_j(\sum_i|\psi_{i,j}|^2)\sup_j(\sum_i|\psi_{i,j}|)^2,$$

thus giving $\sup_d \sum_{d_2=1}^d |A_{d,d_2}^{(j_1,j_2)}|^2 < \infty$ (the first part of (62)) we use a similar method to show $\sup_d |A_{d,d}^{(j_1,j_2)}| < \infty$. $\qquad\square$

We state an analogous result for $\nabla_{\boldsymbol{a}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0)\rfloor_j = \frac{1}{T_m}\{\mathcal{B}_T^{(m,j)} + C_T^{(m,j)}\}$.

Proposition A.2 *Suppose Assumptions 2.1, 5.1 and 5.2 are satisfied and $\mathcal{B}_T^{(m,j)}$ and $C_T^{(m,j)}$ satisfy (58). Then we have*

$$(\mathbb{E}|C_T^{(m,j)}|^2)^{1/2} \leq K \tag{63}$$

and $\mathcal{B}_T^{(m,j)} = \sum_{d=1}^T B_d^{(m,j)}$, where $\{B_d^{(m,j)}\}$ are martingale difference with $B_d^{(m,j)} = \sum_{j_1=1}^{n+1} \eta_{d,j_1} D_d^{(j,j_1)}$ and

$$D_d^{(j,j_1)} = \frac{1}{2\pi m} \sum_{k=1}^m \sum_{s=k}^{T-m+k} \sum_{r=-k}^{m-k} x_{s,j} x_{s+r,j_1} a_{s+m/2-k,m}(r) \psi_{s+d,j_1}. \tag{64}$$

Furthermore $D_d^{(j,j_1)}$ is bounded by

$$\sup_{j,j_2} |D_d^{(j,j_1)}| \leq \frac{1}{2\pi} \sup_{t,j} |x_{t,j}|^2 \sup_s \sum_r |a_{s,m}(r)| \sup_j \sum_{s=1}^\infty |\psi_{s,j}|. \tag{65}$$

PROOF. It is straightforward, hence we omit the details. $\qquad\square$

In order to use the martingale central limit theorem we show that the martingales differences $\{W_d^{(m)}\}$ and $\{B_d^{(m,j)}\}$ have a finite fourth moment.

Lemma A.3 *Suppose Assumptions 2.1, 5.1 and 5.2(i) are satisfied. Let $Z_d^{(m)}$ be defined as in (60) and $B_d^{(m,j)}$ be defined as in (64). Then, for some finite K, we have $\sup_d \mathbb{E}(|Z_d^{(m)}|^4) \leq K$ and $\sup_d \mathbb{E}(|B_d^{(m,j)}|^4) \leq K$.*

PROOF. In order to bound $\mathbb{E}(|Z_d^{(m)}|^4)$ we note that $Z_d^{(m)} = X_d^{(m)} + Y_d^{(m)}$, where $X_d^{(m)} = 2\sum_{j_1,j_2=1}^{n+1} \eta_{d,j_1} \sum_{d_1<d} \eta_{d_1,j_2} A_{d,d_1}^{(j_1,j_2)}$ and $Y_d^{(m)} = \sum_{j_1,j_2=1}^{n+1} \{\eta_{d,j_1}\eta_{d,j_2} - \mathbb{E}[\eta_{d,j_1}\eta_{d,j_2}]\} A_{d,d}^{(j_1,j_2)}$. Now the fourth moment of the inner summand in $X_d^{(m)}$ is

$$\mathbb{E}\{\eta_{d,j_1}\sum_{d_2<d}\eta_{d_2,j_2}A_{d,d_2}^{(j_1,j_2)}\}^4 \leq \mathbb{E}(\eta_{d,j_1}^4)\sum_{d_2<d,d_3<d}\mathbb{E}[\eta_{d_2,j_2}^2\eta_{d_3,j_2}^2](A_{d,d_2}^{(j_1,j_2)})^2(A_{d,d_3}^{(j_1,j_2)})^2$$

$$\leq \max_j(\mathbb{E}[\eta_{d,j}^4]^2, \mathbb{E}[\eta_{d,j}^2]^4)\big(\sum_{d_1<d}(A_{d,d_1}^{(j_1,j_2)})^2\big)^2.$$

Hence $\mathbb{E}(|X_d^{(m)}|^4) \leq 2(n+1)\max_j[(\mathbb{E}(\eta_{d,j}^4))^2, (\mathbb{E}(\eta_{d,j}^2))^4]\big(\sum_{d_1<d}(A_{d,d_1}^{(j_1,j_2)})^2\big)^2$. Since $n < \infty$ by using (62), we have $\mathbb{E}(|X_d^{(m)}|^4) < \infty$. Using the same arguments we can show that $\mathbb{E}(|Y_d^{(m)}|^4) < \infty$, this gives $\mathbb{E}(|Z_d^{(m)}|^4) \leq K$. Finally we note that since $B_d^{(m,j)} = \sum_{j_2=0}^n \eta_{d,j_2} D_d^{(j,j_2)}$, by using (65) we have $\mathbb{E}(|B_d^{(m,j)}|^4) < \infty$, as desired. $\qquad\square$

37

We use the following lemma to prove asymptotic normality of the estimator. In the lemma below we show that the likelihood of the derivative can be written in terms of a martingale term and a term which is of lower order.

From the above we have $\mathcal{Z}_T^{(m)} = O_p(T^{-1/2})$, $\mathcal{B}_T^{(m,j)} = O_p(T^{-1/2})$, and by using (55),

$$
\begin{aligned}
\sqrt{T}\nabla_{\boldsymbol{\theta}}\mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0) &= T^{-1/2}\mathcal{Z}_T^{(m)} + O_p(T^{-1/2}) \\
\sqrt{T}\nabla_{a_j}\mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0) &= T^{-1/2}\mathcal{B}_T^{(m,j)} + O_p(T^{-1/2}),
\end{aligned}
\tag{66}
$$

as $T \to \infty$. We use this to prove Theorem 5.1.

Proposition A.3 *Suppose Assumptions 2.1, 5.1 and 5.2 hold, then we have*

$$
\sqrt{T_m}\nabla\mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0) \xrightarrow{\mathcal{D}} \mathcal{N}(0, W_T^{(m)}), \ \ as \ T \to \infty,
\tag{67}
$$

where $W_T^{(m)}$ is defined in (27).

PROOF. Using (66), Propositions A.1 and A.2 we have

$$
\sqrt{T}\nabla\mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0) = T^{-1/2}\sum_{d=1}^{T}\mathcal{S}_d + O_p(T^{-1/2}),
$$

where $\mathcal{S}_d' = (B_d^{(m,1)}, \ldots, B_d^{(m,n)}, Z_d^{(m)})$ are martingale differences defined in Propositions A.1 and A.2. Clearly $T^{-1/2}\sum_{d=1}^{T}\mathcal{S}_d$ is the dominating term in $\sqrt{T}\nabla\mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0)$, hence we prove the result by showing asymptotic normality of $T^{-1/2}\sum_{d=1}^{T}\mathcal{S}_d$ using the martingale central limit theorem and the Cramer-Wold device. Let $\boldsymbol{c} = (c_1, \ldots, c_{2n+q})$ be any non-zero vector and define the random variable

$$
Q_d = \sum_{j=1}^{n}c_j B_d^{(m,j)} + \sum_{j=n+1}^{2n+q+1}c_j(Z_d^{(m)})_{j-n}.
$$

To apply the martingale central limit theorem we need to verify that the variance of $T^{-1/2}\sum_{d=1}^{T}Q_d$ is finite and that Lindeberg's condition is satisfied (see (Hall & Heyde, 1980), Theorem 3.2). Using Lemma A.3 we have that $\frac{1}{T}\text{var}(\sum_{d=1}^{T}Q_d) = O(1)$. To verify Lindeberg's condition, we require that for all $\delta > 0$, $L_T = \frac{1}{T}\sum_{d=1}^{T}\mathbb{E}(Q_d^2 I(T^{-1/2}|Q_d| > \delta)|\mathcal{F}_{d-1}) \xrightarrow{\mathcal{P}} 0$ as $T \to \infty$, where $I(\cdot)$ is the identity function and $\mathcal{F}_{d-1} = \sigma(Q_{d-1}, Q_{d-2}, \ldots, Q_1)$. By using Hölder's and Markov's inequality and Lemma A.3 we obtain the following bound $L_T \leq \frac{1}{T^2\delta}\sum_{d=1}^{T}\mathbb{E}(Q_d^4|\mathcal{F}_{d-1}) = O_p(\frac{1}{T}) \to 0$, as $T \to \infty$. Finally we need to show that

$$
\frac{1}{T}\sum_{d=1}^{T}\mathbb{E}(Q_d^2|\mathcal{F}_{d-1}) = \frac{1}{T}\sum_{d=1}^{T}\left[\mathbb{E}(Q_d^2|\mathcal{F}_{d-1}) - \mathbb{E}(Q_d^2)\right] + \frac{1}{T}\sum_{d=1}^{T}\mathbb{E}(Q_d^2) \xrightarrow{\mathcal{P}} \underline{c}'W_T^{(m)}\underline{c}
\tag{68}
$$

where $\underline{c}' = (c_1, \ldots, c_{n+q})$. From the definition of Q_d^2 in terms of $Z_d^{(m)}$ and $\{B_d^{(m,j)}\}$ which are truncated versions of $\nabla \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0)$, it is straightforward to show that $|\frac{1}{Tm} \sum_{d=1}^{T} \mathbb{E}(Q_d^2) - \underline{c}' W_T^{(m)} \underline{c}| \to 0$ as $T \to \infty$. Therefore it remains to show

$$P_T = \frac{1}{Tm} \sum_{d=1}^{T} \left(\mathbb{E}(Q_d^2 | \mathcal{F}_{d-1}) - \mathbb{E}(Q_d^2) \right) \xrightarrow{\mathcal{P}} 0,$$

which will give us (68). To do this we note that $\mathbb{E}(P_T) = 0$ and

$$\text{var}(P_T) = \frac{1}{T_m^2} \sum_{d=1}^{T} \text{var}(\mathbb{E}(Q_d^2 | \mathcal{F}_{d-1})) + \frac{2}{T_m^2} \sum_{d_1 > d_2}^{T} \text{cov}(\mathbb{E}(Q_{d_1}^2 | \mathcal{F}_{d_1-1}), \mathbb{E}(Q_{d_2}^2 | \mathcal{F}_{d_2-1})). \qquad (69)$$

Now by using the Cauchy Schwartz inequality and conditional expectation arguments for $\mathcal{F}_{d_2} \subset \mathcal{F}_{d_1}$ we have

$$\text{cov}(\mathbb{E}(Q_{d_1}^2 | \mathcal{F}_{d_1-1}), \mathbb{E}(Q_{d_2}^2 | \mathcal{F}_{d_2-1}))$$
$$\leq \left[\mathbb{E}\left(\mathbb{E}(Q_{d_2}^2 | \mathcal{F}_{d_2-1}) - \mathbb{E}(Q_{d_2}^2) \right)^2 \right]^{1/2} \left[\mathbb{E}\left(\mathbb{E}(Q_{d_1}^2 | \mathcal{F}_{d_2-1}) - \mathbb{E}(Q_{d_1}^2) \right)^2 \right]^{1/2}. \qquad (70)$$

