A note on general quadratic forms of nonstationary stochastic processes

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Abstract

In this paper general quadratic forms of nonstationary, α-mixing time series are considered. Under relatively weak mixing and moment assumptions, asymptotically normality of these forms are derived. The results are applied to the weighted covariance of the Discrete Fourier Transforms of a time series, an important example of a quadratic form.

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1 Introduction

The study of the asymptotic theory of statistics often involves quadratic forms which have the general form

$$Q_T = \frac{1}{T}\sum_{t,\tau=1}^{T} G_{t,\tau} h(X_t, X_\tau), \tag{1}$$

where $\{X_t\}$ is a stochastic process, $h(\cdot)$ is a function and $\{G_{t,\tau}\}$ are weights, which vary according to the application.

In view of its importance in statistics, several authors have studied Q_T for the particular case $h(X_t, X_\tau) = X_t X_\tau$. For example, Mikosch (1990), Götze and Tikhomirov (1999) and the references therein, analyze Q_T under the assumption that $\{X_t\}$ are iid random variables. Kokoszka and Taqqu (1997) and Bhansali, Giraitis, and Kokoszka (2007) relax the independence assumption and establish asymptotic normality of Q_T under the assumption that $\{X_t\}$ is a realisation from stationary, linear time series. Rosenblatt (1984) takes $\{X_t\}$ in a different direction and allows for nonlinear time series, by assuming that $\{X_t\}$ are α-mixing. In particular, under the assumption $\{X_t\}$ is a strictly stationary α-mixing time series and has absolutely summable eight order cumulants, he shows asymptotic normality of Q_T. Gao and Anh (2000) relax the moment assumptions by placing strong geometric mixing assumptions on $\{X_t\}$ and Lin (2009) considers the case $\{X_t\}$ is the sum of stationary α-mixing random variables. It should be mentioned, that there are other methods for measuring dependence. For example, Wu and Shao (2007) show asymptotic normality when $\{X_t\}$ can be written as a function of the innovations and satisfies the assumption of physical dependence. The study of the general quadratic form given in (1) has received less attention. One reason for this is that techniques used in the articles mentioned above cannot be directly applied to (1). A notable exception is Hsing and Wu (2004) who proves several limit theorems for (1) under relatively weak conditions on the underlying process.

The underlying assumption in all the above mentioned results is that the stochastic process $\{X_t\}$ is strictly stationary and ergodic. However, quadratic forms of nonstationary processes also arise in several situations. For example, to test for second order stationarity, the test statistics defined in Dwivedi and Subba Rao (2010) and Jentsch and Subba Rao (2015) are quadratic forms of nonstationary time series under the alternative hypothesis (of nonstationarity). In such a setting it is important to obtain the sampling properties of quadratic forms of nonstationary time series. In a recent paper Zhou (2014) studied generalized quadratic forms of the type

$$W_T = \sum_{k=1}^{T}\sum_{j=1}^{T} w_T(t_k, t_j) h(X_k, X_j) \qquad t_j = j/T, t_k = k/T, \tag{2}$$

where $h(x,y)$ has a semi-multiplicative structure, in the sense that there exists some function $L(x)$ such that $h(x,y) = \widetilde{h}(x,y)L(x)L(y)$ where $\widetilde{h}(x,y)$ is absolutely integrable on $\mathbb{R}^2$ (noting that the property that distinguishes h from $\widetilde{h}$ is that $h(x,y)$ need not satisfy such a condition). He assumes that $\{X_{t,T}\}$ satisfies a piecewise local stationarity assumption (each segment changes slowly over time). However, no closed form expression for the mean and variance of W_T is derived. Therefore despite the results being theoretical very interesting, it is unclear what W_T is estimating (for general functions $w_T(\cdot)$ and $h(\cdot)$).

Our intention in this paper is to analysis quadratic forms which have the form (1), we do not place any locally stationary type assumptions on the process $\{X_{t,T}\}$, thus allowing for other types of nonstationarity such as periodically and quasi-periodically stationary process (see Gladysev (1963), Goodman (1965), Yaglom (1987) Lii and Rosenblatt (2002)), stochastic time-varying processes (introduced in Giraitis, Kapetanios, and Yates (2014)) and parametric stationary process (defined in Azrak and Mélard (2006)). The motivation of this paper is to obtain results that can easily be applied to many realistic problems (the results in this paper have been used in Dwivedi and Subba Rao (2010), Jentsch and Subba Rao (2015) and Subba Rao (2016)).

In Section 2 we show asymptotic normality of the general quadratic form under some moment assumptions and α-mixing of the stochastic process (which includes both nonstationary and nonlinear processes). To understand how quadratic forms of stationary and nonstationary processes may differ, in Section 3, we consider the sampling properties of quadratic forms of locally stationary processes, which are a subclass of nonstationary time series (see Priestley (1965) and Subba Rao (1970) who introduced the model and Dahlhaus (1997) who paved the way for the asymptotic treatment of such models; see also Dahlhaus and Subba Rao (2006), Subba Rao (2006) Zhou and Wu (2009), Vogt (2012) and Dahlhaus (2012)). In Section 4 we prove the central limit theorem by using a classical Bernstein blocking argument. Technical proofs are found in the appendix.

2 The generalized quadratic form

Let us suppose that $\{X_{t,T}\}_{t=1}^{T}$ is a time series which is not necessarily stationary (in order to allow for triangular arrays and various forms of nonstationarity we have subscripted the process with T). We will assume that for all t, $\mathbb{E}(X_{t,T}) = 0$ and $0 < \inf_{t,T} \mathrm{var}(X_{t,T}) \leq \sup_{t,T} \mathrm{var}(X_{t,T}) < \infty$. This condition excludes degenerate cases by ensuring that $\{X_{t,T}\}$ does not converge to a deterministic sequence but always has a bounded variance. We consider the generalized quadratic form

$$Q_T = \frac{1}{T} \sum_{t,\tau=1}^{T} G_{t,\tau,T} h(X_{t,T}, X_{\tau,T}), \tag{3}$$

where $h : \mathbb{R}^2 \to \mathbb{R}$ is any arbitrary function. We assume that $\sup_T |G_{t,\tau,T}| \leq |t-\tau|^{-(2+\eta)} I_{|t-\tau|>0}$, for some $\eta > 0$. This assumption ensures that Q_T is almost surely finite even when $h(X_{t,T}, X_{\tau,T}) = (X_{t,T} + X_{\tau,T})$.

We now state some the conditions required to prove asymptotic normality of Q_T.

Assumption 2.1 *(i) $\{X_{t,T}\}$ is an α-mixing time series such that*

$$\sup_{T} \sup_{1 \leq k \leq T} \sup_{\substack{A \in \sigma(X_{\tau,T};\tau \geq t+k) \\ B \in \sigma(X_{\tau,T};\tau \leq k)}} |P(A \cap B) - P(A)P(B)| \leq \alpha(t),$$

where $\alpha(t)$ are the mixing coefficients which satisfy $\alpha(t) \leq K|t|^{-s}$ for some $s > 0$.

(ii) Let Q_T be defined as in (3). The coefficients satisfy $\sup_T |G_{t,\tau,T}| \leq C|t-\tau|^{-(2+\eta)}$ ($\eta > 0$) and $\frac{c_1}{T} \leq var(Q_T) \leq \frac{c_2}{T}$ (for some $0 < c_1 \leq c_2 < \infty$).

(iii) $s > 3$ *and for some* $r > 2s/(s-3) > 0$, $\sup_{t,\tau,T} \mathbb{E}|h(X_{t,T}, X_{\tau,T})|^r < \infty$.

Several time series, both stationary and nonstationary, satisfy the α-mixing conditions given in Assumption 2.1(i), see, for example, Doukhan (1994), Cline and Pu (1999), Bradley (2007), Fryzlewicz and Subba Rao (2010) and Vogt (2012).

We use the above assumptions to derive the limiting distribution of Q_T.

Theorem 2.1 *Suppose Assumption 2.1 is satisfied. Let* Q_T *be defined in (3) and* $var(Q_T) = V_T$. *Then*

$$V_T^{-1/2} [Q_T - \mathbb{E}(Q_T)] \xrightarrow{D} \mathcal{N}(0,1) \qquad \text{as } T \to \infty.$$

PROOF. See Section 4. □

The above results are for quadratic forms of univariate time series. As multivariate time series arise in several applications we now give an analogous result for multivariate time series. Noting that the proof is identical to the univariate case. Let $\{\underline{X}_{t,T}\}$ be a d-dimensional vector time series and define Q_T

$$Q_T = \frac{1}{T} \sum_{t,\tau=1}^{T} \underline{X}_{t,T}' G_{t,\tau,T} \underline{X}_{\tau,T}, \tag{4}$$

where $G_{t,\tau,T}$ is a $d \times d$ matrix.

Corollary 2.1 *Let us suppose that* $\{\underline{X}_{t,T}\}$ *is a* d*-dimensional vector time series, which is* α*-mixing*

$$\sup_T \sup_{1 \le k \le T} \sup_{\substack{A \in \sigma(\underline{X}_{\tau,T}; \tau \ge t+k) \\ B \in \sigma(\underline{X}_{\tau,T}; \tau \le k)}} |P(A \cap B) - P(A)P(B)| \le \alpha(t),$$

where $\alpha(t)$ *are the mixing coefficients and are such that* $\alpha(t) \le K|t|^{-s}$ *where* $s > 3$. *Suppose there exists some* $r > \frac{2s}{s-3}$, *such that* $\sup_T \sup_{1 \le t \le T} \mathbb{E}\|\underline{X}_{t,T}\|_{2r} < \infty$. *Let* Q_T *be defined as in (4), where the matrices satisfy* $\sup_T |G_{t,\tau,T}| \le K|t-\tau|^{-(2+\eta)}$ $(\eta > 0)$ *(*$|\cdot|$ *is the* ℓ_1 *norm of a matrix). We assume there exists* $0 < c_1 \le c_2 < \infty$ *such that* $c_1/T \le var(Q_T) \le c_2/T$. *Then we have* $V_T^{-1/2}[Q_T - \mathbb{E}(Q_T)] \xrightarrow{D} \mathcal{N}(0,1)$, *where* $V_T = var[Q_T]$.

PROOF. The proof is exactly the same as the proof of Theorem 2.1, hence we omit the details. □

The above results require the fairly strong condition of the rate of decay of the coefficients

$\sup_T |G_{t,\tau,T}| < C|t-\tau|^{-(2+\eta)} I_{|t-\tau|>0}$. We now relax this assumption but assume $h(X_{t,T}, X_{\tau,T}) = X_{t,T}X_{\tau,T}$. Let

$$Q_{T,M} = \frac{1}{T}\sum_{t,\tau=1}^{T} G_{t,\tau,M} X_{t,T} X_{\tau,T}, \tag{5}$$

where $G_{t,\tau,M}$ is a weight function that satisfies $|G_{t,\tau,M}| = 0$ for some $M = T^{\delta}$ where $0 \leq \delta < 1$ (note that we can let $M = CT^{\delta}$, but this makes the notation more cumbersome).

To show asymptotic normality we require the following assumptions.