We now show that $\left[\mathbb{E}\left(\mathbb{E}(Q_{d_1}^2 | \mathcal{F}_{d_2-1}) - \mathbb{E}(Q_{d_1}^2) \right)^2 \right] \to 0$ as $d_1 - d_2 \to \infty$. Define the sigma-algebra $\mathcal{G}_d = \sigma(\eta_{d,1}, \eta_{d,n+1}, \eta_{d-1,1}, \ldots)$. it is straightaforward to show that $\mathcal{F}_d \subset \mathcal{G}_d$. Therefore $\mathbb{E}\left[\mathbb{E}(Q_{d_1}^2 | \mathcal{F}_{d_2-1})^2 \right] \leq \mathbb{E}\left[\mathbb{E}(Q_{d_1}^2 | \mathcal{G}_{d_2-1})^2 \right]$ which gives

$$\mathbb{E}\left[\mathbb{E}(Q_{d_1}^2 | \mathcal{F}_{d_2-1}) - \mathbb{E}(Q_{d_1}^2) \right]^2 \leq \mathbb{E}\left[\mathbb{E}(Q_{d_1}^2 | \mathcal{G}_{d_2-1})^2 \right] - \left[\mathbb{E}(Q_{d_1}^2) \right]^2.$$

From the definition of Q_{d_1} (in terms of $\{\eta_t\}_{t \geq 1}$) and using Propositions A.1 and A.2, we have that $\mathbb{E}\left[\mathbb{E}(Q_{d_1}^2 | \mathcal{G}_{d_2-1})^2 \right] - \left[\mathbb{E}(Q_{d_1}^2) \right]^2 \to 0$ as $d_2 \to \infty$, and

$$\sup_{d_1} \sum_{d_2=1}^{d_1} \left(\mathbb{E}\left[\mathbb{E}(Q_{d_1}^2 | \mathcal{G}_{d_2-1})^2 \right] - \left[\mathbb{E}(Q_{d_1}^2) \right]^2 \right)^{1/2} < \infty.$$

Substituting the above into (70) and (69) we have $\text{var}(P_T) = O(T^{-1})$, hence we have shown (68), and the conditions of the martingale central limit theorem are satisfied. Thus we obtain (67). $\square$

PROOF of Theorem 5.1. To prove (29) we note that by Lemma A.2(ii), $\nabla^2 \mathcal{L}_T^{(m)}$ is pointwise convergent and equicontinuous on probability. Together with the compactness of the parameter spaces Ω and $\Theta_1 \otimes \Theta_2$ this implies that $\sup_{\boldsymbol{a}, \boldsymbol{\theta}} |\nabla^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta}) - \mathbb{E}(\nabla^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\theta}))| \xrightarrow{\mathcal{P}} 0$, therefore $|\nabla^2 \mathcal{L}_T^{(m)}(\bar{\boldsymbol{a}}_T, \bar{\boldsymbol{\theta}}_T) - \mathbb{E}(\nabla^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0))| \xrightarrow{\mathcal{P}} 0$. Given that $\mathbb{E}(\nabla_{\boldsymbol{a}} \nabla_{\boldsymbol{\theta}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0)) = 0$, and $\mathbb{E}(\nabla_{\boldsymbol{\theta}}^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0))$ and $\mathbb{E}(\nabla_{\boldsymbol{a}}^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0))$ are non-singular, then $|\nabla^2 \mathcal{L}_T^{(m)}(\bar{\boldsymbol{a}}_T, \bar{\boldsymbol{\theta}}_T)^{-1} - \mathbb{E}(\nabla^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0))^{-1}| \xrightarrow{\mathcal{P}} 0$. Using this and the mean value theorem element wise to $\nabla \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0)$ gives

$$\begin{pmatrix} \hat{\boldsymbol{a}}_T - \boldsymbol{a}_0 \\ \hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0 \end{pmatrix}_k = \nabla^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0)^{-1} \begin{pmatrix} \nabla_{\boldsymbol{a}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0) \\ \nabla_{\boldsymbol{\theta}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0) \end{pmatrix} + o_p(\frac{1}{\sqrt{T}}). \qquad (71)$$

The above together with (67) proves (29). $\square$

A.3 Proofs in Section 5.3

We now consider the sampling properties of the two-stage scheme. We observe that the derivatives of $\mathcal{L}_T^{(1)}(\boldsymbol{a}, \boldsymbol{\Sigma})$ evaluated at the true parameter are

$$\nabla_{\boldsymbol{\Sigma}} \mathcal{L}_T(\boldsymbol{a}_0, \boldsymbol{\Sigma}_0) = \frac{1}{T} \sum_{t=1}^{T} \frac{(\sigma_t(\boldsymbol{\Sigma}_0) - X_t^2)\mathbf{X}_t}{\sigma_t(\boldsymbol{\Sigma}_0)^2} \quad \text{and} \quad \mathbb{E}(\nabla_{\boldsymbol{\Sigma}}^2 \mathcal{L}_T(\boldsymbol{a}_0, \boldsymbol{\Sigma}_0)) = \frac{1}{T} \sum_{t=1}^{T} \frac{\mathbf{X}_t \mathbf{X}_t'}{\sigma_t(\boldsymbol{\Sigma}_0)^3}$$

where $\mathbf{X}_t = (x_{t,1}^2, \ldots, x_{t,n}^2)'$.

Lemma A.4 *Suppose Assumptions 2.1, 5.1 and 5.2 are satisfied then Let $\mathcal{L}_T^{(1)}(\boldsymbol{a}, \boldsymbol{\Sigma})$ and $\tilde{\boldsymbol{\Sigma}}_T$ be defined as in (12). Then we have*

$$\big((\tilde{\boldsymbol{a}}_T - \boldsymbol{a}_0), (\tilde{\boldsymbol{\Sigma}}_T - \boldsymbol{\Sigma}_0)\big) = O_p(T^{-1/2}), \tag{72}$$

$$\sqrt{T}\nabla \mathcal{L}_T^{(1)}(\boldsymbol{a}_0, \boldsymbol{\Sigma}_0) = \frac{1}{\sqrt{T}} Z_T^{(1)} + \frac{1}{\sqrt{T}} U_T^{(1)}, \tag{73}$$

where $\sup_T \mathbb{E}|U_T^{(1)}| < \infty$, $\mathcal{Z}_T^{(1)} = \sum_{d=1}^{T} Z_d^{(1)}$ and $\{Z_d^{(1)}\}$ are martingale differences with

$$Z_d^{(1)} = \sum_{j_1, j_2 = 0}^{n} \Big([\eta_{d,j_1}\eta_{d,j_2} - \mathbb{E}(\eta_{d,j_1}\eta_{d,j_2})]B_{d,d}^{(j_1, j_2)} + 2\eta_{d,j_1} \sum_{d_2 < d} \eta_{d_2, j_2} B_{d,d_2}^{(j_1, j_2)}\Big) \tag{74}$$

$$B_{d,d_2}^{(j_1, j_2)} = \sum_{s=1}^{T} \psi_{s-d,j_1}\psi_{s-d_2,j_2} \frac{x_{s,j_1} x_{s,j_2}}{\sigma_t(\boldsymbol{\Sigma}_0)} \boldsymbol{X}_s. \tag{75}$$

PROOF. By using analogous results to Lemmas A.1 and A.2 and the same technique of as the proof of Proposition 5.1 it can be shown that $\big((\tilde{\boldsymbol{a}}_T - \boldsymbol{a}_0), (\tilde{\boldsymbol{\Sigma}}_T - \boldsymbol{\Sigma}_0)\big) \xrightarrow{\mathcal{P}} 0$. Further this gives (72). Using Propositions A.1 and A.2 we can show that $\sqrt{T}\nabla \mathcal{L}_T(\boldsymbol{a}_0, \boldsymbol{\Sigma}_0)$ and (73) is dominated by a term which is the sum of martingale differences, which is given in (73). $\qquad \square$

PROOF of Proposition 5.2 By using the same arguments in Propositions A.3 and Theorem 5.1 we have (30).

We now consider the properties of $\mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \tilde{\boldsymbol{\Sigma}}_T)$. To do this we compare $\mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \tilde{\boldsymbol{\Sigma}}_T)$ with $\mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \boldsymbol{\Sigma}_0)$

Lemma A.5 *Suppose Assumptions 2.1, 5.1 and 5.2 are satisfied, $\mathbb{E}(Z_t(\omega)) = 0$ and for any deterministic sequence $\{a_k\}$, with $\sup_k |a_k| < \infty$ we have $\sup_\omega var(\frac{1}{n}\sum_{t=1}^{n} a_t Z_t(\omega)) = O(T^{-1})$, then for any $k \geq 1$ we have*

$$\frac{1}{\sqrt{T}} \int \sum_{t=1}^{T} a_t Z_t(\omega) \big\{ \nabla_{\boldsymbol{\vartheta}} \mathcal{F}_{t,m}(\boldsymbol{\vartheta}_0, \tilde{\boldsymbol{\Sigma}}_T, \omega) - \nabla_{\boldsymbol{\vartheta}} \mathcal{F}_{t,m}(\boldsymbol{\vartheta}_0, \boldsymbol{\Sigma}_0, \omega) \big\} d\omega \xrightarrow{\mathcal{P}} 0 \tag{76}$$

40

$$\frac{1}{\sqrt{n}} \int \sum_{t=1}^{n} a_t Z_t(\omega) \left(\frac{1}{\mathcal{F}_{t,m}(\boldsymbol{\vartheta}_0, \tilde{\boldsymbol{\Sigma}}_T, \omega)^k} - \frac{1}{\mathcal{F}_{t,m}(\boldsymbol{\vartheta}_0, \boldsymbol{\Sigma}_0, \omega)^k} \right) d\omega \xrightarrow{\mathcal{P}} 0 \quad k > 0. \tag{77}$$

PROOF. To prove the result we first recall that from the definition of $\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}, \omega)$, that

$$\sum_{t=1}^{T} a_t Z_t(\omega) \mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}, \omega) = \sum_{j=1}^{n+1} \sigma_j \sum_{t=1}^{T} a_t Z_t(\omega) \int_{-\pi}^{\pi} I_{t,m}^{(j)}(\lambda) f_j(\boldsymbol{\vartheta}, \omega - \lambda) d\lambda. \tag{78}$$

Now by using the above, (72) and differentiating $\mathcal{F}_{t,m}$ with respect to $\boldsymbol{\vartheta}$ we immediately obtain (76). We now prove (77). We observe that

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} a_t Z_t(\omega) \left(\frac{1}{\mathcal{F}_{t,m}(\boldsymbol{\vartheta}_0, \tilde{\boldsymbol{\Sigma}}_T, \omega)^k} - \frac{1}{\mathcal{F}_{t,m}(\boldsymbol{\vartheta}_0, \boldsymbol{\Sigma}_0, \omega)^k} \right) = A + B + C,$$

where

$$
\begin{aligned}
A &= \frac{1}{\sqrt{T}} \sum_{t=1}^{T} a_t Z_t(\omega) \frac{\left[\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \tilde{\boldsymbol{\Sigma}}_T, \omega)^k - \mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0, \omega)^k \right]}{\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0, \omega)^{2k}} \\
B &= \frac{1}{\sqrt{T}} \sum_{t=1}^{T} a_t Z_t(\omega) \frac{\left[\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \tilde{\boldsymbol{\Sigma}}_T, \omega)^k - \mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0, \omega)^k \right]^2}{\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0, \omega)^{3k}} \\
C &= \frac{1}{\sqrt{T}} \sum_{t=1}^{T} a_t Z_t(\omega) \frac{\left[\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \tilde{\boldsymbol{\Sigma}}_T, \omega)^k - \mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0, \omega)^k \right]^3}{\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0, \omega)^{3k} \mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \tilde{\boldsymbol{\Sigma}}_T, \omega)^{3k}}.
\end{aligned}
$$

Now by using (78) and (72), and the stated assumptions $\mathbb{E}(\frac{1}{T} \sum_{t=1}^{T} a_t Z_t(\omega)) = 0$ and $\sup_\omega \operatorname{var}(\frac{1}{T} \sum_{t=1}^{T} a_t Z_t(\omega)) = O(T^{-1})$ and that $\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0, \omega) \geq \delta$ (due to Assumption 5.1(i)), we have $A = O_p(T^{-1/2})$ and $B = O_p(T^{-1})$. Finally, to obtain a bound for C we note that

$$|C| \leq \frac{1}{\delta^{6k} \sqrt{n}} \sum_{t=1}^{n} |a_t| \cdot |Z_t(\omega)| \cdot \left| \left[\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \tilde{\boldsymbol{\Sigma}}_T, \omega)^k - \mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0, \omega)^k \right]^3 \right|,$$

now by using the same methods as above it can be shown that $C = O_p(T^{-1})$. Thus we obtain (77). $\square$

We use the result above to prove the results below.

Lemma A.6 *Suppose the assumptions in Theorem 5.2 are satisfied. Then we have*

$$\sup_{\tilde{\boldsymbol{\theta}}} \left| \frac{1}{T_m} \int \sum_{t=m/2}^{T-m/2} \left(\log[\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \tilde{\boldsymbol{\Sigma}}_T, \omega)] - \log[\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0, \omega)] \right) d\omega \right| \xrightarrow{\mathcal{P}} 0 \tag{79}$$

$$\sup_{\tilde{\boldsymbol{\theta}}, \boldsymbol{a}} \left| \frac{1}{T_m} \int \sum_{t=m/2}^{T-m/2} I_{t,m}(\boldsymbol{a}, \omega) \left(\frac{\nabla^{k_1} \mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0, \omega)}{\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0, \omega)^{k_2}} - \frac{\nabla^{k_1} \mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0, \omega)}{\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \tilde{\boldsymbol{\Sigma}}_T, \omega)^{k_2}} \right) \right| \xrightarrow{\mathcal{P}} 0 \tag{80}$$

41

$$\frac{1}{\sqrt{T_m}} \int \sum_{t=m/2}^{T-m/2} \Re[J_{t,m}^{(X)}(\omega) J_{t,m}^{(j)}(-\omega)] \Big(\frac{1}{\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \tilde{\boldsymbol{\Sigma}}_T, \omega)} - \frac{1}{\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0, \omega)} \Big) d\omega \xrightarrow{\mathcal{P}} 0 \tag{81}$$

$$\frac{1}{\sqrt{T_m}} \int \sum_{t=m/2}^{T-m/2} \big(I_{t,m}(\boldsymbol{a}_0,\omega) - \mathcal{F}_{t,m}(\boldsymbol{\vartheta}_0, \tilde{\boldsymbol{\Sigma}}_T, \omega)\big) \Big(\frac{1}{\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \tilde{\boldsymbol{\Sigma}}_T, \omega)^2} - \frac{1}{\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0, \omega)^2} \Big) \nabla_{\tilde{\boldsymbol{\theta}}} \mathcal{F}_{t,m}(\boldsymbol{\vartheta}_0, \tilde{\boldsymbol{\Sigma}}_T, \omega) d\omega \xrightarrow{\mathcal{P}} 0 \tag{82}$$

and

$$\frac{1}{\sqrt{T_m}} \int \sum_{t=m/2}^{T-m/2} \frac{\big(I_{t,m}(\boldsymbol{a}_0,\omega) - \mathcal{F}_{t,m}(\boldsymbol{\vartheta}_0, \tilde{\boldsymbol{\Sigma}}_T, \omega)\big)}{\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0, \omega))} \big\{ \nabla_{\tilde{\boldsymbol{\theta}}} \mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \tilde{\boldsymbol{\Sigma}}_T, \omega) - \nabla_{\tilde{\boldsymbol{\theta}}} \mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0, \omega) \big\} d\omega \xrightarrow{\mathcal{P}} 0 \tag{83}$$

PROOF. To prove (79) we use the mean value theorem and that $\mathcal{F}_{t,m} \geq \delta > 0$ to obtain

$$\big| \log(\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \tilde{\boldsymbol{\Sigma}}_T, \omega)) - \log(\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0, \omega)) \big| \leq \frac{1}{\delta} \big| \mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0, \omega) - \mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \tilde{\boldsymbol{\Sigma}}_T, \omega) \big|.$$

Now by substituting the above into the sum in (79) and using (72) we obtain (79). Using a similar argument to the proof given in Lemma A.5 together with (72) we obtain (80). Now noting that $\mathbb{E}\Re[J_{t,m}^{(X)}(\omega) J_{t,m}^{(j)}(-\omega)] = 0$, $\mathbb{E}\big(I_{t,m}(\boldsymbol{a}_0,\omega) - \mathcal{F}_{t,m}(\boldsymbol{\vartheta}_0, \tilde{\boldsymbol{\Sigma}}_T, \omega)\big) = 0$ and using Lemma 5.1 we can apply (77) to obtain (81) and (82). Finally by using (76) we have (83). □

We now obtain approximations of $\mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}, \tilde{\boldsymbol{\Sigma}}_T)$ and its derivatives. In the following lemma we show that $\nabla_{\boldsymbol{a}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \tilde{\boldsymbol{\Sigma}}_T)$ is uneffected by whether the estimated $\tilde{\boldsymbol{\Sigma}}_T$ or true $\boldsymbol{\Sigma}_0$ is used. On the other hand, estimating $\tilde{\boldsymbol{\Sigma}}_T$, changes of the asymptotic variance of $\nabla_{\boldsymbol{\vartheta}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \tilde{\boldsymbol{\Sigma}}_T)$.

Lemma A.7 *Suppose the assumptions Theorem 5.2 are satisfied and let $\mathcal{Q}_T(\boldsymbol{a}_0, \boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0)$ be defined as in (32). Then we have*

$$\sup_{\boldsymbol{a}, \boldsymbol{\vartheta}} \big| \mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\vartheta}, \tilde{\boldsymbol{\Sigma}}_T) - \mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0) \big| \xrightarrow{\mathcal{P}} 0, \tag{84}$$

$$\sup_{\boldsymbol{a}, \boldsymbol{\vartheta}} \big| \nabla^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\vartheta}, \tilde{\boldsymbol{\Sigma}}_T) - \nabla^2 \mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0) \big| \xrightarrow{\mathcal{P}} 0, \tag{85}$$

$$\sqrt{T} \nabla_{\boldsymbol{a}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \tilde{\boldsymbol{\Sigma}}_T) = \sqrt{T} \nabla_{\boldsymbol{a}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \boldsymbol{\Sigma}_0) + o_p(1) \tag{86}$$

$$\sqrt{T} \nabla_{\boldsymbol{\vartheta}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \tilde{\boldsymbol{\Sigma}}_T) = \sqrt{T} \nabla_{\boldsymbol{\vartheta}} \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \boldsymbol{\Sigma}_0) + \sqrt{T} \mathcal{Q}_T \nabla_{\boldsymbol{\Sigma}} \mathcal{L}_T(\boldsymbol{a}_0, \boldsymbol{\Sigma}_0) + o_p(1) \tag{87}$$

where $\mathcal{Q}_T$ is defined in (32).

PROOF. By using (79) and (80) we immediately obtain (84) and (85). By using (81) we immediately obtain (86). To prove (87) we observe that by expanding $\nabla_{\boldsymbol{\vartheta}}\mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \boldsymbol{\Sigma})$ and using (82) and (81) we have

$$\sqrt{T}\nabla_{\boldsymbol{\vartheta}}\mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \tilde{\boldsymbol{\Sigma}}_T) = \sqrt{T}\nabla_{\boldsymbol{\vartheta}}\mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \boldsymbol{\Sigma}_0) + \sqrt{T}\mathcal{K}_T(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \boldsymbol{\Sigma}_0) + o_p(1)$$

where

$$
\begin{aligned}
\sqrt{T}\mathcal{K}_T(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \boldsymbol{\Sigma}_0) &= \frac{1}{\sqrt{T}}\int \sum_{t=m/2}^{T-m/2} \frac{\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\theta}_{2,T}, \omega) - \mathcal{F}_{t,m}(\boldsymbol{\vartheta}_0, \boldsymbol{\Sigma}_0, \omega)}{\mathcal{F}_{t,m}(\boldsymbol{\vartheta}_0, \boldsymbol{\Sigma}_0, \omega)^2} \nabla_{\boldsymbol{\vartheta}}\mathcal{F}_{t,m}(\boldsymbol{\theta}_{1,0}, \boldsymbol{\Sigma}_0, \omega) \\
&= \frac{1}{\sqrt{T}}\sum_{j=1}^{n+1}(\hat{\sigma}_j - \sigma_{j,0})\int \sum_{t=m/2}^{T-m/2} \frac{h_j^{(t,m)}(\omega)}{\mathcal{F}_{t,m}(\boldsymbol{\vartheta}_0, \boldsymbol{\Sigma}_0, \omega)^2} \nabla_{\boldsymbol{\vartheta}}\mathcal{F}_{t,m}(\boldsymbol{\vartheta}, \boldsymbol{\Sigma}_0, \omega) \\
&= \sqrt{T}\mathcal{Q}_T\nabla_{\boldsymbol{\Sigma}}\mathcal{L}_T^{(1)}(\boldsymbol{a}_0, \boldsymbol{\Sigma}_0) + o_p(1),
\end{aligned}
$$

where $\mathcal{Q}_T$ is defined in (32). Thus we obtain the desired result. $\qquad\square$

Theorem A.1 *Suppose the assumptions Theorem 5.2 are satisfied, and let $\tilde{W}_T^{(m)}$ be defined as in Theorem 5.2. Then we have*

$$\sqrt{T}\nabla\mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \tilde{\boldsymbol{\Sigma}}_T) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tilde{W}_T^{(m)})$$

PROOF. To prove the result we substitute (66) into Lemma A.7 to obtain

$$
\begin{aligned}
\sqrt{T}\nabla_{\boldsymbol{a}}\mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \tilde{\boldsymbol{\Sigma}}_T) &= T^{-1/2}\sum_{d=1}^{T} B_d^{(m,j)} + o_p(1) \\
\sqrt{T}\nabla_{\tilde{\boldsymbol{\theta}}}\mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\vartheta}_0, \tilde{\boldsymbol{\Sigma}}_T) &= T^{-1/2}\sum_{d=1}^{T} Z_d^{(m)} + T^{-1/2}\sum_{d=1}^{T}\mathcal{Q}_T Z_d^{(1)} + o_p(1),
\end{aligned}
$$

where $B_d^{(m,j)}$ and $Z_d^{(1)}$ are defined as in (64) and (75), and $Z_d^{(m)}$ is defined as in (60), except it does not include the derivatives with respect to $\boldsymbol{\Sigma}$. Now from the definitions we observe that $\{B_d^{(m,j)}\}$ and $\{Z_d^{(m)} + \mathcal{Q}_T Z_d^{(1)}\}$ are martingale differences, hence by using the same method of proof given in Proposition A.3, we have the result. $\qquad\square$

PROOF of Theorem 5.2 The proof follows by making a Taylor series expansion of $\mathcal{L}_T^{(m)}(\boldsymbol{a}, \boldsymbol{\vartheta}, \tilde{\boldsymbol{\Sigma}}_T)$ with respect to $\boldsymbol{a}$ and $\boldsymbol{\vartheta}$ and using Theorem A.1. $\qquad\square$