Assumption 2.2 *(i) Let $Q_{T,M}$ be defined as in (5). Suppose M is a function of T. The coefficients satisfy $\sup_{t,\tau,T}|G_{t,\tau,T}| < \infty$, if $|t-\tau| < M$ then $G_{t,\tau,T} = 0$ and $\frac{c_1 M}{T} \leq var(Q_{T,M}) \leq \frac{c_2 M}{T}$ (for some $0 < c_1 \leq c_2 < \infty$).*

(ii) For $2 \leq j \leq 8$

$$\sup_{1\leq t\leq T} \sum_{1\leq t_2,\ldots,t_j\leq T} \left| cum\left(X_{t,T}, X_{t_2,T}, \ldots, X_{t_j,T}\right)\right| < \infty.$$

Note that sufficient conditions for Assumption 2.2(ii) can be derived in terms of mixing and moment conditions (see Lemma A.3). Furthermore, Rosenblatt (1984) uses similar conditions to show asymptotic normality of the spectral density estimator based on stationary time series.

Theorem 2.2 *Suppose Assumptions 2.1(i) (with $s > 2$) and 2.2 are satisfied. Let $Q_{T,M}$ be defined in (5) with $M = T^{\delta}$ (where $0 \leq \delta < T$) and $var(Q_T) = V_{T,M}$. Then*

$$V_{T,M}^{-1/2}\left[Q_{T,M} - \mathbb{E}(Q_{T,M})\right] \stackrel{D}{\to} \mathcal{N}(0,1) \qquad \text{as } T \to \infty.$$

PROOF. The proof is very similar to the proof of Theorem 2.2 and can be found in Appendix B. $\square$

The above results are for any general nonstationary time series. In the following section, we apply the above results to so called locally stationary processes (a special class of nonstationary time series) and to compare these results to those for stationary processes.

3 Sampling properties of Quadratic forms of locally stationary time series

Our main motivation for considering quadratic forms of nonstationary time series is to obtain the asymptotic sampling properties of the weighted discrete Fourier transform in the case of local stationarity. These typically have the form

$$\widehat{A}_T(r) = \frac{1}{T}\sum_{k=1}^{T}\phi(\omega_k)J_T(\omega_k)\overline{J_T(\omega_{k+r})}, \tag{6}$$

where $J_T(\omega_k) = \frac{1}{\sqrt{2\pi T}}\sum_{t=1}^{T}X_{t,T}\exp(it\omega_k)$, $\omega_k = \frac{2\pi k}{T}$ and $\phi : [0,2\pi] \to \mathbb{R}$ is a periodic function with $\phi(\omega) = \sum_{s\in\mathbb{Z}} G_s \exp(is\omega)$. We assume that the Fourier coefficients $|G_s| \leq K|s|^{-(2+\eta)}I_{|s|>0}$, which implies that $\sup_\omega |\phi''(\omega)| < \infty$.

We now show that $\widehat{A}_T(r)$ can be represented as (3). From the definition of $J_T(\omega_k)$ it is easily seen that

$$\widehat{A}_T(r) = \frac{1}{T}\sum_{t,\tau=1}^{T}X_{t,T}X_{\tau,T}\exp(-i\omega_r\tau)\left(\frac{1}{2\pi T}\sum_{k=1}^{T}\phi(\omega_k)\exp(i\omega_k(t-\tau))\right). \tag{7}$$

Let $G_{t-\tau,T} = \frac{1}{2\pi T}\sum_{k=1}^{T}\phi(\omega_k)\exp(i\omega_k(t-\tau))$ and $G_{t-\tau} = \int_0^{2\pi}\phi(\omega)\exp(-i(t-\tau)\omega)d\omega$. Under the assumption $\sup_\omega |H''(\omega)| < \infty$, then $|G_{t-\tau,T} - G_{t-\tau}| = O(T^{-2})$. Thus we have

$$\widehat{A}_T(r) = A_T(r) + D_{T,1} \quad \text{where } A_T(r) = T^{-1}\sum_{t,\tau=1}^{T}G_{t-\tau}X_{t,T}X_{\tau,T}\exp(-i\omega_r\tau), \tag{8}$$

and $\mathbb{E}|D_{T,1}| = O(T^{-1})$, thus $\widehat{A}_T(r) = A_T(r) + O_p(T^{-1})$. Both $\widehat{A}_T(r)$ and $A_T(r)$ take the form (3), with $G_{t,\tau,T} = \frac{1}{2\pi T}\sum_{k=1}^{T}\phi(\omega_k)\exp(i\omega_k(t-\tau) - i\omega_r\tau)$ and $G_{t,\tau,T} = G_{t-\tau}\exp(-i\omega_r\tau)$ respectively.

Suppose we drop the suffix T and assume $X_{t,T} = X_t$ is a stationary time series with $\sum_r |r| \cdot |\mathrm{cov}(X_0, X_r)| < \infty$. Theorems 2.1 and 2.2, Subba Rao (2016) states that

$$\mathbb{E}[\widehat{A}_T(r)] = \begin{cases} O(T^{-1}) & r \neq 0 \\ \int_0^{2\pi}\phi(\omega)f(\omega)d\omega + O(T^{-1}) & r = 0 \end{cases}, \tag{9}$$

$\mathrm{cov}[\sqrt{T}\widehat{A}_T(r_1), \sqrt{T}\widehat{A}_T(r_2)] = O(T^{-1})$ (for $0 \leq r_1 < r_2 < T/2$) and

$$\begin{aligned} T\mathrm{var}[\widehat{A}_T(r)] &= \frac{1}{2\pi}\int_0^{2\pi} f(\omega)f(\omega+\omega_r)|\left(|\phi(\omega)|^2 + \phi(\omega)\overline{\phi(-\omega-\omega_r)}\right)d\omega + \\ &\quad \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\phi(\omega_1)\overline{\phi(\omega_2)}f_4(\omega_1, -\omega_1-\omega_r, \omega_2)d\omega_1 d\omega_2 + o(1), \end{aligned} \tag{10}$$

where f and f_4 denote the spectral density the fourth order spectral density of $\{X_t\}$, respectively. In time series analysis $\widehat{A}_T(0)$ (the weighted average of the periodogram) is a commonly used statistic. However, $\widehat{A}_T(r)$ for $r > 0$ is also of interest and below we give a few examples where it is of interest.

Example 3.1 *(i) For $1 \leq r \leq T/2$, the expectation of $\widehat{A}_T(r)$ for stationary and nonstationary processes is very different (compare (9) with (19) below). Dwivedi and Subba Rao (2010) and Jentsch and Subba Rao (2015) use this dichtomy to test for second order stationarity of a time series.*

(ii) As mentioned above, in the case that the time series is stationary, several statistics of interest can be written in the form $\widehat{A}_T(0)$. However as can be seen from (10) the variance of $\widehat{A}_T(0)$ is unwieldy and difficult to directly estimate. One method for estimating this variance using $\{\widehat{A}_T(r)\}$ is proposed in Subba Rao (2016); from (9) we see that for $r << T$ that the variance of $\widehat{A}_T(0)$ is close to the variance of $\widehat{A}_T(r)$. Based on this observation, we use $\{\widehat{A}_T(r); 1 \leq r \leq M\}$ as the 'orthogonal sample' to $\widehat{A}_T(0)$ to estimate the variance of $\widehat{A}_T(0)$.

As mentioned in Example 3.1 $\widehat{A}_T(r)$ exhibits very different behaviours under stationarity and local stationarity. In the remainder of this section we study the behavious of $\widehat{A}_T(r)$ under local stationarity. To do so, we define precisely what is meant by local stationarity.

Definition 3.1 (Locally stationary time series) *A time series $\{X_{t,T}\}_{t=1}^T$ is called locally stationary if there exists a series of "sister" processes $\{X_t(u)\}$, where for fixed $u \in [0,1]$, $\{X_t(u); t \in \mathbb{Z}\}$ is strictly stationary and for $1 \leq t \leq T$ and $u \in [0,1]$*

$$\left|X_{t,T} - X_t(u)\right| \leq \left(\left|\frac{t}{T} - u\right| + \frac{1}{T}\right) V_t, \tag{11}$$

where $\{V_t\}$ is a positive time series that does not depend on T or u and $\sup_t \mathbb{E}|V_t| < \infty$.

This definition is commonly used, cf. Dahlhaus and Subba Rao (2006), Subba Rao (2006), Vogt (2012) and Dahlhaus (2012). Statistical inference is done using rescaled time, where $\{X_{t,T}\}_{t=1}^T$ is the Tth row on a triangular array and as $T \to \infty$ we move down the triangular array.

3.1 Properties of the Fourier transforms

In order to study $\widehat{A}_T(r)$ we derive some expressions for the cumulants of the DFT. This extends the results on cumulants of DFTs derived in Brillinger (1981) to locally stationary time series. In the case the time series is stationary, Brillinger (1981) makes the assumption that the nth order cumulants of the time series satisfies $\sum_{t_1,\ldots,t_{n-1}\in\mathbb{Z}}(1+|t_j|)|\kappa_n(t_1,\ldots,t_{n-1})| < \infty$

where $\kappa_n(t_1,\ldots,t_{n-1}) = \mathrm{cum}(X_0, X_{t_1},\ldots,X_{t_{n-1}})$. If this condition holds, then Brillinger (1981) Theorem 4.3.2 states that

$$\begin{aligned} &\mathrm{cum}\left[J_T(\omega_{j_1}),\ldots,J_T(\omega_{j_n})\right] \\ &= \left\{ \begin{array}{cc} O(\frac{1}{T^{n/2}}) & \text{if} \quad \sum_{s=1}^{n} j_s \neq T\mathbb{Z} \\ \left(\frac{2\pi}{T}\right)^{n/2-1} f_n(\omega_{j_1},\ldots,\omega_{j_{n-1}}) + O(\frac{1}{T^{n/2}}) & \text{otherwise} \end{array} \right. \end{aligned} \tag{12}$$

where $\omega_{j_k} = \frac{2\pi j_k}{T}$, $f_n(\omega_1,\ldots,\omega_{n-1}) = \frac{1}{(2\pi)^{n-1}}\sum_{t_1,\ldots,t_{n-1}=-\infty}^{\infty}\kappa_n(t_1,\ldots,t_{n-1})\exp(i\sum_{j=1}^{n-1}t_j\omega_j)$ denotes the nth order spectral density of the stationary process $\{X_t\}$. In the case that $\{X_t\}$ is nonstationary the above result does not hold true, however we show that for locally stationary processes some succinct expressions can still be obtained.

We recall from Definition 3.1 if $u \in [0,1]$ is fixed then $\{X_t(u)\}_t$ is a stationary process. Therefore its corresponding spectral density and higher order spectral densities can be defined. We define the covariance and nth order cumulant of the process as $c(u,r) = \mathrm{cov}[X_t(u), X_{t+r}(u)]$ and $\kappa_n(u; t_2 - t_1,\ldots,t_n - t_1) = \mathrm{cum}\big[X_{t_1}(u),\ldots,X_{t_n}(u)\big]$ (noting that due to stationarity these terms do not depend on t or t_1). Using this notation we define the spectral density $f(u;\omega) = \frac{1}{2\pi}\sum_{r=-\infty}^{\infty} c(u;r)\exp(ir\omega)$ and the nth order spectral spectral density

$$f(u;\omega_1,\ldots,\omega_{n-1}) = \frac{1}{(2\pi)^{n-1}}\sum_{t_1,\ldots,t_{n-1}=-\infty}^{\infty}\kappa_n(u;t_1,\ldots,t_{n-1})\exp(i\sum_{j=1}^{n-1}t_j\omega_j), \tag{13}$$

where $\omega_1,\ldots,\omega_{n-1} \in [0,2\pi]$.

Despite the process being locally stationary in order to obtain an analytic expression for the mean and variance of $\widehat{A}_T(r)$ we require the following smoothness conditions on the cumulants.

Assumption 3.1 (Lipschitz conditions on the cumulants) *Let $\{X_{t,T}\}$ be a locally stationary process and $\{X_t(u)\}$ be its stationary approximation (as defined in (11)).*

Further, assume there exists processes $E_{t,T}$ and $E_t(u,v)$ such that

$$\text{(i) } X_{t,T} = X_t\left(\frac{t}{T}\right) + E_{t,T} \qquad X_t(u) = X_t(v) + (u-v)E_t(u,v).$$

For all $u_1,\ldots,u_n, v \in [0,1]$ the nth order joint cumulants of $\{X_{t,T}\}$, $\{X_t(u_i)\}$, $\{E_{t,T}\}$ and $\{E_t(u_i,v)\}$ satisfy

$$\text{(ii) } \left|cum\left[X_{t_1,T}, X_{t_2,T},\ldots,X_{t_{n-1},T}, E_{t_n,T}\right]\right| \leq T^{-1}\nu_n(t_2-t_1,\ldots,t_n-t_2)$$

$$\text{(iii) } \sup_{u_1,\ldots,u_n,v\in[0,1]}\left|cum\left[X_{t_1}(u_1),\ldots,X_{t_{n-1}}(u_{n-1}), E_{t_n}(u_n,v)\right]\right| \leq \nu_n(t_2-t_1,\ldots,t_n-t_2)$$

(iv) $\sup_u |cum[X_{t_1}(u),\ldots,X_{t_n}(u)]| \leq \nu_n(t_2-t_1,\ldots,t_n-t_1)$

$$(v)\ \sup_u \left|\frac{\partial cum[X_{t_1}(u),\ldots,X_{t_n}(u)]}{\partial u}\right| \leq \nu_n(t_2-t_1,\ldots,t_n-t_1)$$

for some positive sequence $\{\nu_n(t_1,\ldots,t_{n-1}); t_1,\ldots,t_{n-1} \in \mathbb{Z}\}$ *which is independent of* T *and satisfies* $\sum_{t_1,\ldots,t_{n-1}\in\mathbb{Z}}(1+|t_j|)\nu_n(t_1,\ldots,t_{n-1}) < \infty$ *(for all* $1 \leq j \leq (n-1)$*).*

Remark 3.1 *The above assumptions are technical, however if the nth moment of a process exists, then it can be shown that several locally stationary time series satisfy these assumptions (see for example, Dahlhaus and Polonik (2006) for the time varying MA(*∞*) model and Dahlhaus and Subba Rao (2006) for the time-varying ARCH process).*

These conditions are also satisfied if the following mixing conditions hold. Define the sigma-algebras $\mathcal{F}^k_{-\infty,T} = \sigma(X_{s,T}, \{X_s(u_i), E_s(u_i,u_j); 1 \leq i < j \leq n\}, E_{s,T}; s \leq k)$ *and* $\mathcal{F}^{\infty}_{t+k,T} = \sigma(X_{s,T}, \{X_s(u_i), E_s(u_i,u_j); 1 \leq i < j \leq n\}, E_{s,T}; s \geq (t+k))$*. Suppose that*

$$\sup_T \sup_{1\leq k\leq T} \sup_{\substack{A\in\mathcal{F}^k_{-\infty,T}\\ B\in\mathcal{F}^{\infty}_{t+k,T}}} |P(A\cap B) - P(A)P(B)| \leq \alpha(t),$$

where for $t \neq 0$ $\alpha(t) \leq C|t|^{-s}$*. Let* $t_1 \leq t_2 \leq \ldots < t_n$ *and* $V_{t_s} \in \{X_{t_s,T}, \{X_{t_s}(u_i), E_{t_s}(u_i,u_j); 1 \leq i < j \leq n\}, E_{t_s,T}\}$*. Let* $\|X\|_r = \mathbb{E}[|X^r|]^{1/2}$*. If there exists an* $r > n/(s+1-2n)$ *such that* $\sup_{t,T}\|X_{t,T}\|_r < \infty$, $\sup_u \|X_t(u)\|_r < \infty$, $\sup_{1\leq t\leq T}\|E_{t,T}\|_r \leq CT^{-1}$ *and* $\sup_{u,v\in[0,1]}\|E_t(u,v)\|_r < \infty$*. Then by using Statulevicius and Jakimavicius (1988), Theorem 3, part (2), we have*

$$|cum[V_{t_1}, V_{t_2},\ldots,V_{t_n}]| \leq C\|V_{t_1}\|_r \prod_{i=2}^{n}\|V_{t_i}\|_r \prod_{i=2}^{k}\alpha(t_i - t_{i-1})^{\frac{1-k/r}{k-1}}$$

which immediately implies that Assumption 3.1(i-iv) is satisfied.

Lemma 3.1 *Let* $\{X_{t,T}\}$ *be a locally stationary process which satisfies Assumption 3.1 and* $f_n(\cdot)$ *be defined as in (13).*

(i) If Assumption 3.1(i,ii,iii) holds. Then

$$\left|cum[X_{t_1,T},\ldots,X_{t_n,T}] - \kappa_n\left(\frac{t_1}{T}, t_2-t_1,\ldots,t_n-t_1\right)\right| \leq \left(\frac{n}{T} + \sum_{j=2}^{n}\frac{|t_j-t_1|}{T}\right)\nu_n(t_2-t_1,\ldots,t_n-t_1) \quad (14)$$

(ii) If Assumption 3.1(iv) holds. Then $\sup_{u\in[0,1],\omega_1,\ldots,\omega_{n-1}\in[0,2\pi]} |f_n(u;\omega_1,\ldots,\omega_{n-1})| < \infty$*.*

(iii) If Assumption 3.1(v) holds. Then

$$\sup_{u\in[0,1],\omega_1,\ldots,\omega_{n-1}\in[0,2\pi]}\left|\frac{\partial f_n(u;\omega_1,\ldots,\omega_{n-1})}{\partial u}\right| < \infty. \tag{15}$$

PROOF. In Appendix A.1. □

Define the frequency dependent Fourier coefficients of the local nth order spectral density as

$$F_n(r;\omega_1,\ldots,\omega_{n-1}) = \int_0^1 f_n(u;\omega_1,\ldots,\omega_{n-1})\exp(i2\pi ru)du. \tag{16}$$

We use the following lemma to derive an expression for the cumulants of the DFT.

Lemma 3.2 *Suppose Assumption 3.1 holds and $F_n(\cdot)$ be defined as in (16). Then*

$$\sup_{\omega_1,\ldots,\omega_{n-1}\in[0,2\pi]}|F_n(r;\omega_1,\ldots,\omega_{n-1})| \leq C \sup_{u\in[0,1],\omega_1,\ldots,\omega_{n-1}\in[0,2\pi]}\left|\frac{\partial f_n(u;\omega_1,\ldots,\omega_{n-1})}{\partial u}\right|\frac{1}{|r|} \tag{17}$$

were C is a finite constant that does not depend on r.

PROOF. In Appendix A.1. □

Remark 3.2 **($F_n(r;\cdot)$)** *If $\{X_t\}$ is a stationary time series (we have dropped the T suffix), then $F_n(0,\omega_1,\ldots,\omega_{n-1}) = f_n(\omega_1,\ldots,\omega_{n-1})$ ($f_n(\cdot)$ is the nth order spectral density function) for $r \neq 0$ $F_n(r;\cdot) = 0$.*

Using the following lemma, we derive a generalisation of Brillinger (1981), Theorem 4.3.2.

Lemma 3.3 *Suppose Assumption 3.1 holds and $F_n(\cdot)$ be defined as in (16). Then*

$$cum\left[J_T(\omega_{j_1}),\ldots,J_T(\omega_{j_n})\right] = \frac{(2\pi)^{(n/2)-1}}{T^{(n/2)-1}}F_n\left[\sum_{k=1}^{n} j_k;\omega_{j_2},\ldots,\omega_{j_n}\right] + O\left(T^{-n/2}\right). \tag{18}$$

PROOF. In Appendix A.1. □

Comparing the above result to the cumulants of DFTs of stationary time series in (12) leads to some interesting conclusions. In the case that $\{X_{t,T}\}$ is second order stationary we have $\mathrm{cov}[J_T(\omega_{k_1}), J_T(\omega_{k_2})] = o(1)$, whereas if $\{X_{t,T}\}$ were locally stationary there is an 'ordering' in correlation between the DFTs. More precisely, $\left|\mathrm{cov}(J_T(\omega_{k_1}), J_T(\omega_{k_2}))\right| \leq C|k_1 - k_2|^{-1}$, where C is a finite constant. Hence the correlation between the DFTs decay the further apart the frequencies, with the rate of decay resembling that of a long memory time series. If $\{X_{t,T}\}$ were nonstationary but not locally stationary this is not necessarily true; for example in the case of periodically stationary time series of order P, there is "significant correlation" between the DFTs when the lags are separated by multiples of T/P. It is possible that these differing behaviours in the correlations of the DFT, could be used as a means of discriminating local stationarity from general nonstationary behaviour.

3.2 Sampling properties of $\widehat{A}_T(r)$ and W_T

In this section consider the sampling properties $\widehat{A}_T(r)$ and W_T (defined in (6) and (2) respectively) under the assumption of local stationarity (see Definition 3.1).

We first consider $\widehat{A}_T(r)$ and later show that W_T shares similar properties to $\widehat{A}_T(0)$. We will show that Lemma 3.3 can easily be applied to obtain the first and second moment of Q_T. However, to use Theorem 2.1 to prove asymptotic normality we need to use the quadratic form representation of Q_T given in (7).

Theorem 3.1 *$\{X_{t,T}\}$ is a zero mean locally stationary time series which satisfies Assumption 3.1 for $n = 2$ and 4. Let $F_n(\cdot)$ be defined as in (16) and $\widehat{A}_T(r)$ be defined as in (6) where $\phi(\omega)$ has a bounded second derivative. Then*

$$\mathbb{E}[\widehat{A}_T(r)] = \frac{1}{2\pi}\int_0^{2\pi}\phi(\omega)F_2(-r,\omega)d\omega + O\left(T^{-1}\right) \tag{19}$$

and cov$[\sqrt{T}\widehat{A}_T(r_1), \sqrt{T}\widehat{A}_T(r_2)] = V_{r_1,r_2} + O(T^{-1}\log T)$ *where*

$$\begin{aligned} & V_{r_1,r_2} \\ = & \sum_{s\in\mathbb{Z}}\frac{1}{2\pi}\int_0^{2\pi}\Big[|\phi(\omega)|^2 F_2(s,\omega)F_2(-s-r_1+r_2,-\omega-\omega_{r_1}) \\ & +\phi(\omega)\overline{\phi(-\omega)}F_2(s+r_2,\omega)F_2(-s-r_1,-\omega-\omega_{r_1})\Big]d\omega \\ & \quad +\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\phi(\omega_1)\overline{\phi(\omega_2)}F_4(-r_1+r_2,\omega_1,-\omega_1-\omega_{r_1},-\omega_2)d\omega_1 d\omega_2. \end{aligned} \tag{20}$$

If, in addition, $\{X_{t,T}\}$ satisfies Assumption 2.1(i,iii) (where there exists an $r > 2s/(s-3)$ such that $\sup_{t,T}\|X_{t,T}\|_{2r} < \infty$). Then

$$V_{r,r}^{-1/2}\left(\widehat{A}_T(r) - \frac{1}{2\pi}\int_0^{2\pi}\phi(\omega)F_2(-r,\omega)d\omega\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1). \tag{21}$$

PROOF. In Appendix A.1. □

We now compare Lemma 3.1 with the mean and variance of $\widehat{A}_T(r)$ when the process is stationary (see (9) and (10)). We observe in contrast to the case that $\{X_t\}$ is second order stationary, if $r \neq 0$ then $\lim_{T\to\infty} \mathbb{E}[\widehat{A}_T(r)]$ is not necessarily non-zero. Comparing the variance (10) with (20) we observe that an important difference is that asymptotically there are significant correlations between $\{\widehat{A}_T(r); 0 \leq r \leq T/2-1\}$, which is not the case when the process is stationary.