A.4 The covariance operator, approximations and the proof of Section 5.4

A.4.1 An expression for $W_T^{(m)}$

An expression for $V_T^{(m)}$ in terms of $\mathcal{F}_{t,m}$ is given in (26) we now obtain a similar expression for $W_T^{(m)}$. The expression for $W_T^{(m)}$ is quite cumbersome, to make it more interpretable we state it in

terms of operators. Let ℓ_2 denote the space of all square summable (vector) sequences. We define the general operator $\Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(\boldsymbol{j})}(\cdot)$ which acts on the d_1 and d_2-dimensional column vector sequences $\underline{g} = \{\underline{g}(k)\} \in \ell_2$ and $\underline{h} = \{\underline{h}(k)\} \in \ell_2$ (to reduce notation we do not specify the dimensions of the vectors in ℓ_2)

$$\Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(\boldsymbol{j})}(\underline{g},\underline{h})(s,u) = \sum_{r_1=-k_1}^{m-k_1} \sum_{r_2=-k_2}^{m-k_2} \underline{g}(r_1)\underline{h}(r_2)' \mathcal{K}_{\mathbf{x}_s,\mathbf{x}_{s+r_1},\mathbf{x}_u,\mathbf{x}_{u+r_2}}^{(\boldsymbol{j})}(s-u,r_1,r_2), \tag{88}$$

where $\mathbf{x}_s = (x_{s,1},\ldots,x_{s,n},1)$ are the regressors observed at time s and the kernel $\mathcal{K}^{(\boldsymbol{j})}$ is defined as either

$$\mathcal{K}_{\mathbf{x}_{s_1},\mathbf{x}_{s_2},\mathbf{x}_{s_3},\mathbf{x}_{s_4}}^{(0)}(k,r_1,r_2) = \sum_{j=1}^{n+1} x_{s_1,j}x_{s_2,j}x_{s_3,j}x_{s_4,j}\mathrm{cov}(\alpha_{0,j}\alpha_{r_1,j},\alpha_{k,j}\alpha_{k+r_2,j})$$

$$\mathcal{K}_{\mathbf{x}_{s_1},\mathbf{x}_{s_2},\mathbf{x}_{s_3},\mathbf{x}_{s_4}}^{(j_1)}(k,r_1,r_2) = \sum_{j=1}^{n+1} x_{s_1,j}x_{s_2,j}x_{s_3,j}x_{s_4,j_1}\mathrm{cov}(\alpha_{0,j}\alpha_{r_1,j},\alpha_{k,j}) \quad 1 \le j_1 \le n$$

$$\text{or } \mathcal{K}_{\mathbf{x}_{s_1},\mathbf{x}_{s_2},\mathbf{x}_{s_3},\mathbf{x}_{s_4}}^{(j_1,j_2)}(k,r_1,r_2) = \sum_{j=1}^{n+1} x_{s_1,j}x_{s_2,j_1}x_{s_3,j}x_{s_4,j_2}\mathrm{cov}(\alpha_{0,j},\alpha_{k,j}) \quad 1 \le j_1,j_2 \le n \tag{89}$$

where for notational convenience we set $\alpha_{t,n+1} := \varepsilon_t$ and $x_{t,n+1} := 1$. We will show that $\mathrm{var}(\nabla\mathcal{L}_T^{(m)}(\boldsymbol{a}_0,\boldsymbol{\theta}_0)$ can be written in terms of $\Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(\boldsymbol{j})}$. Suppose $\mathcal{G}_s(\cdot)$ and $\mathcal{H}_u(\cdot)$ are square integrable vector functions, then it can be shown

$$\mathrm{var}\left(\begin{array}{c} \int_0^{2\pi} \mathcal{H}_u(\omega)X_u \sum_{r_2=-k_2}^{m-k_2} x_{u+r_1,j_1}\exp(ir_2\omega)d\omega \\ \int_0^{2\pi} \mathcal{G}_s(\omega)X_s \sum_{r_1=-k_1}^{m-k_1} X_{s+r_1}\exp(ir_1\omega)d\omega \end{array}\right)$$

$$= \left(\begin{array}{cc} \Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_2,k_2),(j_1,j_1)}(\underline{h}_u,\bar{\underline{h}}_u)(u,u) & \Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(j_1)}(\bar{\underline{g}}_s,\underline{h}_u)(s,u) \\ \Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(j_1)}(\underline{g}_s,\bar{\underline{h}}_u)(s,u) & \Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_1),(0)}(\underline{g}_s,\bar{\underline{g}}_s)(s,s) \end{array}\right) \tag{90}$$

where $\underline{g}_s = \{\underline{g}_s(r) = \int \mathcal{G}_s(\omega)\exp(ir\omega)d\omega\}_r$, $\underline{h}_s = \{\underline{h}_s(r) = \int \mathcal{H}_s(\omega)\exp(r\omega)d\omega\}_r$, $\bar{\underline{g}}_s = \{\underline{g}_s(-r)\}_r$ and $\bar{\underline{h}}_s = \{h_s(-r)\}_r$.

Using the above notation we can write

$$W_T^{(m)} = T\mathrm{var}(\nabla\mathcal{L}_T^{(m)}(\boldsymbol{a}_0,\boldsymbol{\theta}_0)) = \frac{1}{m^2T_m} \sum_{k_1,k_2=1}^m \sum_{s=k_1}^{T-m+k_1} \sum_{u=k_2}^{T-m+k_2} \tag{91}$$

$$\left(\begin{array}{cc} \Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(\boldsymbol{j},\boldsymbol{j})}(a_{s+m/2-k_1},a_{u+m/2-k_2})(s,u) & \Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(\boldsymbol{j})}(\underline{b}_{s+m/2-k_1},a_{u+m/2-k_2})(s,u) \\ \Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(\boldsymbol{j})}(\bar{\underline{b}}_{s+m/2-k_1},a_{u+m/2-k_2})(s,u) & \Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(0)}(\underline{b}_{s+m/2-k_1},\bar{\underline{b}}_{u+m/2-k_2})(s,u) \end{array}\right),$$

where $\Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(\boldsymbol{j},\boldsymbol{j})} = \{\Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(j_1,j_2)}\}_{j_1,j_2=1,\ldots,n}$ and $\Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(\boldsymbol{j})} = \{\Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(j)}\}_{j=1,\ldots,n}$, $\underline{b}_s = \{\underline{b}_s(k)\}$, $a_s = \{a_s(k)\}$ with

$$a_s(k) = \int \mathcal{F}_{s,m}(\boldsymbol{\theta}_0,\omega)^{-1}\exp(ik\omega)d\omega, \quad \underline{b}_s(k) = \int \nabla_{\boldsymbol{\theta}}\mathcal{F}_{s,m}(\boldsymbol{\theta}_0,\omega)^{-1}\exp(ik\omega)d\omega, \tag{92}$$

44

(noting that $\mathcal{F}_{s,m}(\boldsymbol{\theta}_0, \omega)^{-1}$ is a symmetric function hence $a_s = \bar{a}_s$). We will use (91) when we compare Estimator 1 with the Gaussian maximum likelihood estimator.

A.4.2 General approximations

The results in this section are used to prove Proposition 5.3. These results may also be of independent interest.

In this section we will consider functions which admit the representation $G(\omega) = \sum_{j=-\infty}^{\infty} g(k) \exp(ik\omega)$, where $\{g(k)\}$ is a square summable sequence, hence $G(\cdot) \in L_2[0, 2\pi]$, where $L_2[0, 2\pi]$ denotes all functions defined on $[0, 2\pi]$. Many of the results below, require that the sequence $\{g(k)\}$ is absolutely summable. Sufficient conditions for this can be obtained in terms of $G(\cdot)$. More precisely, since $(ik)^r g(k) = \int_0^{2\pi} \frac{d^r G(\omega)}{d\omega^r} \exp(ik\omega) d\omega$, if $\frac{d^r G(\omega)}{d\omega^r} \in L_2[0, 2\pi]$, then $\sum_k |k^{r-1}| \cdot |g(k)| < \infty$. Hence if the derivative of $G(\cdot)$ is square integrable, then this implies absolute summability of $\{g(k)\}$. Let $\|G\|_2 = \int |G(x)|^2 dx$, where $G : [0, 2\pi] \to \infty$.

We begin by considering some of the properties of the operators $\Gamma_{\mathbf{x}_1,...,\mathbf{x}_T}^{(k_1,k_2),(\mathbf{j})}$, defined in (88), and

$$\Gamma_{\mathbf{x}_s,\mathbf{x}_u}^{(\mathbf{j})}(\underline{g}, \underline{h})(k) = \sum_{r_1=-\infty}^{\infty} \sum_{r_2=-\infty}^{\infty} \underline{g}(r_1)\underline{h}(r_2)\mathcal{K}_{\mathbf{x}_s,\mathbf{x}_s,\mathbf{x}_u,\mathbf{x}_u}^{(\mathbf{j})}(k, r_1, r_2), \tag{93}$$

where $\underline{g} = \{\underline{g}(r)\}$ and $\underline{h} = \{\underline{h}(r)\}$ are d_1 and d_2-dimensional column vector sequences which are in ℓ_2 and $\mathcal{K}^{(\mathbf{j})}$ is defined in (89).

We recall that $\mathrm{var}(\nabla \mathcal{L}_T^{(m)}(\boldsymbol{a}_0, \boldsymbol{\theta}_0))$ can be written in terms of $\Gamma_{\mathbf{x}_1,...,\mathbf{x}_T}^{(k_1,k_2),(\mathbf{j})}$ (see (91)). The operator $\Gamma_{\mathbf{x}_s,\mathbf{x}_u}$ has a similar interpretation. Define the (unobserved) time series $X_t(s) = \sum_{j=1}^{n} \alpha_{t,j} x_{s,j} + \varepsilon_t$ and $J_X(s, \omega) = \frac{1}{\sqrt{2\pi m}} \sum_{k=1}^{m} X_{s-m/2+k}(s) \exp(ik\omega)$, using this, it can be shown that

$$\mathrm{cov}\left(\int_0^{2\pi} \mathcal{G}_s(\omega) \sum_{r_1=-\infty}^{\infty} X_s(s) X_{s+r_1}(s) \exp(ir_1\omega) d\omega, \int_0^{2\pi} \mathcal{H}_u(\omega) \sum_{r_2=-\infty}^{\infty} X_u(u) X_{u+r_2}(u) \exp(ir_2\omega) d\omega \right)$$
$$= \Gamma_{\mathbf{x}_s,\mathbf{x}_u}^{(0)}(g_s, \bar{h}_u)(s - u) \tag{94}$$

where $g_s = \{g_s(r) = \int \mathcal{G}_s(\omega) \exp(ir\omega) d\omega\}$, $\bar{h}_s = \{h_s(r) = \int \mathcal{H}_s(\omega) \exp(r\omega) d\omega\}$. Similar interpretations are true for $\Gamma_{\mathbf{x}_s,\mathbf{x}_u}^{(j_1,j_2)}(g_s, h_u)(s - u)$ and $\Gamma_{\mathbf{x}_s,\mathbf{x}_u}^{(j)}(g_s, h_u)(s - u)$.

The proof of Proposition 5.3 requires the proposition below, where we show that if the kernels $\mathcal{K}^{(\mathbf{j})}$ satisfy a type of Lipschitz condition, then we can approximate $\Gamma_{\mathbf{x}_1,...,\mathbf{x}_T}^{(k_1,k_2)}$ with $\Gamma_{\mathbf{x}_s,\mathbf{x}_u}$. We start by obtain upper bounds for the kernels $\mathcal{K}^{(\mathbf{j})}$. Recalling the definition of $\rho_2(\cdot)$ and $\rho_4(\cdot)$ in (24), let $\rho_3(k_1, k_2) = \kappa_3 \sup_{j_1,j_2,j_3} \sum_i \cdot |a_{i,j_1}| \cdot |a_{i+k_1,j_2}| \cdot |a_{i+k_2,j_3}|$. Using these bounds it is clear that for

$\mathbf{j} \in \{0, j, (j_1, j_2)\}$ we have $\sup |\mathcal{K}^{(\mathbf{j})}_{\mathbf{x}_{s_1}, \mathbf{x}_{s_2}, \mathbf{x}_{s_3}, \mathbf{x}_{s_4}}(k, r_1, r_2)| \leq \tilde{\mathcal{K}}^{(\mathbf{j})}(k, r_1, r_2)$, where

$$\mathcal{K}^{(0)}(k, r_1, r_2) = (n+1) \sup_{s,j} |x_{s,j}|^4 \left(\rho_2(k)\rho_2(k + r_1 - r_2) + \rho_2(k - r_2)\rho_2(k + r_2) + \rho_4(r_1, k, k - r_2) \right)$$

$$\tilde{\mathcal{K}}^{(j)}(k, r_1, r_2) = (n+1) \sup_{s,j} |x_{s,j}|^4 \rho_3(k, r) \text{ and } \tilde{\mathcal{K}}^{(j_1, j_2)}(k, r_1, r_2) = (n+1) \sup_{s,j} |x_{s,j}|^4 \rho_2(k).$$

For all $\mathbf{j} \in \{0, j, (j_1, j_2)\}$, these kernels can be used as the Lipschitz bounds

$$|\mathcal{K}^{(\mathbf{j})}_{\mathbf{x}_{s_1}, \mathbf{x}_{s_2}, \mathbf{x}_{s_3}, \mathbf{x}_{s_4}}(k, r_1, r_2) - \mathcal{K}^{(\mathbf{j})}_{\mathbf{x}_{s_5}, \mathbf{x}_{s_6}, \mathbf{x}_{s_7}, \mathbf{x}_{s_8}}(k, r_1, r_2)| \leq \frac{1}{\hat{N}} \left\{ \sum_{i=1}^{4} |s_i - s_{i+4}| \right\} \tilde{\mathcal{K}}^{(\mathbf{j})}(k, r_1, r_2). \quad (95)$$

In the proofs below we require the following bounds, which are finite under Assumption 5.1

$$\mathcal{K}_1^{(\mathbf{j})} = \begin{cases} \sum_{k,r_1} |\tilde{\mathcal{K}}^{(0)}(k, r_1, r_2)| & \mathbf{j} = 0 \\ \sum_{k,r_1} |\tilde{\mathcal{K}}^{(j)}(k, r_1, r_2)| & \mathbf{j} = j \\ \sum_{k} |\tilde{\mathcal{K}}^{(j_1, j_2)}(k, r_1, r_2)| & \mathbf{j} = (j_1, j_2) \end{cases} \qquad \mathcal{K}_2^{(\mathbf{j})} = \begin{cases} \sum_{k,r_1} |k\tilde{\mathcal{K}}^{(0)}(k, r_1, r_2)| & \mathbf{j} = 0 \\ \sum_{k,r_1} |k\tilde{\mathcal{K}}^{(j)}(k, r_1, r_2)| & \mathbf{j} = j \\ \sum_{k} |k\tilde{\mathcal{K}}^{(j_1, j_2)}(k, r_1, r_2)| & \mathbf{j} = (j_1, j_2) \end{cases} \quad (96)$$

Proposition A.4 *Let* $\Gamma^{(k_1, k_2), (\mathbf{j})}_{\mathbf{x}_1, \dots, \mathbf{x}_T}$ *and* $\Gamma^{(\mathbf{j})}_{\mathbf{x}_s, \mathbf{x}_u}$ *be defined as in (88) and (93). Let* $H_s(\omega) = \sum_{k=-\infty}^{\infty} h_s(k) \exp(ik\omega)$ *and* $G_s(\omega) = \sum_{k=-\infty}^{\infty} g_s(k) \exp(ik\omega)$. *Then we have*

$$\left| \frac{1}{m^2 T_m} \sum_{k_1, k_2 = 1}^{m} \sum_{s=k_1}^{T-m+k_1} \sum_{u=k_2}^{T-m+k_2} \left(\Gamma^{(k_1, k_2), (\mathbf{j})}_{\mathbf{x}_1, \dots, \mathbf{x}_T}(g_{s+k_1}, h_{u+k_2})(s, u) - \Gamma^{(\mathbf{j})}_{\mathbf{x}_s, \mathbf{x}_u}(g_{s+k_1}, h_{u+k_2})(s - u) \right) \right|$$

$$\leq C \left\{ \frac{1}{\hat{N}} + \frac{1}{m} \right\} \left\{ \sup_s \left\| \frac{d^2 H_s(\omega)}{d\omega^2} \right\|_2 \sup_u \left\| \frac{d^2 G_u(\omega)}{d\omega^2} \right\|_2 \right\} \mathcal{K}_1^{(\mathbf{j})},$$

where $\mathbf{j} \in \{0, j, (j_1, j_2)\}$ *and* $C < \infty$.

PROOF. The main differences between $\Gamma^{(k_1, k_2), (\mathbf{j})}_{\mathbf{x}_1, \dots, \mathbf{x}_T}$ and $\Gamma^{(\mathbf{j})}_{\mathbf{x}_s, \mathbf{x}_u}$ are the kernels used and the limits of the summands. Systematically replacing the kernels $\mathcal{K}^{(\mathbf{j})}_{\mathbf{x}_s, \mathbf{x}_{s+r_1}, \mathbf{x}_u, \mathbf{x}_{u+r_2}}(k, r_1, r_2)$ with $\mathcal{K}^{(\mathbf{j})}_{\mathbf{x}_s, \mathbf{x}_s, \mathbf{x}_u, \mathbf{x}_u}(k, r_1, r_2)$ and the summands $\sum_{r_1=-k_1}^{m-k_1} \sum_{r_2=-k_2}^{m-k_2}$ with $\sum_{r_1=-\infty}^{\infty} \sum_{r_2=-\infty}^{\infty}$, and using (95) gives

$$\left| \frac{1}{m^2 T_m} \sum_{k_1, k_2 = 1}^{m} \sum_{s=k_1}^{T-m+k_1} \sum_{u=k_2}^{T-m+k_2} \left(\Gamma^{(k_1, k_2), (\mathbf{j})}_{\mathbf{x}_1, \dots, \mathbf{x}_T}(g_{s-k_1}, h_{u-k_2})(s, u) - \Gamma^{(\mathbf{j})}_{\mathbf{x}_s, \mathbf{x}_u}(g_{s-k_1}, h_{u-k_2})(s - u) \right) \right| \leq I + II + III$$

$$(97)$$

where

$$I = \frac{1}{m^2 T_m \hat{N}} \sum_{k_1, k_2 = 1}^{m} \sum_{s=k_1}^{T-m+k_1} \sum_{u=k_2}^{T-m+k_2} \sum_{r_1, r_2 = -\infty}^{\infty} (|r_1| + |r_2|)|g_{s-k_1}(r_1)| \cdot |h_{u-k_2}(r_2)| \cdot \tilde{\mathcal{K}}^{(\mathbf{j})}(s - u, r_1, r_2)$$

$$II = \frac{1}{m^2 T_m} \sum_{k_1, k_2 = 1}^{m} \sum_{d_1=0}^{T-m} \sum_{d_2=0}^{T-m} \sum_{r_1=-\infty}^{\infty} \sum_{r_2 < -k_2, r_2 > m-k_2} |g_{d_1}(r_1)| \cdot |h_{d_2}(r_2)| \cdot \tilde{\mathcal{K}}^{(\mathbf{j})}(d_1 + k_1 - d_2 - k_2, r_1, r_2)$$

$$III = \frac{1}{m^2 T_m} \sum_{k_1, k_2 = 1}^{m} \sum_{d_1=0}^{T-m} \sum_{d_2=0}^{T-m} \sum_{r_1 < -k_1, r_1 > m-k_1} \sum_{r_2=-\infty}^{\infty} |g_{d_1}(r_1)| \cdot |h_{d_2}(r_2)| \cdot \tilde{\mathcal{K}}^{(\mathbf{j})}(d_1 + k_1 - d_2 - k_2, r_1, r_2)$$

46

We will derive bounds for I, II and III in the case $\mathbf{j} = 0$, similar bounds can be obtained for the $\mathbf{j} = j$ and $\mathbf{j} = (j_1, j_2)$. We first bound I. By using the Cauchy-Schwartz inequality we have

$$\sum_{r_1, r_2 = -\infty}^{\infty} |r_1| \cdot |g_{s-k_1}(r_1)| \cdot |h_{u-k_2}(r_2)| \cdot \tilde{\mathcal{K}}^{(0)}(s - u, r_1, r_2)$$

$$\leq \left(\sum_{r_1 = -\infty}^{\infty} |r_1 \cdot g_{s-k_1}(r_1)|^2 \right)^{1/2} \left(\sum_{r_1 = -\infty}^{\infty} |\tilde{\mathcal{K}}^{(0)}(s - u, r_1, r_2)|^2 \right)^{1/2} \leq \sup_s \| \frac{dG_s(\omega)}{d\omega} \|_2 \left(\sum_{r_1 = -\infty}^{\infty} |\tilde{\mathcal{K}}^{(0)}(s - u, r_1, r_2)|^2 \right)^{1/2}.$$

The same bound can be obtained for $\sum_{r_1, r_2 = -\infty}^{\infty} |r_1| \cdot |g_{s-k_1}(r_2)| \cdot |h_{u-k_2}(r_2)| \cdot \tilde{\mathcal{K}}^{(0)}(s - u, r_1, r_2)$, substituting both these bounds into I and noting that $\sum_{u=-\infty}^{\infty} \left(\sum_{r_2=-\infty}^{\infty} |\tilde{\mathcal{K}}^{(0)}(s-u, r_1, r_2)|^2 \right)^{1/2} \leq \sum_{u=-\infty}^{\infty} \sum_{r_2=-\infty}^{\infty} |\tilde{\mathcal{K}}^{(0)}(s - u, r_1, r_2)| = \mathcal{K}_1^{(\mathbf{j})}$, we have

$$I \leq \frac{1}{m^2 T_m \hat{N}} \sum_{k_1, k_2 = 1}^{m} \sum_{s = k_1}^{T - m + k_1} \sup_s \| \frac{dG_s(\omega)}{d\omega} \|_2 \sup_u \| \frac{dH_s(\omega)}{d\omega} \|_2 \sum_{u = k_2}^{T - m + k_2} \left(\sum_{r_1 = -\infty}^{\infty} + \sum_{r_1 = -\infty}^{\infty} \right) \tilde{\mathcal{K}}^{(0)}(s - u, r_1, r_2)$$

$$\leq \frac{K}{\hat{N}} \sup_s \| \frac{dG_s(\omega)}{d\omega} \|_2 \sup_u \| \frac{dH_s(\omega)}{d\omega} \|_2 \mathcal{K}_1^{(0)},$$

where K is a finite constant. We now bound II. By using (96) we have

$$II \leq \frac{1}{mT_m} \sup_{s,r} |g_s(r)| \mathcal{K}_1^{(0)} \sum_{d_2 = 0}^{T - m} \sum_{k_2 = 1}^{m} \sum_{r_2 < -k_2, r_2 > m - k_2} |h_{d_2}(r_2)|. \tag{98}$$