Let us return to the generalized quadratic form defined in (2). Zhou (2014), Corollary 1 proves asymptotic normality of W_T, but no expression for the mean and variance is given. We bridge this gap, and use Theorem 3.1 to obtain an expression for the mean and variance of the generalized quadratic form defined in (2). In particular we focus on

$$W_T = T^{-1}\sum_{\tau=1}^{T}\sum_{t=1}^{T} G_{t-\tau} h(X_{t,T}, X_{\tau,T}) \tag{22}$$

where $\{X_{t,T}\}$ is a locally stationary time series that satisfies (11), $|G_{t-\tau}| \leq K|t-\tau|^{-(2+\eta)}$ ($\eta > 0$). We do not assume that $h(\cdot)$ satisfies any integrability condition, however, we do assume that there exists a function $L(\cdot)$ such that $h(X,Y) = \widetilde{h}(X,Y)L(X)L(Y)$ and $\widetilde{h}$ has the representation

$$\widetilde{h}(X,Y) = \int_{\mathbb{R}^2} \Gamma(x,y)\exp(ixY - iyY)dxdy \text{ with } \int_{\mathbb{R}^2}(1+|x|+|y|)|\Gamma(x,y)|dxdy < \infty. \tag{23}$$

Thus $h(\cdot)$ is not necessarily integrable, but $\widetilde{h}(\cdot)$ is. Basic algebra gives

$$W_T = \int_{\mathbb{R}^2} \Gamma(x,y) W_T(x,y) + D_{T,2} \tag{24}$$

with $\mathbb{E}|D_{T,2}| = O((\log T)T^{-1})$ and

$$W_T(x,y) = \frac{2\pi}{T}\sum_{k=1}^{T} \phi(\omega_k) J_T(x;\omega_k)\overline{J_T(y;\omega_k)},$$

where $\phi(\omega) = \frac{1}{2\pi}\sum_{s\in\mathbb{Z}} G_s \exp(-is\omega)$ (noting that $G_s = \int_0^{2\pi}\phi(\omega)\exp(is\omega)d\omega$ and $\sup_\omega |H'(\omega)| < \infty$) and

$$J_T(x;\omega_k) = \frac{1}{\sqrt{2\pi T}}\sum_{t=1}^{T} L(X_{t,T})\exp(ixX_{t,T} + it\omega_k).$$

Our objective is to apply Theorem 3.1 to (24) to obtain an expression for the mean and variance of W_T. To do so we require the following assumption on the function $L(\cdot)$ and process $\{X_{t,T}\}$ and $\{X_t(u)\}$

$$|L(X_{t,T}) - L(X_t(u))| \leq V_t|X_{t,T} - X_t(u)| \qquad |L(X_t(u)) - L(X_t(v))| \leq V_t|X_t(u) - X_t(v)| \tag{25}$$

where the process $\{V_t\}$ is independent T, u and v and $\sup_t \mathbb{E}|V_t| < \infty$. Observe that this condition is satisfied if $L(\cdot)$ is a Lipschitz continuous function. Let

$$Y_{x,t,T} = L(X_{t,T})\exp(ixX_{t,T}) \text{ and } Y_{x,t}(u) = L(X_t(u))\exp(ixX_t(u)). \tag{26}$$

If $\{X_{t,T}\}$ is locally stationary process as defined in Definition 3.1 and $L(\cdot)$ satisfies (25). Then

$$\begin{aligned} |Y_{x,t,T} - Y_{x,t}(u)| \leq\ & |\{L(X_{t,T}) - L(X_t(u))\}\exp(ixX_{t,T})| + \\ & |\{\exp(ixX_{t,T}) - \exp(ixX_t(u))\}L(X_t(u))| \leq \left(\left|\frac{t}{T} - u\right| + \frac{1}{T}\right)(1+|x|)\,V_{2,t} \end{aligned}$$

where $Y_{x,t}(u) = L(X_t(u))\exp(ixX_t(u))$ and $\mathbb{E}|V_{2,t}| < \infty$.

In order to obtain an expression for the mean and variance of W_T we define the moments and cumulants of the stationary process. Let $\mu_x(u) = \mathbb{E}[Y_{x,0}(u)]$, $c_{x,y}(u;r) = \text{cov}[Y_{x,0}(u), Y_{y,r}(u)]$ and for $n = 3,4$ $\kappa_{n,x_1,\ldots,x_n}(u;t_1,\ldots,t_{n-1}) = \text{cum}[Y_{x_1,0}(u), Y_{x_2,t_1}(u), \ldots, Y_{x_n,t_{n-1}}(u)]$. We now state some cumulant assumptions on $Y_{x,t,T}$ and $Y_{x,t}(u)$ that are analogous to Assumption 3.1.

Assumption 3.2 *Let $\{X_{t,T}\}$ be a locally stationary process and $\{X_t(u)\}$ be its stationary approximation (as defined in (11)). Let $Y_{x,t,T}$ and $Y_{x,t}(u)$ be defined as in (26).*

For $x \in \mathbb{R}$ assume there exists processes $E_{x,t,T}$ and $E_{x,t}(u,v)$ such that

$$Y_{x,t,T} = Y_{x,t,T}\left(\frac{t}{T}\right) + E_{x,t,T} \qquad Y_{x,t}(u) = Y_{x,t}(v) + (u-v)E_{x,t}(u,v).$$

The nth order joint cumulants of $\{Y_{x,t,T}\}$, $\{Y_{x,t}(u)\}$, $\{E_{x,t,T}\}$ and $\{E_{x,t}(u,v)\}$ satisfy

(ii) $\left|cum\left[Y_{x_1,t_1,T}, X_{x_2,t_2,T}, \ldots, X_{x_{n-1},t_{n-1},T}, E_{x_n,t_n,T}\right]\right| \leq T^{-1}(1+|x_n|)\nu_n(t_2-t_1,\ldots,t_n-t_2)$

(iii) $\sup_{u_1,\ldots,u_n,v\in[0,1]}\left|cum\left[Y_{x_1,t_1}(u_1),\ldots,Y_{x_{n-1},t_{n-1}}(u_{n-1}), E_{x_n,t_n}(u_n,v)\right]\right| \leq (1+|x_n|)\nu_n(t_2-t_1,\ldots,t_n-t_2)$

(iv) $\sup_u |cum[Y_{x_1,t_1}(u),\ldots,Y_{x_n,t_n}(u)]| \leq \nu_n(t_2-t_1,\ldots,t_n-t_1)$

(v) $\sup_u \left|\frac{\partial cum[Y_{x_1,t_1}(u),\ldots,Y_{x_n,t_n}(u)]}{\partial u}\right| \leq \left[1+\sum_{i=1}^n |x_i|\right]\nu_n(t_2-t_1,\ldots,t_n-t_1)$

for some positive sequence $\{\nu_n(t_1,\ldots,t_{n-1}); t_1,\ldots,t_{n-1} \in \mathbb{Z}\}$ which is independent of T and satisfies $\sum_{t_1,\ldots,t_{n-1}\in\mathbb{Z}}(1+|t_j|)\nu_n(t_1,\ldots,t_{n-1}) < \infty$ (for all $1 \leq j \leq (n-1)$).

It is worth noting that Remark 3.1 also applies to the above set of assumptions.

We define the joint spectral densities as

$$\begin{aligned} f_{2,x,y}(u;\omega) &= \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} c_{x,y}(u;k)e^{ik\omega} \\ f_{n,x_1,x_2,\ldots,x_n}(u,\omega_1,\omega_2,\ldots,\omega_{n-1}) &= \frac{1}{(2\pi)^{n-1}}\sum_{t_1,\ldots,t_{n-1}\in\mathbb{Z}} \kappa_{n,x_1,\ldots,x_n}(u;t_1,\ldots,t_{n-1})e^{i\sum_{j=1}^{n-1}t_j\omega_j} \end{aligned} \tag{27}$$

and the analogous version of F_n (defined in (28)) as

$$\begin{aligned} F_{1,x}(k) &= \int_0^1 \mu_x(u)\exp(i2\pi ku)du \\ F_{n,x_1,\ldots,x_n}(k;\omega_1,\ldots,\omega_{n-1}) &= \int_0^1 f_{n,x_1,x_2,\ldots,x_n}(u,\omega_1,\omega_2,\ldots,\omega_{n-1})\exp(i2\pi ku)du. \end{aligned} \tag{28}$$

Theorem 3.2 *Suppose $\{X_{t,T}\}$ is a locally stationary process which satisfies (11). Let W_T be defined as in (22) where $h(x,y) = \widetilde{h}(x,y)L(x)L(y)$, and $\widetilde{h}$ and L satisfy (23) and Assumption 3.2 for $n = 2,3$ and 4. Then*

$$\mathbb{E}[W_T] = \int_{\mathbb{R}^2}\Gamma(x,y)E(x,y)dxdy + O(T^{-1}\log T) \tag{29}$$

$$var[\sqrt{T}W_T] = \int_{\mathbb{R}^4}\Gamma(x_1,y_1)\overline{\Gamma(x_2,y_2)}v(x_1,y_1,x_2,y_2)dx_1dy_1dx_2dy_2 + O(T^{-1}\log T) \tag{30}$$

where

$$E(x,y) = \int_0^{2\pi}\phi(\omega)F_{2,x,y}(\omega)d\omega + \phi(0)\sum_{k=0}^{\infty}F_{1,x}(k)F_{1,y}(-k)$$

and

$$\begin{aligned} & v(x_1,y_1,x_2,y_2) \\ = & \sum_{s\in\mathbb{Z}}\int_0^{2\pi}\Big\{|\phi(\omega)|^2F_{2,x_1,x_2}(s,\omega)F_{2,y_1,y_2}(-s,-\omega) + \phi(\omega)\overline{\phi(-\omega)}F_{2,x_1,y_2}(s,\omega)F_{2,y_1,x_2}(-s,-\omega)\Big\}d\omega \\ & + \int_0^{2\pi}\int_0^{2\pi}\phi(\omega_1)\overline{\phi(\omega_2)}F_{4,x_1,y_1,x_2,y_2}(0,\omega_1,-\omega_1,-\omega_2)d\omega_1 d\omega_2 \\ & + \sum_{s=0}^{\infty}\int_0^{2\pi}\phi(0)\overline{\phi(\omega)}\left\{F_{1,x_1}(s)F_{3,y_1,x_2,y_2}(s;0,-\omega) + F_{1,y_1}(-s)F_{3,x_1,x_2,y_2}(-s;0,-\omega)\right\}d\omega \\ & \sum_{s=0}^{\infty}\int_0^{2\pi}\phi(\omega)\overline{\phi(0)}\left\{F_{1,x_2}(-s)F_{3,x_1,y_1,y_2}(s;\omega,0) + F_{1,y_2}(s)F_{3,x_1,y_1,x_2}(-s;\omega,0)\right\}d\omega. \end{aligned}$$

PROOF. See the Appendix. □

Comparing Theorem 3.1 with Lemma 3.2 we observe that the mean and variance of $\widehat{A}_T(0)$ and W_T are similar. There are two main differences (a) the mean and variance of Lemma 3.1 are in terms of the spectrums of $\{X_t(u)\}$ whereas from Theorem 3.2 we observe that the equivalent quantities are in terms of the spectrums of $\{L(X_t(u))\exp(ixX_t(u))\}$ (b) because the mean of $L(X_t(u))\exp(ixX_t(u))$ may not be zero, Thereom 3.2 includes additional terms.