Studying the inner sum of the above we observe that

$$\sum_{k_2 = 1}^{m} \sum_{r_2 < -k_2, r_2 > m - k_2} |h_{d_2}(r_2)| \leq \sum_{r_2 = -\infty}^{\infty} |r_2 h_{d_2}(r_2)| \leq \sup_s \| \frac{d^2 H_s(\omega)}{d\omega^2} \|_2.$$

Substituting the above and $\sup_{s,r} |g_s(r)| \leq \sup_s \| \frac{dG_s(\omega)}{d\omega} \|_2$ into (98) gives

$$II \leq \frac{1}{m} \mathcal{K}_1^{(0)} \left\{ \sup_s \| \frac{dH_s(\omega)}{d\omega} \|_2 \sup_u \| \frac{dG_u(\omega)}{d\omega} \|_2 \right\}.$$

Using a similar argument we obtain the same bound for II. Altogether these bounds for I, II and III give (97). To prove the result for $\mathbf{j} \in \{j, (j_1, j_2)\}$ is similar and we omit the details. $\qquad \square$

Proposition A.5 *Suppose the assumptions in Proposition A.4 hold, $\sup_{s,\omega} | \frac{dH_s(\omega)}{d\omega} | < \infty$ and for all s and u we have*

$$\sup_\omega |H_s(\omega) - H_u(\omega)| \leq K \frac{|s - u|}{\hat{N}} \quad \sup_\omega |G_s(\omega) - G_u(\omega)| \leq K \frac{|s - u|}{\hat{N}}, \tag{99}$$

where $K < \infty$. Then we have

$$\left| \frac{1}{m^2 T_m} \sum_{k_1, k_2 = 1}^{m} \sum_{s = k_1}^{T - m + k_1} \left(\sum_{u = k_2}^{T - m + k_2} \Gamma_{\mathbf{x}_s, \mathbf{x}_u}^{(\mathbf{j})}(g_{s-k_1}, h_{u-k_2})(s - u) - \sum_{u = -\infty}^{\infty} \Gamma_{\mathbf{x}_s, \mathbf{x}_s}^{(\mathbf{j})}(g_{s-k_1}, h_{s-k_2})(s - u) \right) \right|$$

$$\leq K \left\{ \frac{1}{\hat{N}} + \frac{1}{m} \right\} \left\{ (2 \sup_{s,\omega} |H_s(\omega)| \sup_{s,\omega} | \frac{dH_s(\omega)}{d\omega} |)^{1/2} + \sup_s \| \frac{H_s(\omega)}{d\omega} \|_2 \right\} \sup_u \| \frac{dG_u(\omega)}{d\omega} \|_2 \mathcal{K}_2^{(\mathbf{j})} \tag{100}$$

47

and

$$\left| \frac{1}{m^2 T_m} \sum_{k_1,k_2=1}^{m} \sum_{s=k_1}^{T-m+k_1} \sum_{u=-\infty}^{\infty} \Gamma_{\mathbf{x}_s,\mathbf{x}_s}^{(\mathbf{j})}(g_{s-k_1}, h_{s-k_2})(s-u) - \frac{1}{T} \sum_{s=1}^{T} \sum_{u=-\infty}^{\infty} \Gamma_{\mathbf{x}_s,\mathbf{x}_s}^{(\mathbf{j})}(g_s, h_s)(s-u) \right|$$

$$\leq \quad K\left\{ \frac{m}{\hat{N}} \right\} \left\{ \sup_{s,\omega} |\frac{dH_s(\omega)}{d\omega}|^{1/2} \sup_{u} \| \frac{dG_u(\omega)}{d\omega} \|_2 \mathcal{K}_2^{(\mathbf{j})}. \right. \tag{101}$$

PROOF. We first prove (100), where the summand $\sum_{u=k_2}^{T-m+k_2}$ changes to $\sum_{u=-\infty}^{\infty}$ and $\Gamma_{\mathbf{x}_s,\mathbf{x}_u}$ changes to $\Gamma_{\mathbf{x}_s,\mathbf{x}_s}$. We first observe

$$\left| \frac{1}{m^2 T_m} \sum_{k_1,k_2=1}^{m} \sum_{s=k_1}^{T-m+k_1} \left\{ \sum_{u=k_2}^{T-m+k_2} \Gamma_{\mathbf{x}_s,\mathbf{x}_u}^{(\mathbf{j})}(g_{s-k_1}, h_{u-k_2})(s-u) - \sum_{u=-\infty}^{\infty} \Gamma_{\mathbf{x}_s,\mathbf{x}_s}^{(\mathbf{j})}(g_{s-k_1}, h_{s-k_2})(s-u) \right\} \right|$$

$$\leq \quad I + II, \tag{102}$$

where

$$I \quad = \quad \frac{1}{m^2 T_m} \sum_{k_1,k_2=1}^{m} \sum_{s=k_1}^{T-m+k_1} \sum_{u=k_2}^{T-m+k_2} \left| \Gamma_{\mathbf{x}_s,\mathbf{x}_u}^{(\mathbf{j})}(g_{s-k_1}, h_{u-k_2})(s-u) - \Gamma_{\mathbf{x}_s,\mathbf{x}_s}^{(\mathbf{j})}(g_{s-k_1}, h_{s-k_2})(s-u) \right|$$

$$II \quad = \quad \frac{1}{m^2 T_m} \sum_{k_1,k_2=1}^{m} \sum_{s=k_1}^{T-m+k_1} \sum_{u<k_2, u>T-m+k_2} \left| \Gamma_{\mathbf{x}_s,\mathbf{x}_s}^{(\mathbf{j})}(g_{s-k_1}, h_{s-k_2})(s-u) \right|.$$

We now bound I. To bound inside the summation of I we use (95) and the definition of $\Gamma_{\mathbf{x}_s,\mathbf{x}_s}^{(\mathbf{j})}$ to obtain

$$\left| \Gamma_{\mathbf{x}_s,\mathbf{x}_u}^{(\mathbf{j})}(g_{s-k_1}, h_{u-k_2})(s-u) - \Gamma_{\mathbf{x}_s,\mathbf{x}_s}^{(\mathbf{j})}(g_{s-k_1}, h_{s-k_2})(s-u) \right|$$

$$\leq \quad \sum_{r_1=-\infty}^{\infty} |g_{s+k_1}(r_1)| \sum_{r_2=-\infty}^{\infty} \left\{ |h_{u-k_2}(r_2) - h_{s-k_2}(r_2)| + \frac{|s-u|}{\hat{N}} |h_{u-k_2}(r_2)| \right\} \tilde{\mathcal{K}}^{(\mathbf{j})}(s-u, r_1, r_2) \tag{103}$$

We start by examining $\sum_{r_2=-\infty}^{\infty} |h_{u-k_2}(r_2) - h_{s-k_2}(r_2)| \tilde{\mathcal{K}}(s-u, r_1, r_2)$, which is the first term in (103). By using the Cauchy-Schwartz inequality we have and that $\left\{ \sum_{r_2=-\infty}^{\infty} \tilde{\mathcal{K}}^{(\mathbf{j})}(s-u, r_1, r_2)^2 \right\}^{1/2} \leq \sum_{r_2=-\infty}^{\infty} \tilde{\mathcal{K}}^{(\mathbf{j})}(s-u, r_1, r_2)$ gives

$$\sum_{r_2=-\infty}^{\infty} |h_{u-k_2}(r_2) - h_{s-k_2}(r_2)| \tilde{\mathcal{K}}^{(\mathbf{j})}(s-u, r_1, r_2) \quad \leq \quad \left\{ \sum_{r_2=-\infty}^{\infty} |h_{u-k_2}(r_2) - h_{s-k_2}(r_2)|^2 \right\}^{1/2} \sum_{r_2=-\infty}^{\infty} \tilde{\mathcal{K}}^{(\mathbf{j})}(s-u, r_1, r_2)$$

Recalling that $h_s(r) = \int H_s(\omega) \exp(ir\omega) d\omega$ we have by Parseval's theorem and (99) that

$$\sum_{r=-\infty}^{\infty} |h_{u+k_2}(r) - h_{s+k_2}(r)|^2 = \int_0^{2\pi} |H_{s+k_2}(\omega) - H_{u+k_2}(\omega)|^2 d\omega \leq H \frac{|u-s|}{\hat{N}},$$

where $H = (2 \sup_{s,\omega} |H_s(\omega)| \sup_{s,\omega} |\frac{dH_s(\omega)}{d\omega}|)^{1/2}$. Substituting the above into (103) gives

$$\left| \Gamma_{\mathbf{x}_s,\mathbf{x}_u}^{(\mathbf{j})}(g_{s-k_1}, h_{u-k_2})(s-u) - \Gamma_{\mathbf{x}_s,\mathbf{x}_s}^{(\mathbf{j})}(g_{s-k_1}, h_{s-k_2})(s-u) \right|$$

$$\leq \quad \frac{|s-u|}{\hat{N}} \sum_{r_1=-\infty}^{\infty} |g_{s-k_1}(r_1)| \left(H + \sum_{r_2=-\infty}^{\infty} |h_{u-k_2}(r_2)| \right) \tilde{\mathcal{K}}^{(\mathbf{j})}(s-u, r_1, r_2).$$

48

Substituting the above into I, and noting that $\sum_{r_2=-\infty}^{\infty}|h_{u-k_2}(r_2)| \leq \sup_u\|\frac{dH_u(\omega)}{d\omega}\|$ and $\sum_{r_1=-\infty}^{\infty}|g_{s-k_1}(r_2)| \leq \sup_s\|\frac{dG_s(\omega)}{d\omega}\|$ we have

$$
\begin{aligned}
I \;&\leq\; \frac{1}{m^2\hat{N}T_m}\sum_{k_1,k_2=1}^{m}\sum_{s=k_1}^{T-m+k_1}\sum_{r_1=-\infty}^{\infty}|g_{s-k_1}(r_1)|\left(H+\sum_{r_2=-\infty}^{\infty}|h_{u-k_2}(r_2)|\right)\sum_{u=k_2}^{T-m+k_2}|s-u|\tilde{\mathcal{K}}^{(\mathbf{j})}(s-u,r_1,r_2)\\
&\leq\; \hat{N}^{-1}\sup_s\|\frac{dG_s(\omega)}{d\omega}\|_2\left(H+\sup_s\|\frac{dH_s(\omega)}{d\omega}\|_2\right)\sum_{u=-\infty}^{\infty}|s-u|\tilde{\mathcal{K}}^{(\mathbf{j})}(s-u,r_1,r_2)\\
&\leq\; \hat{N}^{-1}\sup_s\|\frac{dG_s(\omega)}{d\omega}\|_2\left(H+\sup_s\|\frac{dH_s(\omega)}{d\omega}\|_2\right)\mathcal{K}_2^{(\mathbf{j})}.
\end{aligned}
$$

Now we bound II. Using similar methods to those used in the proof of Proposition A.4 we have

$$
\begin{aligned}
II \;&\leq\; \frac{1}{m^2T_m}\sum_{k_1,k_2=1}^{m}\sum_{s=k_1}^{T-m+k_1}\sum_{u=-\infty}^{\infty}|u|\sum_{r_1,r_2=-\infty}^{\infty}|g_{s+k_1}(r_1)||h_{s+k_2}(r_2)|\tilde{\mathcal{K}}(s-u,r_1,r_2)\\
&\leq\; \frac{1}{m}\sup_s\|\frac{dH_s(\omega)}{d\omega}\|_2\sup_s\|G_s(\omega)\|_2\mathcal{K}_2^{(\mathbf{j})}.
\end{aligned}
$$

Substituting the bounds for I and II into (102) and noting that $\|G_s(\omega)\|_2 \leq \|\frac{dG_s(\omega)}{d\omega}\|_2$ gives (100).