If $\{X_t\}$ is a stationary time series, then for $k \neq 0$ $F_{n,x_1,\ldots,x_{n-1}}(k;\omega_1,\ldots,\omega_{n-1}) = 0$, which leads to substantial simplifications for the mean and variance of W_T.

4 Proof of Theorem 2.1

In order to prove Theorem we will use a Bernstein blocking argument, which is achieved through a series of approximations. The first is to truncate the support of the weight function $G_{t,\tau,T}$. Let $\widehat{G}_{t,\tau,\widehat{M}} = G_{t,\tau,T} I_{|t-\tau|\leq \widehat{M}}$ (hence $\widehat{G}_{t,\tau,\widehat{M}} = 0$ if $|t-\tau| > \widehat{M}$). Then we have

$$Q_T = \widehat{Q}_{T,\widehat{M}} + D_3$$

where

$$\widehat{Q}_{T,\widehat{M}} = \frac{1}{T}\sum_{t,\tau=1}^{T} \widehat{G}_{t,\tau,\widehat{M}} h(X_{t,T}, X_{\tau,T}) \text{ and } D_3 = \frac{1}{T}\sum_{|t-\tau|>\widehat{M}} G_{t,\tau,T} h(X_{t,T}, X_{\tau,T}).$$

Using that $|G_{t,\tau,T}| \leq C|t-\tau|^{-(2+\eta)} I_{|t-\tau|>0}$ and $\sup_{t,T} \mathbb{E}|g(X_{t,T}, X_{\tau,T})|^2 < \infty$ it is straightforward to show

$$\|D_3\|_2 = O(\widehat{M}^{-(1+\eta)}).$$

From now onwards we set $\widehat{M} = T^{1/2}$. Using this we obtain the approximation

$$Q_T = \widehat{Q}_{T,1} + O_p(T^{-1/2-\eta}), \tag{31}$$

where $\widehat{Q}_{T,1} = \widehat{Q}_{T,T^{1/2}}$. To facilitate the classical Bernstein blocking argument we define

$$Y_{t,T} = \widehat{G}_{t,t,T^{1/2}} h(X_{t,T}, X_{t,T}) + \sum_{\tau=1}^{t-1} [\widehat{G}_{t,\tau,T^{1/2}} h(X_{t,T}, X_{\tau,T}) + \widehat{G}_{\tau,t,T^{1/2}} h(X_{\tau,T}, X_{t,T})]. \tag{32}$$

Thus it is clear that $\widehat{Q}_{T,1} = \sum_{t=1}^{T} Y_{t,T}$ and

$$Q_T = \frac{1}{T}\sum_{t=1}^{T} Y_{t,T} + O_p(T^{-1/2-\eta}). \tag{33}$$

Define the block sums

$$U_{j,T} = \sum_{t=jr_T+1}^{jr_T+p_T} [Y_{t,T} - \mathbb{E}(Y_{t,T})] \quad \text{and} \quad V_{j,T} = \sum_{t=jr_T+p_T+1}^{(j+1)r_T} [Y_{t,T} - \mathbb{E}(Y_{t,T})], \tag{34}$$

where $r_T = (p_T + q_T)$. By setting $p_T >> q_T$, $U_{j,T}$ and $V_{j,T}$ are "big" and "small" blocks respectively. Using this notation we have

$$\widehat{Q}_{T,M} - \mathbb{E}[\widehat{Q}_{T,M}] \;=\; \frac{1}{T}\sum_{j=1}^{k_T}(U_{j,T}+V_{j,T}) \text{ and } Q_T - \mathbb{E}[Q_T] = \frac{1}{T}\sum_{j=1}^{k_T}(U_{j,T}+V_{j,T}) + D_3, \tag{35}$$

where $k_T = T/(p_T + q_T)$.

To prove the result we let p_T and q_T but such that $q_T - T^{1/2} \to \infty$, $q_T/(p_T + q_T) \to 0$, $\frac{T}{(q_T - T^{1/2})^{s-1}} \to 0$ (where s denotes the mixing rate of $\{X_{t,T}\}$ defined in Assumption 2.1(i)) and $p_T/T \to 0$ as $T \to \infty$. In the next few lemmas we use the notation

$$\Gamma(r) = \sum_{j=1}^{\infty} j^{-r}. \tag{36}$$

Lemma 4.1 *Suppose Assumption 2.1 holds. Let Q_T and $\{U_{j,T}\}$ be defined as in (3) and (34) respectively. Then*

$$\sqrt{T}\,(Q_T - \mathbb{E}[Q_T]) = \frac{1}{\sqrt{T}}\sum_{j=1}^{k_T} U_{j,T} + D_4, \tag{37}$$

where $\|D_4\|_2 = O\left(T^{-\eta} + (\frac{q_T}{p_T+q_T})^{1/2}\right)$

PROOF. It is clear from (35) that $D_4 = \sqrt{T}D_3 + \frac{1}{\sqrt{T}}\sum_{j=1}^{k_T} V_{j,T}$. We know from (31) that $\sqrt{T}\|D_3\|_2 = O(T^{-\eta})$ therefore we need to bound $\left\|\frac{1}{\sqrt{T}}\sum_{j=1}^{k_T} V_{j,T}\right\|_2$. Since $\mathbb{E}[V_{j,T}] = 0$ the above is simply the variance

$$\left\|\frac{1}{\sqrt{T}}\sum_{j=1}^{k_T} V_{j,T}\right\|_2^2 = \frac{1}{T}\sum_{j=1}^{k_T}\text{var}[V_{j,T}] + \frac{2}{T}\sum_{j_1=1}^{k_T}\sum_{j_2=j_1+1}^{k_T}\text{cov}\left(V_{j_1,T}, V_{j_2,T}\right).$$

By using Lemma A.7 we can bound the above

$$\left\| \frac{1}{\sqrt{T}} \sum_{j=1}^{k_T} V_{j,T} \right\|_2^2 \leq C T^{-1} \sup_{1 \leq j \leq k_T} \|V_{j,T}\|_{2+\delta}^2 \, k_T \left[1 + \sum_{j=1}^{k_T} \alpha \left(j p_T - T^{1/2} \right)^{1-2/(2+\delta)} \right].$$

Using Lemma A.5 we have

$$\sup_{1 \leq j \leq k_T} \|V_{j,T}\|_{2+\delta} \leq C q_T^{1/2} \sup_{t,\tau,T} \|g(X_{t,T}, X_{\tau,T})\|_r \left[\Gamma(1+\eta) + \Gamma\left(\frac{s}{2+\delta} - \frac{s}{r} \right) \right].$$

To reduce notation let $\sup_{t,\tau,T} \|g(X_{t,T}, X_{\tau,T})\|_r^2 = |g|_r$, this gives

$$\begin{aligned}
\left\| \frac{1}{\sqrt{T}} \sum_{j=1}^{k_T} V_{j,T} \right\|_2^2 &\leq \frac{C k_T q_T}{T} |g|_r \left[\Gamma(1+\eta) + \Gamma\left(\frac{s}{2+\delta} - \frac{s}{r} \right) \right]^2 \left(1 + \sum_{j=1}^{k_T} \frac{1}{[j(p_T - T^{1/2})]^{s(1-2/(2+\delta))}} \right) \\
&\leq \frac{C k_T q_T}{T} |g|_r \left[\Gamma(1+\eta) + \Gamma\left(\frac{s}{2+\delta} - \frac{s}{r} \right) \right]^2 \Gamma\left(s \left[1 - \frac{2}{2+\delta} \right] \right)
\end{aligned}$$

In order that $\Gamma\left(s\left[1-\frac{2}{2+\delta}\right]\right)$ *and* $\Gamma\left(\frac{s}{2+\delta}-\frac{s}{r}\right)$ are finite for some δ, we require that

$$\frac{1}{2+\delta} - \frac{1}{r} > \frac{1}{s} \text{ and } 1 - \frac{2}{2+\delta} > \frac{1}{s}.$$

This is equivalent to

$$\frac{1}{s} + \frac{1}{r} < \frac{1}{2+\delta} < \frac{1}{2}\left(\frac{s-1}{s} \right).$$

Note that for $s > 1$, $\frac{1}{2}(\frac{s-1}{s}) < \frac{1}{2}$. Therefore as the above inequality need only hold for some $\delta > 0$ we simply require that s and r satisfies

$$\frac{1}{s} + \frac{1}{r} < \frac{1}{2}\left(\frac{s-1}{s} \right).$$

The above inequality holds if $s > 3$ and there exists an $r > 2s/(s-3)$ such that $\sup_{t,\tau,T} \|g(X_{t,T}, X_{\tau,T})\|_r < \infty$, which is the condition stated in Assumption 2.1(iii). Altogether this implies that

$$\left\| \frac{1}{\sqrt{T}} \sum_{j=1}^{k_T} V_{j,T} \right\|_2 = O\left(\frac{k_T^{1/2} q_T^{1/2}}{T^{1/2}} \right) = O\left(\left(\frac{q_T}{p_T + q_T} \right)^{1/2} \right)$$

Thus we obtain (37). □

Lemma 4.2 *Suppose Assumption 2.1 holds. Let Q_T and $\{U_{j,T}\}$ be defined as in (3) and (34)* *respectively. Then*

$$var\left[\sqrt{T}Q_T\right] = \frac{1}{T}\sum_{j=1}^{k_T} var[U_{j,T}] + O\left(T^{-2\eta} + \frac{q_T}{p_T+q_T} + \frac{1}{q_T - T^{1/2}}\right).$$

PROOF. By using Lemma 4.1 we have

$$\text{var}\left[\sqrt{T}Q_T\right] = \text{var}\left(\frac{1}{\sqrt{T}}\sum_{j=1}^{k_T} U_{j,T}\right) + O\left(T^{-2\eta} + \frac{q_T}{p_T+q_T}\right).$$

In the remainder of the proof we show that

$$\text{var}\left(\frac{1}{\sqrt{T}}\sum_{j=1}^{k_T} U_{j,T}\right) = \frac{1}{T}\sum_{j=1}^{k_T}\text{var}[U_{j,T}] + O\left(\frac{1}{q_T - T^{1/2}}\right).$$

Taking differences

$$I = \text{var}\left(\frac{1}{\sqrt{T}}\sum_{j=1}^{k_T} U_{j,T}\right) - \frac{1}{T}\sum_{j=1}^{k_T}\text{var}[U_{j,T}] = \frac{1}{T}\sum_{j_1\neq j_2}^{k_T}\text{cov}\left(U_{j_1,T}, U_{j_2,T}\right).$$

The remainder of the proof is very similar to the proof of Lemma 4.1, following the same steps we have

$$\begin{aligned} I &\leq \frac{Ck_Tp_T}{T}|g|_r\left[\Gamma(1+\eta) + \Gamma\left(\frac{s}{2+\delta} - \frac{s}{r}\right)\right]^2 \sum_{j=1}^{k_T}\frac{1}{[j(q_T - T^{1/2})]^{s(1-2/(2+\delta))}} \\ &\leq \frac{Cp_T}{p_T+q_T}|g_r|\left[\Gamma(1+\eta) + \Gamma\left(\frac{s}{2+\delta} - \frac{s}{r}\right)\right]^2 \Gamma\left(s\left[1 - \frac{2}{2+\delta}\right]\right)[(q_T - T^{1/2})]^{-s(1-2/(2+\delta))}. \end{aligned}$$

Under Assumption 2.1(ii) (see the arguments in Lemma 4.1) there exists an $r > 3$ and $\delta > 0$ such that $\Gamma\left(s\left[1 - \frac{2}{2+\delta}\right]\right) < \infty$ and $\Gamma\left(\frac{s}{2+\delta} - \frac{s}{r}\right) < \infty$. Since δ is such that $s\left[1 - \frac{2}{2+\delta}\right] > 1$ this implies that $[(q_T - T^{1/2})]^{-s(1-2/(2+\delta))} < [(q_T - T^{1/2})]^{-1}$. Altogether this gives the bound

$$I \leq C\frac{p_T}{p_T+q_T}[(q_T - T^{1/2})]^{-1} \leq \frac{C}{q_T - T^{1/2}},$$

thus we obtain the desired bound. □

Let

$$\sigma_T^2 = \text{var}(\sqrt{T}Q_T) \text{ and } \sigma_{U,T}^2 = \frac{1}{T}\sum_{j=1}^{k_T}\text{var}[U_{j,T}]. \tag{38}$$

Then assuming that $q_T/(p_T+q_T) \to 0$ and $q_T - T^{1/2} \to \infty$ as $T \to \infty$, Lemmas 4.1 and 4.2 imply

$$\sqrt{\frac{T}{\sigma_T^2}}\left(Q_T - \mathbb{E}(Q_T)\right) = \sqrt{\frac{1}{T\sigma_T^2}}\sum_{j=1}^{k_T} U_{j,T} + o_p(1) = \sqrt{\frac{T}{\sigma_{U,T}^2}} S_{k_T} + o_p(1), \text{ where } S_{k_T} = \frac{1}{T}\sum_{j=1}^{k_T} U_{j,T}. \quad (39)$$

Thus the limiting distribution of $\sqrt{\frac{T}{\sigma_T^2}}\left(Q_T - \mathbb{E}(Q_T)\right)$ is determined by $\sqrt{\frac{T}{\sigma_{U,T}^2}} S_{k_T}$. To show normality of $\sqrt{T} S_{k_T}$ we define an "unobserved" sum $\widetilde{S}_{k_T}$ where

$$\widetilde{S}_{k_T} = \frac{1}{T}\sum_{j=1}^{k_T} \widetilde{U}_{j,T}, \quad (40)$$

and $\widetilde{U}_{j,T}$ and $U_{j,T}$ have identical distributions, but $\{\widetilde{U}_{j,T}\}$ are independent random variables. In the following lemma we show that asymptotically the distributions of S_{k_T} and $\widetilde{S}_{k_T}$ are equivalent. This result requires the following theorem, which gives a bound on the difference of characteristic functions of sums of mixing random variables and their independent counterpart.

Lemma 4.3 *Suppose Assumption 2.1 is satisfied. Let $U_{j,T}$ and $\widetilde{U}_{j,T}$ be defined in (34) and (40) respectively. If $q_T > T^{1/2}$, then*

$$\left|\mathbb{E}\left(\exp(ix\sum_{j=1}^{k_T} U_{j,T})\right) - \prod_{j=1}^{k_T}\mathbb{E}\left(\exp(ix\widetilde{U}_{j,T})\right)\right| \leq C k_T \alpha(q_T - T^{1/2}), \text{ for all } x \in \mathbb{R} \quad (41)$$

where C is a finite constant.

Further, if $\frac{T}{(q_T - T^{1/2})^{s-1}} \to 0$, as $q_T \to \infty$ and $T \to \infty$ where s denotes the mixing rate of $\{X_{t,T}\}$ defined in Assumption 2.1(i), then the distribution function of $\sqrt{T} S_{k_T}$ converges to the distribution function of $\sqrt{T}\widehat{S}_{k_T}$.

PROOF. Let $A_j = \exp(ixU_{j,T})$. By taking differences we have

$$\begin{aligned} & \mathbb{E}\left(\exp(ix\sum_{j=1}^{k_T} U_{j,T})\right) - \prod_{j=1}^{k_T}\mathbb{E}\left(\exp(ix\widetilde{U}_{j,T})\right) \\ = & \ \mathrm{cov}\left(A_1, \prod_{j=2}^{k_T} A_j\right) + \sum_{s=2}^{k_T-1}\left(\prod_{i=1}^{s-1}\mathbb{E}[A_i]\right)\mathrm{cov}\left(A_s, \prod_{i=s+1}^{k_T} A_i\right). \end{aligned}$$

Therefore

$$\left|\mathbb{E}\left(\exp(ix\sum_{j=1}^{k_T} U_{j,T})\right) - \prod_{j=1}^{k_T}\mathbb{E}\left(\exp(ix\widetilde{U}_{j,T})\right)\right| \leq \sum_{s=1}^{k_T-1}\left|\mathrm{cov}\left(A_s, \prod_{i=s+1}^{k_T} A_i\right)\right|. \quad (42)$$

To bound the covariance we use (57), which gives $|\text{cov}\left(A_s, \prod_{i=s+1}^{k_T} A_i\right)| \leq C\alpha(q_T - T^{1/2})$. Substituting this into (42) gives

$$\left|\mathbb{E}\left(\exp(ix\sum_{j=1}^{k_T} U_{j,T})\right) - \prod_{j=1}^{k_T}\mathbb{E}\left(\exp(ix\widetilde{U}_{j,T})\right)\right| \leq Ck_T\alpha(q_T - T^{1/2}). \tag{43}$$

Thus giving (41).

Therefore by (43), if

$$k_T\alpha(q_T - T^{1/2}) \leq C\frac{T}{(q_T - T^{1/2})^{s-1}} \to 0,$$

then

$$\left|\Phi_{\sqrt{T}\widehat{S}_{k_T}}(x) - \Phi_{\sqrt{T}S_{k_T}}(x)\right| \to 0, \text{ for all } x \in \mathbb{R}$$

where $\Phi_Y(x) = \exp[ixY]$ is the characteristic function of Y. Since $\Phi_{\sqrt{T}\widehat{S}_{k_T}}(x) \to \Phi_{\sqrt{T}\widehat{S}_{k_T}}(x)$, by the inversion theorem $|F_{\sqrt{T}S_{k_T}}(x) - F_{\sqrt{T}\widehat{S}_{k_T}}(x)| \to 0$, where $F_{\sqrt{T}S_{k_T}}(x)$ and $F_{\sqrt{T}\widehat{S}_{k_T}}(x)$ denote the distribution functions of $\sqrt{T}S_{k_T}$ and $\sqrt{T}\widehat{S}_{k_T}$ respectively. □

Finally, we show asymptotic normality of $\sqrt{T}\widetilde{S}_{k_T}$.

Lemma 4.4 *Suppose Assumption 2.1 holds. Let $\sigma^2_{U,T}$ and $\widetilde{S}_{k_T}$ be defined as in (38) and (40) respectively. Then*

$$\sqrt{\frac{T}{\sigma^2_{U,T}}}\widetilde{S}_{k_T} \overset{D}{\to} \mathcal{N}(0,1),$$

with $p_T/T \to 0$ as $T \to \infty$.

PROOF. The proof involves verification of the classical Lindeberg condition (see, for example, (Davidson, 1994), Theorem 23.6), that is for every $\varepsilon > 0$

$$L_C = \sum_{j=1}^{k_T}\mathbb{E}\left[T^{-1}U^2_{j,T}I(T^{-1/2}|U_{j,T}| > \varepsilon)\right] \to 0$$

as $T \to \infty$. By the Hölder's and Chebyshev's inequality we have

$$\mathbb{E}\left(T^{-1}U^2_{j,T}I(T^{-1/2}|U_{j,T}| > \varepsilon\right) \leq T^{-1}\|U^2_{j,T}\|_{1+\delta}[P(|U_{j,T}| > T^{1/2}\varepsilon)]^{\frac{\delta}{1+\delta}} \leq \frac{1}{\widehat{\varepsilon}T^{1+\frac{\delta}{1+\delta}}}\|U^2_{j,T}\|_{1+\delta}\|U_{j,T}\|_2^{2(\frac{\delta}{1+\delta})}.$$

where $\widehat{\varepsilon} = \varepsilon^{2\delta/(1+\delta)}$. By using Lemma A.5 and under Assumption 2.1(iii), there exists a $\delta > 0$ such that $\sup_{1\leq j\leq k_T}\|U_{j,T}^2\|_{1+\delta} \leq Cp_T$ (which immediately implies $\|U_{j,T}\|_2^{2(\frac{\delta}{1+\delta})} \leq p_T^{\frac{\delta}{1+\delta}}$). Substituting this into the above implies

$$\sup_{1\leq j\leq k_T} \mathbb{E}\left(T^{-1}U_{j,T}^2 I(T^{-1/2}|U_{j,T}| > \widehat{\varepsilon}\right) \;\leq\; C\widehat{\varepsilon}^{-1}\left(\frac{p_T}{T}\right)^{1+\frac{\delta}{1+\delta}}.$$

Finally substituting this into L_C gives

$$\Rightarrow L_C \;\leq\; Ck_T\frac{p_T^{1+\frac{\delta}{1+\delta}}}{\widehat{\varepsilon}T^{1+\frac{\delta}{1+\delta}}} = C\frac{T}{p_T+q_T}\frac{p_T^{1+\frac{\delta}{1+\delta}}}{\widehat{\varepsilon}T^{1+\frac{\delta}{1+\delta}}} = \frac{C}{\widehat{\varepsilon}}\left(\frac{p_T}{p_T+q_T}\right)\left(\frac{p_T}{T}\right)^{\frac{\delta}{1+\delta}}.$$

Since $p_T/T \to 0$ as $T \to \infty$, the Lindeberg condition is satisfied and we obtain the result. □

PROOF of Theorem 2.1 If q_T and p_T are chosen such that $q_T/(p_T+q_T) \to 0$ and $q_T - T^{1/2} \to \infty$ as $T \to \infty$, then by using (39) we have

$$\sqrt{\frac{T}{\sigma_T^2}}\left(Q_T - \mathbb{E}(Q_T)\right) \;=\; \sqrt{\frac{T}{\sigma_{U,T}^2}}S_{k_T} + o_p(1).$$

From Lemma 4.3, if $T/(q_T - T^{1/2})^{s-1} \to 0$ as $q_T \to \infty$ and $T \to \infty$ (where s denotes the mixing rate) then

$$\sqrt{\frac{T}{\sigma_T^2}}\left(Q_T - \mathbb{E}(Q_T)\right) \;\stackrel{D}{\to}\; \sqrt{\frac{T}{\sigma_{U,T}^2}}\widetilde{S}_{k_T},$$

where $A_n \stackrel{D}{\to} B_n$ (means the distribution of A_n converges to the distribution of B_n as $T \to \infty$) $\widetilde{S}_{k_T}$ is defined in (40). Finally by Lemma 4.4 we have

$$\sqrt{\frac{T}{\sigma_T^2}}\left(Q_T - \mathbb{E}(Q_T)\right) \;\stackrel{D}{\to}\; \mathcal{N}(0,1),$$

as $p_T/T \to 0$ as $T \to \infty$.