The proof of (101) uses the same methods used to bound I, hence we omit the details. $\qquad\square$

A.4.3   Proofs in Section 5.4

We first outline the proof of Lemma 5.2.

PROOF of Lemma 5.2 The result can be shown by using the same arguments as those used in Dahlhaus (2000), where it is shown that a version of the Whittle likelihood containing the pre-periodogram and the localised spectrum is asymptotically equivalent to the Gaussian likelihood. As the proof is identical to the proof given in Dahlhaus (2000), we omit the details. $\qquad\square$

In the following proposition we obtain a series of approximations to (91) which we use to approximate $\operatorname{var}(\sqrt{T}\nabla\mathcal{L}_T^{(m)}(\boldsymbol{\alpha}_0,\boldsymbol{\theta}_0))$. The proof of Proposition 5.3 follows immediately from the proposition below and

$$
\frac{N}{T}\int_0^{T/N}\sum_{k=-\infty}^{\infty}\Gamma_{\mathbf{x}(v),\mathbf{x}(v)}^{(j_1,j_2)}(a(v),a(v))(k)dv = \frac{N}{T}\int_0^{T/N}x_{j_1}(v)x_{j_2}(v)\mathcal{F}(v,\boldsymbol{\theta}_0,0)^{-1}dv,
$$

$$
\frac{N}{T}\int_0^{T/N}\sum_{k=-\infty}^{\infty}\Gamma_{\mathbf{x}(v),\mathbf{x}(v)}^{(0)}(\underline{b}(v),\bar{\underline{b}}(v))(k)dv = \frac{N}{T}\int_0^{T/N}\int_0^{2\pi}\frac{\nabla_{\boldsymbol{\theta}}\mathcal{F}(v,\boldsymbol{\theta}_0,\omega)(\nabla_{\boldsymbol{\theta}}\mathcal{F}(v,\boldsymbol{\theta}_0,-\omega))'}{|\mathcal{F}(v,\boldsymbol{\theta}_0,\omega)|^2}d\omega dv.
$$

Proposition A.6 *Suppose Assumptions 2.1, 5.1 and 5.2 hold, $\hat{N}$ is defined as in (35), the operators $\Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(\mathbf{j})}$ and $\Gamma_{\mathbf{x}_s,\mathbf{x}_u}^{(\mathbf{j})}$ are defined as in (88) and (93) respectively and*

$$
\sup_j\int|\frac{d^2f_j(\boldsymbol{\vartheta}_0,\omega)}{d\omega^2}|^2d\omega < \infty \quad\text{and}\quad \sup_j\int|\frac{d^2\nabla_{\boldsymbol{\theta}}f_j(\boldsymbol{\vartheta}_0,\omega)}{d\omega^2}|^2d\omega < \infty. \tag{104}
$$

Then we have

$$\left| \frac{1}{m^2 T_m} \sum_{k_1,k_2=1}^{m} \sum_{s=k_1}^{T-m+k_1} \left\{ \sum_{u=k_2}^{T-m+k_2} \Gamma_{\mathbf{x}_1,\dots,\mathbf{x}_T}^{(k_1,k_2),(\mathbf{j})}(d_{s-m/2+k_1}^{(1)}, d_{u-m/2+k_2}^{(2)})(s,u) - \right.\right.$$

$$\left.\left. \sum_{u=-\infty}^{\infty} \Gamma_{\mathbf{x}_s,\mathbf{x}_s}^{(\mathbf{j})}(d_{s-m/2+k_1}^{(1)}, d_{s-m/2+k_2}^{(2)})(s-u) \right\} \right| \leq K\left\{ \frac{1}{\hat{N}} + \frac{1}{m} + \frac{1}{T_m} \right\} \tag{105}$$

$$\left| \frac{1}{m^2 T_m} \sum_{k_1,k_2=1}^{m} \sum_{s=k_1}^{T-m+k_1} \sum_{u=-\infty}^{\infty} \Gamma_{\mathbf{x}_s,\mathbf{x}_s}^{(\mathbf{j})}(d_{s-m/2+k_1}^{(1)}, d_{s-m/2+k_2}^{(2)})(s-u) - \right.$$

$$\left. \frac{1}{T} \sum_{s=1}^{T} \sum_{u=-\infty}^{\infty} \Gamma_{\mathbf{x}_s,\mathbf{x}_s}^{(\mathbf{j})}(d_s^{(1)}, d_s^{(2)})(s-u) \right| \leq K\left(\frac{m}{\hat{N}} + \frac{m}{T} \right) \tag{106}$$

$$\left| \frac{1}{T} \sum_{s=1}^{T} \sum_{u=-\infty}^{\infty} \Gamma_{\mathbf{x}_s,\mathbf{x}_s}^{(\mathbf{j})}(d_s^{(1)}, d_s^{(2)})(s-u) - \frac{N}{T} \int_0^{T/N} \sum_{u=-\infty}^{\infty} \Gamma_{\mathbf{x}(v),\mathbf{x}(v)}^{(\mathbf{j})}(d(v)^{(1)}, d(v)^{(2)})(u)dv \right| \leq K\left(\frac{m}{\hat{N}} + \frac{1}{\hat{N}} + \frac{1}{m} \right) \tag{107}$$

$$\left| \frac{1}{T_m} \sum_{s=m/2}^{T-m/2} \frac{\nabla \mathcal{F}_{s,m}(\boldsymbol{\theta}_0,\omega) \nabla \mathcal{F}_{s,m}(\boldsymbol{\theta}_0,\omega)'}{(\mathcal{F}_{s,m}(\boldsymbol{\theta}_0,\omega))^2} - \frac{N}{T} \int_0^{T/N} \frac{\nabla \mathcal{F}(u,\boldsymbol{\theta}_0,\omega) \nabla \mathcal{F}(u,\boldsymbol{\theta}_0,\omega)'}{(\mathcal{F}(u,\boldsymbol{\theta}_0,\omega))^2} du \right| \leq K\frac{m}{\hat{N}} \tag{108}$$

where $\mathbf{j} \in \{0, j, (j_1, j_2)\}$, $d_s^{(i)}(k) \in \{a_s(k), \underline{b}_s(k)\}$ *(see (92) for their definition),* $d^{(i)}(u,k) \in \{a(u,k), \underline{b}(u,k)\}$ *(see (34) for their definition) and* K *is a finite constant.*

PROOF. The proof follows from Propositions A.4 and A.5 We need to verify the conditions in these propositions which we do below. $\quad\square$

Under Assumptions 5.1 and 5.2 we have $\sup_{t,m,\omega} |\mathcal{F}_{s,m}(\boldsymbol{\theta}_0,\omega)| < \infty$, $\sup_{t,m,\omega} |\mathcal{F}_{s,m}(\boldsymbol{\theta}_0,\omega)^{-1}| < \infty$ and $\sup_{t,m,\omega} |\nabla_{\boldsymbol{\theta}} \mathcal{F}_{s,m}(\boldsymbol{\theta}_0,\omega)^{-1}| < \infty$, therefore $\mathcal{F}_{s,m}(\boldsymbol{\theta}_0,\cdot)$, $\mathcal{F}_{s,m}(\boldsymbol{\theta}_0,\cdot)^{-1}$ and $\nabla_{\boldsymbol{\theta}} \mathcal{F}_{s,m}(\boldsymbol{\theta}_0,\cdot)^{-1}$ belong to $L_2[0,2\pi]$. We recall that Proposition A.6 is based on the sequences $\{a_s(k)\}_k$ and $\{\underline{b}_s(k)\}_k$ (defined in (92)), the above means

$$\mathcal{F}_{s,m}(\boldsymbol{\theta}_0,\omega)^{-1} = \sum_{k=-\infty}^{\infty} a_s(k) \exp(ik\omega) \quad \text{and} \quad \nabla_{\boldsymbol{\theta}} \mathcal{F}_{s,m}(\boldsymbol{\theta}_0,\omega)^{-1} = \sum_{k=-\infty}^{\infty} \underline{b}_s(k) \exp(ik\omega). \tag{109}$$

We use the following lemma to prove Lemma A.1.

Lemma A.8 *Suppose that Assumptions 2.1, 5.1 and 5.2 hold. Then we have*

$$\sup_{t,m,\boldsymbol{\theta},\omega} \left| \frac{d\nabla_{\boldsymbol{\theta}}^k \mathcal{F}_{t,m}(\boldsymbol{\theta},\omega)}{d\omega} \right| < \infty, \tag{110}$$

$$\sup_{t,m,\boldsymbol{\theta},\omega} \left| \frac{d\nabla_{\boldsymbol{\theta}}^k \mathcal{F}_{t,m}(\boldsymbol{\theta},\omega)^{-1}}{d\omega} \right| < \infty, \tag{111}$$

and

$$\sup_{t,m,\boldsymbol{\theta}} \sum_{r=-\infty}^{\infty} \left| \int \nabla_{\boldsymbol{\theta}}^k \mathcal{F}_{t,m}(\boldsymbol{\theta},\omega)^{-1} \exp(ir\omega) d\omega \right| < \infty. \tag{112}$$

PROOF. In order to prove the result we will use that Assumption 5.1(iii) implies

$$\sup_j \sum_{r=0}^{\infty} |r^k \nabla_{\boldsymbol{\theta}} c_j(\boldsymbol{\theta},r)| < \infty. \tag{113}$$

To prove (110), we observe $\nabla_{\boldsymbol{\theta}} \mathcal{F}_{t,m}(\boldsymbol{\theta},\omega) = \sum_{j=1}^{n+1} \sum_{r=-(m-1)}^{m-1} \exp(ir\omega) \nabla_{\boldsymbol{\theta}} c_j(\boldsymbol{\theta},r) \frac{1}{m} \sum_{k=1}^{m-|r|} x_{t-m/2+k} x_{t-m/2+k+r}$,
now differentiating this wrt ω gives

$$\frac{d\nabla_{\boldsymbol{\theta}}^k \mathcal{F}_{t,m}(\boldsymbol{\theta},\omega)}{d\omega} = \sum_{j=1}^{n+1} \sum_{r=-(m-1)}^{m-1} ir \exp(ir\omega) \nabla_{\boldsymbol{\theta}}^k c_j(\boldsymbol{\theta},r) \frac{1}{m} \sum_{k=1}^{m-|r|} x_{t-m/2+k} x_{t-m/2+k+r}.$$

Hence using (113) gives (110). We now prove (111) for $k=0$. Under Assumption 5.1(i), $\inf_{\boldsymbol{\theta},\omega} \mathcal{F}_{t,m}(\boldsymbol{\theta},\omega) > \delta > 0$ (for some δ). Therefore, this together with (110) gives

$$\sup_{t,m,\boldsymbol{\theta},\omega} \left| \frac{d\mathcal{F}_{t,m}(\boldsymbol{\theta},\omega)^{-1}}{d\omega} \right| = \sup_{t,m,\boldsymbol{\theta},\omega} \left| \frac{1}{\mathcal{F}_{t,m}(\boldsymbol{\theta},\omega)^2} \frac{d\mathcal{F}_{t,m}(\boldsymbol{\theta},\omega)}{d\omega} \right| < \infty,$$

hence (111) holds for $k=0$. The proof of (111) for $k > 0$, is similar and we omit the details. To prove (112), we note that (111) implies $\frac{d\nabla_{\boldsymbol{\theta}}^k \mathcal{F}_{t,m}(\boldsymbol{\theta},\cdot)^{-1}}{d\omega} \in L_2[0,2\pi]$, hence the Fourier coefficients of $\nabla_{\boldsymbol{\theta}}^k \mathcal{F}_{t,m}(\boldsymbol{\theta},\omega)^{-1}$ are absolutely summable, thus giving (112). $\qquad\square$

In order for us to apply Propositions A.4 and A.5 to $\Gamma_{\mathbf{x}_1,\ldots,\mathbf{x}_T}^{(k_1,k_2),(\mathbf{j})}(\underline{b}_{s+m/2-k_1}, \underline{b}_{u+m/2-k_2})(s,u)$ we require boundedness of $\frac{d^2 \mathcal{F}_{s,m}(\boldsymbol{\theta}_0,\omega)^{-1}}{d\omega^2}$ and $\frac{d^2 \nabla_{\boldsymbol{\theta}} \mathcal{F}_{s,m}(\boldsymbol{\theta}_0,\omega)^{-1}}{d\omega^2}$, which is shown in the following lemma.