Therefore for the result to hold we require that there exists a p_T and q_T such that $q_T/(p_T+q_T) \to 0$, $q_T - T^{1/2} \to \infty$, $T/(q_T - T^{1/2})^{s-1} \to 0$ and $p_T/T \to 0$ as $T \to \infty$. Since $s > 3$, these hold if p_T and q_T are chosen such that $p_T = T^{1/2+\delta_1}$ and $q_T = T^{1/2+\delta_2}$ where $0 < \delta_2 < \delta_1 < 1/2$. Thus proving the result. □

Note the proof of Theorem 2.2 is similar and given in Appendix B

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A Appendix

A.1 Proof of results in Section 3

PROOF Lemma 3.1 Taking differences gives

$$\begin{aligned}
& \mathrm{cum}\left[X_{t_1,T},\ldots,X_{t_n,T}\right] - \mathrm{cum}\left[X_{t_1}\left(\frac{t_1}{T}\right),\ldots,X_{t_n}\left(\frac{t_n}{T}\right)\right] \\
= \; & \mathrm{cum}\left[X_{t_1,T} - X_{t_1}\left(\frac{t_1}{T}\right), X_{t_2,T},\ldots,X_{t_n,T}\right] + \sum_{j=2}^{n}\mathrm{cum}\left[X_{t_1}\left(\frac{t_1}{T}\right),\ldots,X_{t_j,T} - X_{t_j}\left(\frac{t_j}{T}\right),\ldots,X_{t_n,T}\right].
\end{aligned}$$

Thus under Assumption 3.1(i,ii) we have

$$\left|\mathrm{cum}\left[X_{t_1,T},\ldots,X_{t_n,T}\right] - \mathrm{cum}\left[X_{t_1}\left(\frac{t_1}{T}\right),\ldots,X_{t_n}\left(\frac{t_n}{T}\right)\right]\right| \leq KnT^{-1}\nu_n(t_2-t_1,\ldots,t_n-t_1).$$

By using the same method and Assumption 3.1(i,iii) we have

$$\begin{aligned}
& \left|\mathrm{cum}\left[X_{t_1}\left(\frac{t_1}{T}\right),\ldots,X_{t_n}\left(\frac{t_n}{T}\right)\right] - \kappa_n\left(\frac{t_1}{T}, t_2-t_1,\ldots,t_n-t_1\right)\right| \\
\leq \; & \sum_{j=2}^{n}\frac{|t_j-t_1|}{T}\nu_n(t_2-t_1,\ldots,t_n-t_1).
\end{aligned}$$

Altogether this gives (i). The proof of (ii) follows immediately from Assumption 3.1(iv).

To prove (iii) we apply dominated convergence and take derivative into summand

$$\frac{\partial f_n(u;\omega_1,\ldots,\omega_{n-1})}{\partial u} = \sum_{t_1,\ldots,t_{n-1}\in\mathbb{Z}}\frac{\partial\kappa_n(u,t_1,\ldots,t_{n-1})}{\partial u}\exp\left(i\sum_{j=1}^{n-1}t_j\omega_j\right).$$

Finally taking the absolute value of $\frac{\partial f_n}{\partial u}$ and using Assumption 3.1(v) we obtain

$$\sup_u \left| \frac{\partial f_n(u;\omega_1,\ldots,\omega_{n-1})}{\partial u} \right| \le \sum_{t_1,\ldots,t_{n-1}\in\mathbb{Z}} \sup_u \left| \frac{\partial \kappa_n(u,t_1,\ldots,t_{n-1})}{\partial u} \right| < \infty,$$

thus proving (15). □

PROOF of Lemma 3.2 We integrating the integral $\int_0^1 f_n(u;\omega_1,\ldots,\omega_{n-1})\exp(i2\pi ru)du$ by parts and using (15), the result immediately follows. □

PROOF of Lemma 3.3 Expanding the cumulant term we have

$$\mathrm{cum}\left[J_T(\omega_{j_1}),\ldots,J_T(\omega_{j_n})\right] = \frac{1}{(2\pi T)^{n/2}} \sum_{t_1,\ldots,t_n=1}^{T} \mathrm{cum}(X_{t_1,T},\ldots,X_{t_n,T})\exp(it_1\omega_{j_1}+\ldots+it_n\omega_{j_n}).$$

We replace $\mathrm{cum}(X_{t_1,T},\ldots,X_{t_n,T})$ with $\kappa_n(\frac{t_1}{T},t_2-t_1,\ldots,t_n-t_1)$ and use Lemma 3.1 to obtain

$$\begin{aligned}&\mathrm{cum}\left[J_T(\omega_{j_1}),\ldots,J_T(\omega_{j_n})\right]\\ =\ & \frac{1}{(2\pi T)^{n/2}} \sum_{t_1,\ldots,t_n=1}^{T} \kappa_n\left(\frac{t_1}{T},t_2-t_1,\ldots,t_n-t_1\right)\exp(it_1\omega_{j_1}+\ldots+it_n\omega_{j_n}) + O\left(T^{-n/2}\right).\end{aligned}$$

Replacing the above summand with $f_n(\frac{t}{T};\omega_{j_1},\ldots,\omega_{j_n})$ (where $f_n(\cdot)$ is defined in (13)) we have

$$\begin{aligned}&\mathrm{cum}\left[J_T(\omega_{j_1}),\ldots,J_T(\omega_{j_n})\right]\\ =\ & \frac{(2\pi)^{(n/2)-1}}{T^{n/2}} \sum_{t=1}^{T} f_n\left(\frac{t}{T},\omega_{j_2},\ldots,\omega_{j_n}\right)\exp(it(\omega_{j_1}+\omega_{j_2}+\ldots+\omega_{j_s})) + O(T^{-n/2}) \end{aligned} \quad (44)$$

and replacing the sum with the integral gives

$$\mathrm{cum}\left[J_T(\omega_{j_1}),\ldots,J_T(\omega_{j_n})\right] = \left(\frac{2\pi}{T}\right)^{n/2-1} \int_0^1 f_n(u;\omega_{j_2},\ldots,\omega_{j_n})\exp\left[iu\left(\sum_{s=1}^{n} j_s\right)\right]du + O(T^{-n/2}).$$

Finally, substituting (17) into the above gives (18). □

It is interesting to note that Paparoditis (2009), Lemma 6.2, derives a similar result to (44) for time-varing MA(∞) processes.

PROOF of Theorem 3.1 By using Lemma 3.3 we have

$$\mathbb{E}(\widehat{A}_T(r)) = \frac{1}{T}\sum_{k=1}^{T}\phi(\omega_k)\mathrm{cov}[J_T(\omega_k),J_T(\omega_{k+r})] = \frac{1}{T}\sum_{k=1}^{T}\phi(\omega_k)F_2(-r,\omega_k) + O(T^{-1}).$$

Since $\phi(\cdot)$ has a bounded first derivative, we replace summand with integrals to obtain

$$\mathbb{E}[\widehat{A}_T(r)] = \frac{1}{2\pi}\int_0^1\int_0^{2\pi}\phi(\omega)f(u,\omega)\exp(-i2\pi ru)d\omega du + O\left(T^{-1}\right)$$

and thus obtain (19). To prove (20) we use the cumulant expansion of $\mathrm{cov}(Y_1Y_2, Y_3Y_4)$ together with Lemma 3.3 to obtain $\mathrm{cov}[\sqrt{T}\widehat{A}_T(r_1), \sqrt{T}\widehat{A}_T(r_2)] = \widetilde{V}_{T,r_1,r_2} + O(T^{-1})$ where

$$\begin{aligned}\widetilde{V}_{T,r_1,r_2} &= \frac{1}{T}\sum_{k_1,k_2=1}^{T}\phi(\omega_{k_1})\overline{\phi(\omega_{k_2})}\Big[F_2(k_1-k_2,\omega_{k_1})F_2(-k_1+k_2-r_1+r_2,-\omega_{k_1+r_1}) + \\ &\quad F_2(k_1+k_2+r_2,\omega_{k_1})F_2(-(k_1+k_2+r_1),-\omega_{k_1+r_1}) + \frac{(2\pi)}{T}F_4(-r_1+r_2,\omega_{k_1},-\omega_{k_1+r_1},-\omega_{k_2})\Big].\end{aligned}$$

In the first summand we change variables with $s = k_1 - k_2$ and in the second summand we let $s = k_2 + k_1$ this gives

$$\begin{aligned}\widetilde{V}_{T,r_1,r_2} &= \frac{1}{T}\sum_{k_1}\sum_{s}\phi(\omega_{k_1})\overline{\phi(\omega_{k_1-s})}F_2(s,\omega_{k_1})F_2(-s-r_1+r_2,-\omega_{k_1+r_1}) \\ &+\frac{1}{T}\sum_{k_1}\sum_{s}\phi(\omega_{k_1})\overline{\phi(\omega_{s-k_1})}F_2(s+r_2,\omega_{k_1})F_2(-(s+r_1),-\omega_{k_1+r_1}) \\ &+\frac{(2\pi)}{T^2}\sum_{k_1}\sum_{k_2}\phi(\omega_{k_1})\overline{\phi(\omega_{k_2})}F_4(-r_1+r_2,\omega_{k_1},-\omega_{k_1+r_1},-\omega_{k_2}).\end{aligned}$$

We replace $\phi(\omega_{k_1-s})$ with $\phi(\omega_{k_1})$ (leading to an error of $O(T^{-1}\log T)$) and $\phi(\omega_{s-k_1})$ with $\phi(\omega_{-k_1})$ (leading to an error of $O(T^{-1}\log T)$). For the first two summands, we replace the sum over k_1 with an integral and in the last summand we replace the sum over k_1 and k_2 with a double integral. Finally, we extend the sum over $\sum_s$ to $\sum_{s\in\mathbb{Z}}$ which gives (20).

To prove asymptotic normality we use $\widehat{A}_T(r) = A_T(r) + O_p(T^{-1})$ where $A_T(r)$ is defined in (8). Thus asymptotic normality of $\sqrt{T}[\widehat{A}_T(r) - \mathbb{E}[\widehat{A}_T(r)]$ follows from asymptotic normality of $\sqrt{T}[A_T(r) - \mathbb{E}[A_T(r)]$. From (20) we observe that Assumption 3.1 for $n = 2$ and 4 implies that Assumption 2.1(ii) is satisfied. We have assumed that the mixing Assumptions2.1(i,iii) are satisfied. Thus all the conditions in Theorem 2.1 are satisfied which implies that $A_T(r)$ is asymptotic normality and thus (21) holds. □

We now prove Theorem 3.2. The proof requires the following two lemmas, which are analogous to Lemma 3.1-3.3 (thus we state them without proof).