Lemma A.9 *Suppose Assumptions 2.1, 5.1 and 5.2 and (104) hold. Then we have*

$$\sup_{t,m,\omega} \left| \frac{d^2 \mathcal{F}_{t,m}(\boldsymbol{\theta}_0,\omega)^{-1}}{d\omega^2} \right| < \infty \qquad \sup_{t,m,\omega} \left| \frac{d^2 \nabla_{\boldsymbol{\theta}} \mathcal{F}_{t,m}(\boldsymbol{\theta}_0,\omega)^{-1}}{d\omega^2} \right| < \infty$$

PROOF. We first note that (104) implies

$$\sup_{j_1,j_2} \sum_r (r^2 |c_{j_1,j_2}(\boldsymbol{\theta}_0,r)|)^2 < \infty \quad \text{and} \quad \sup_{j_1,j_2} \sum_r |r^2 \nabla_{\boldsymbol{\theta}} c_{j_1,j_2}(\boldsymbol{\theta}_0,r)|^2 < \infty,$$

which implies

$$\sup_{t,m,\omega} \left| \frac{d^2 \mathcal{F}_{t,m}(\boldsymbol{\theta}_0,\omega)}{d\omega^2} \right| < \infty \quad \text{and} \quad \sup_{t,m,\omega} \left| \frac{d^2 \nabla_{\boldsymbol{\theta}} \mathcal{F}_{t,m}(\boldsymbol{\theta}_0,\omega)}{d\omega^2} \right| < \infty. \tag{114}$$

By expanding $\frac{d^2 \mathcal{F}_{t,m}(\boldsymbol{\theta}_0,\omega)^{-1}}{d\omega^2}$ and $\frac{d^2 \nabla_{\boldsymbol{\theta}} \mathcal{F}_{t,m}(\boldsymbol{\theta}_0,\omega)^{-1}}{d\omega^2}$ and using (110) and (114) we obtain the result.
$\square$

51

Lemma A.10 *Suppose Assumptions 5.1(i,iii) and 5.2(i) holds. Let $\mathcal{F}_{t,m}$ and $\hat{N}$ be defined as in (8) and (35). Then we have*

$$\left| \frac{1}{\mathcal{F}_{t,m}(\boldsymbol{\theta}_0, \omega)} - \frac{1}{\mathcal{F}_{s,m}(\boldsymbol{\theta}_0, \omega)} \right| \leq \frac{K}{\hat{N}} |s - t|, \tag{115}$$

$$\left| \nabla \mathcal{F}_{t,m}(\boldsymbol{\theta}_0, \omega)^{-1} - \nabla \mathcal{F}_{s,m}(\boldsymbol{\theta}_0, \omega)^{-1} \right| \leq \frac{K}{\hat{N}} |s - t| \tag{116}$$

PROOF. To prove (115), we use (8), Assumptions 5.1(iii) and 5.2(i) to obtain

$$\left| \mathcal{F}_{t,m}(\boldsymbol{\theta}_0, \omega) - \mathcal{F}_{s,m}(\boldsymbol{\theta}_0, \omega) \right| = \left| \sum_{j=1}^{n+1} \int_{-\pi}^{\pi} \left\{ I_{t,m}^{(j)}(\lambda) - I_{s,m}^{(j)}(\lambda) \right\} f_j(\boldsymbol{\theta}_0, \omega - \lambda) d\lambda \right| \leq K \frac{|t - s|}{\hat{N}}, \tag{117}$$

for some finite constant K. Now, under Assumption 5.1(i), $\mathcal{F}_{t,m}(\boldsymbol{\theta}_0, \omega)$ is uniformly bounded away from zero over all ω, t and m. Therefore by using (117) we obtain (115). We use a similar method to prove (116). $\qquad \square$

Proof of Proposition A.6(i,ii) To prove Proposition A.6(i) we apply Propositions A.4 and A.5, equation (100), where depending on the block matrix in $W_T^{(m)}$ we are considering, we let $G_s(\omega)$ and $H_s(\omega)$ take either one of the functions

$$\mathcal{F}_{s,m}(\boldsymbol{\theta}_0, \omega)^{-1} = \sum_{k=-\infty}^{\infty} a_s(k) \exp(ik\omega), \quad \nabla_{\boldsymbol{\theta}} \mathcal{F}_{s,m}(\boldsymbol{\theta}_0, \omega)^{-1} = \sum_{k=-\infty}^{\infty} \underline{b}_s(k) \exp(ik\omega).$$

To verify the conditions in Propositions A.4 and A.5, for $\mathcal{F}_{s,m}(\boldsymbol{\theta}_0, \omega)^{-1}$ and $\nabla_{\boldsymbol{\theta}} \mathcal{F}_{s,m}(\boldsymbol{\theta}_0, \omega)^{-1}$, we note that Lemma A.9 implies $\|\mathcal{F}_{s,m}(\boldsymbol{\theta}_0, \cdot)^{-1}\|_2 \in L_2[0, \pi]$ and $\|\nabla_{\boldsymbol{\theta}} \mathcal{F}_{s,m}(\boldsymbol{\theta}_0, \cdot)^{-1}\|_2 \in L_2[0, \pi]$ (verfying the conditions in Propositions A.4) and Lemma A.10 verifies the condition in Proposition A.5. Altogether this gives Proposition A.6(i).

To prove Proposition A.6(ii) we apply Proposition A.5, (101). The proof uses identical arguments to those given above, hence we omit the details. $\qquad \square$

We use the following lemma to prove Proposition A.6(iii,iv).

Lemma A.11 *Suppose Assumptions 5.1(i,iii) and 5.2(i) holds. Let $\mathcal{F}_{t,m}$, $\hat{N}$ and $\mathcal{F}(u, \cdot)$ be defined as in (8), (35) and (34). Then we have*

$$\left| \mathcal{F}_{t,m}(\boldsymbol{\theta}_0, \omega) - \mathcal{F}(\frac{t}{N}, \boldsymbol{\theta}_0, \omega) \right| \leq K \left(\frac{m}{\hat{N}} + \frac{1}{m} \right), \tag{118}$$

$$\left| \frac{1}{\mathcal{F}_{t,m}(\boldsymbol{\theta}_0, \omega)} - \frac{1}{\mathcal{F}(\frac{t}{N}, \boldsymbol{\theta}_0, \omega)} \right| \leq K \left(\frac{m}{\hat{N}} + \frac{1}{m} \right), \quad \left| \nabla \mathcal{F}_{t,m}(\boldsymbol{\theta}_0, \omega)^{-1} - \nabla \mathcal{F}_{t,m}(\boldsymbol{\theta}_0, \omega)^{-1} \right| \leq K \left(\frac{m}{\hat{N}} + \frac{1}{m} \right) \tag{119}$$

PROOF. We first prove (118). We note that by using Assumption 5.1(iii) (with $k = 0$) we have $\sum_{r=-\infty}^{\infty} |r\nabla_{\boldsymbol{\theta}}^k c_{j_1,j_2}(\boldsymbol{\theta}_0, r)| < \infty$. Therefore by using this, the definitions of $\mathcal{F}_{t,m}$ and $\mathcal{F}(u, \cdot)$ given in (8) and (34) and a similar method of proof to Lemma A.10 we have (118). To prove (119) we use that under Assumption 5.1(i), $\mathcal{F}_{t,m}(\boldsymbol{\theta}_0, \omega)$ is uniformly bounded away from zero over all ω, t and m and similar proof to the proof of (118), we omit the details. $\qquad\square$

Proof of Proposition A.6(iii,iv) We first prove (iii). We will use similar methods to those used in (103). We will prove it for $d_s^{(1)}(k) := a_s(k)$, $d_s^{(2)}(k) := a_s(k)$, $d^{(1)}(u, k) := a(u, k)$ and $d^{(2)}(u, k)^{(2)}$ (the proof for the other cases are identical). The difference between the two operators is

$$\Gamma_{\mathbf{x}_s,\mathbf{x}_s}^{(j)}(a_s, a_s)(k) - \Gamma_{\mathbf{x}(\frac{s}{N}),\mathbf{x}(\frac{s}{N})}^{(j)}(a(\frac{s}{N}), a(\frac{s}{N}))(k)$$

$$\leq \sum_{r_1=-\infty}^{\infty} \left(|a_s(r_1)| + |a(\frac{s}{N}, r_1)| \right) \sum_{r_2=-\infty}^{\infty} |a_s(r_2) - a(\frac{s}{N}, r_2)| |\tilde{\mathcal{K}}^{(j)}(k, r_1, r_2)$$

$$\leq \sum_{r_1=-\infty}^{\infty} \left(|a_s(r_1)| + |a(\frac{s}{N}, r_2)| \right) \left\{ \sum_{r_2=-\infty}^{\infty} |a_s(r_2) - a(\frac{s}{N}, r_2)|^2 \right\}^{1/2} \left\{ \sum_{r_2=-\infty}^{\infty} \tilde{\mathcal{K}}^{(j)}(s - u, r_1, r_2)^2 \right\}^{1/2}.$$

Therefore by using the same arguments given in the proof of Proposition A.4, and using (119) we have

$$\sum_{r=-\infty}^{\infty} |a_s(r) - \tilde{a}(\frac{s}{N}, r)|^2 = \int_0^{2\pi} \left| \frac{1}{\mathcal{F}_{s,m}(\boldsymbol{\theta}_0, \omega)} - \frac{1}{\mathcal{F}(\frac{s}{N}, \boldsymbol{\theta}_0, \omega)} \right|^2 d\omega \leq K \left(\frac{m}{\hat{N}} + \frac{1}{m} \right)^2.$$

Substituting the above into the difference below gives

$$\left| \frac{1}{T} \sum_{s=1}^{T} \sum_{u=-\infty}^{\infty} \left(\Gamma_{\mathbf{x}_s,\mathbf{x}_s}^{(j)}(a_s, a_s)(s - u) - \Gamma_{\mathbf{x}(\frac{s}{N}),\mathbf{x}(\frac{s}{N})}^{(j)}(a(u), a(u))(k) \right) \right|$$

$$\leq K \left| \frac{m}{\hat{N}} + \frac{1}{m} \right| \left(\sup_s \left\| \frac{d\mathcal{F}_{t,m}(\boldsymbol{\theta}_0, \omega)}{d\omega} \right\|_2 + \left\| \frac{d\mathcal{F}(u, \boldsymbol{\theta}_0, \omega)}{d\omega} \right\|_2 \right) \tilde{\mathcal{K}}_1^{(j)},$$

finally replacing $\sum_{s=1}^{T}$ with and integral yields (iii). It is straightfoward to prove (iv) by using (116), we omit the details. $\qquad\square$

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