Lemma A.1 *Let $\mu_x(\cdot)$, $f_{n,x_1,\ldots,x_n}(\cdot)$ and $F_{n,x_1,\ldots,x_n}(\cdot)$ be defined as in (27) and (28) respectively.*

(i) If Assumption 3.2(i,ii,iii) is satisfied. Then

$$\left|\mathbb{E}[Y_{x,t,T}] - \mu_x\left(\frac{t}{T}\right)\right| \le C\frac{(1+|x|)}{T}$$

for $n \ge 2$

$$\left|cum\left[Y_{x_1,t_1,T},\ldots,Y_{x_n,t_n,T}\right] - \kappa_{n,x_1,\ldots,x_n}\left(\frac{t_1}{T},t_2-t_1,\ldots,t_n-t_1\right)\right|$$
$$\le \ C\left(\frac{n}{T}+\sum_{j=2}^{n}\frac{|t_j-t_1|}{T}\right)\nu_n(t_2-t_1,\ldots,t_n-t_1).$$

(iv) If Assumption 3.2(iv) is satisfied then $\sup_{u\in[0,1],\omega_1,\ldots,\omega_{n-1}\in[0,2\pi]}|f_{n,x_1,\ldots,x_n}(u;\omega_1,\ldots,\omega_{n-1})| < C\left(1+\sum_{i=1}^{n}|x_i|\right)$.

(iii) If Assumption 3.2(v) is satisfied then

$$\sup_{u\in[0,1],\omega_1,\ldots,\omega_{n-1}\in[0,2\pi]}\left|\frac{\partial f_{n,x_1,\ldots,x_n}(u;\omega_1,\ldots,\omega_{n-1})}{\partial u}\right| < C\left(1+\sum_{i=1}^{n}|x_i|\right).$$

and

$$\sup_{\omega_1,\ldots,\omega_{n-1}\in[0,2\pi]}|F_{n,x_1,\ldots,x_n}(r;\omega_1,\ldots,\omega_{n-1})| \le C\sup_{u\in[0,1],\omega_1,\ldots,\omega_{n-1}\in[0,2\pi]}\left|\frac{\partial f_n(u;\omega_1,\ldots,\omega_{n-1})}{\partial u}\right|\frac{1}{|r|}$$

Lemma A.2 *Suppose Assumption 3.2 holds and let* $F_{n,x_1,\ldots,x_n}(\cdot)$ *be defined as in (28). Then*

$$\frac{1}{\sqrt{2\pi}}\mathbb{E}[J_n(x;\omega_k)] = F_{1,x}(k) + O\left(\frac{1+|x|}{T}\right)$$

$$cum\left[J_T(x_1,\omega_{j_1}),\ldots,J_T(x_n,\omega_{j_n})\right] = \frac{(2\pi)^{(n/2)-1}}{T^{(n/2)-1}}F_{n,x_1,\ldots,x_n}\left[\sum_{k=1}^{n}j_k;\omega_{j_2},\ldots,\omega_{j_n}\right] + O\left(\left[1+\sum_{i=1}^{n}|x_i|\right]\frac{1}{T^{n/2}}\right).(45$$

Using the above lemmas we prove the result.

PROOF of Theorem 3.2 To prove (29) we use (24) to give

$$\mathbb{E}[W_T] = \int_{\mathbb{R}^2}\Gamma(x,y)\mathbb{E}\left[W_T(x,y)\right]dxdy + O\left(\frac{\log T}{T}\right).$$

Focusing on $W_T(x,y)$ we have

$$\begin{aligned}\mathbb{E}[W_T(x,y)] &= \frac{2\pi}{T}\sum_{k=1}^{T}\phi(\omega_k)\mathbb{E}[J_T(x;\omega_k)\overline{J_T(y;\omega_k)}] \\ &= \frac{2\pi}{T}\sum_{k=1}^{T}\phi(\omega_k)\left\{\mathrm{cov}[J_T(x;\omega_k), J_T(y;\omega_k)] + \mathbb{E}[J_T(x;\omega_k)]\mathbb{E}[\overline{J_T(y;\omega_k)}]\right\}.\end{aligned}$$

By using Lemma A.2 and replacing sum with integral it is straightforward to show that $\mathbb{E}[W_T(x,y)] = E_T(x,y) + O([1+|x|+|y|]T^{-1})$ where

$$E_T(x,y) = \int_0^{2\pi}\phi(\omega)F_{2,x,y}(\omega)d\omega + \sum_{k=1}^{T}\phi(\omega_k)F_{1,x}(k)F_{1,y}(-k).$$

Replacing $\phi(\omega_k)$ with $\phi(0)$

$$\left|\sum_{k=1}^{T}\phi(\omega_k)F_{1,x}(k)F_{1,y}(-k) - \phi(0)\sum_{k=1}^{T}F_{1,x}(k)F_{1,y}(-k)\right| \leq \frac{C\,(1+|x|+|y|)}{T}.$$

Thus

$$\mathbb{E}[W_T(x,y)] = \int_0^{2\pi}\phi(\omega)F_{2,x,y}(\omega)d\omega + \phi(0)\sum_{k=1}^{T}F_{1,x}(k)F_{1,y}(-k) + O\left(\frac{[1+|x|+|y|]}{T}\right).$$

Finally by using (23) we have (29).

To prove (30) we use (24) to give

$$\mathrm{var}[\sqrt{T}W_T] = \int_{\mathbb{R}^4}\Gamma(x_1,y_1)\overline{\Gamma(x_2,y_2)}\mathrm{cov}\left[\sqrt{T}W_T(x_1,y_1), \sqrt{T}W_T(x_2,y_2)\right]dxdy + O\left(\frac{\log T}{T}\right). \quad (46)$$

We note that

$$\begin{aligned}&\mathrm{cov}\left[\sqrt{T}W_T(x_1,y_1), \sqrt{T}W_T(x_2,y_2)\right] \\ =\ & \frac{1}{T}\sum_{k_1,k_2=1}^{T}\phi(\omega_{k_1})\overline{\phi(\omega_{k_2})}\mathrm{cov}\left[J_T(x_1,\omega_{k_1})\overline{J_T(y_1,\omega_{k_1})}, J_T(x_2,\omega_{k_2})\overline{J_T(y_2,\omega_{k_2})}\right].\end{aligned}$$

Applying indecomposable partitions to $\mathrm{cov}[J_T(x_1,\omega_{k_1})\overline{J_T(y_1,\omega_{k_1})}, J_T(x_2,\omega_{k_2})\overline{J_T(y_2,\omega_{k_2})}]$ and using Lemma A.2 we have

$$T\mathrm{cov}[W_T(x_1,y_1), W_T(x_2,y_2)] = \widetilde{v}_T(x_1,y_1,x_2,y_2) + O\left(\frac{(1+|x_1|+|y_1|)(1+|x_2|+|y_2|)}{T}\right)$$

where

$$\frac{1}{(2\pi)^2}\widetilde{v}_T(x_1,y_1,x_2,y_2) = \qquad \frac{1}{T}\sum_{k_1,k_2=1}^{T}\phi(\omega_{k_1})\overline{\phi(\omega_{k_2})}\Big[F_{2,x_1,x_2}(k_1-k_2,\omega_{k_1})F_{2,y_1,y_2}(-k_1+k_2,-\omega_{k_1})+$$

$$F_{2,x_1,y_2}(k_1+k_2,\omega_{k_1})F_{2,y_1,x_2}(-(k_1+k_2),-\omega_{k_1})\Big]$$

$$+\frac{(2\pi)}{T^2}\sum_{k_1,k_2=1}^{T}\phi(\omega_{k_1})\overline{\phi(\omega_{k_2})}F_{4,x_1,y_1,x_2,y_2}(0,\omega_{k_1},-\omega_{k_1},-\omega_{k_2})$$

$$+\frac{1}{T}\sum_{k_1,k_2=1}^{T}\phi(\omega_{k_1})\overline{\phi(\omega_{k_2})}\Big\{F_{1,x_1}(k_1)F_{3,y_1,x_2,y_2}(k_1;-\omega_{k_1},-\omega_{k_2})+F_{1,y_1}(-k_1)F_{3,x_1,x_2,y_2}(-k_1;-\omega_{k_1},-\omega_{k_2})$$

$$F_{1,x_2}(-k_2)F_{3,x_1,y_1,y_2}(k_2;\omega_{k_1},-\omega_{k_2})+F_{1,y_2}(k_2)F_{3,x_1,y_1,x_2}(-k_2;\omega_{k_1},-\omega_{k_2})\Big\}.$$

Substituting this into (46) and using (23) gives (30). □

A.2 Cumulants and moment bounds

In this section we state some bounds on the sums of moments and cumulants. These results will be used to prove Theorem 2.1.

The following lemma and corollary link mixing conditions to summability of cumulants. These results can be used provide sufficient conditions for Assumption 2.2(ii) to hold. Note that a similar result was discussed in Remark 3.1, which gave sufficient conditions for Assumption 3.1(i-iv) to hold.

We first state a bound for the sum of cumulants based on the mixing rate. This result is motivated by Neumann (1996), Remark 3.1.

Lemma A.3 *Let us suppose that $\{X_{t,T}\}$ is an α-mixing time series which satisfies Assumption 2.1(i).*

(i) If $t_1 \leq t_2 \leq \ldots \leq t_k$, then $|cum(X_{t_1,T},\ldots,X_{t_k,T})| \leq C_k \sup_{t,T}\||X_{t,T}\|_r^k \prod_{i=2}^{k}\alpha(t_i-t_{i-1})^{\frac{1-k/r}{k-1}}$.

(ii) If s and r are such that $s(r-k) > r(k-1)$ then

$$\sup_{t_1}\sum_{t_2,\ldots,t_k=1}^{\infty}|cum(X_{t_1,T},\ldots,X_{t_k,T})| \leq C_k \sup_{t,T}\|X_{t,T}\|_r^k\left[\Gamma\left(\frac{1-k/r}{k-1}\right)\right]^{k-1} < \infty.$$

(iii) If s and r are such that $(s-1)(r-k) > r(k-1)$ then for all $2 \leq j \leq k$

$$\sup_{t_1} \sum_{t_2,\ldots,t_k=1}^{\infty} (1+|t_j|)|cum(X_{t_1,T},\ldots,X_{t_k,T})| \leq C_k \sup_{t,T} \|X_{t,T}\|_r^k \left[\Gamma\left(\frac{1-k/r}{k-1}-1\right)\right]^{k-1} < \infty.$$

C_k is a finite constant which depends only on k.

PROOF. To prove the lemma we apply a result from Statulevicius and Jakimavicius (1988), Theorem 3, part (2), which states that if $t_1 \leq t_2 \leq \ldots \leq t_k$, then for all $2 \leq i \leq k$ we have $\left|\text{cum}(X_{t_1,T}, X_{t_2,T}, \ldots, X_{t_k,T})\right| \leq 3(k-1)!2^{k-1}\alpha(t_i - t_{i-1})^{1-\frac{k}{r}} \prod_{i=1}^{k} \|X_{t_i,T}\|_r$.

To prove (i) we use a method similar to the proof of Neumann (1996), Remark 3.1. By taking the $(k-1)$th root of the above for all $2 \leq i \leq k$ we have

$$\left|\text{cum}(X_{t_1,T}, X_{t_2,T}, \ldots, X_{t_k,T})\right|^{\frac{1}{k-1}} \leq C_k^{1/(k-1)} \alpha(t_i - t_{i-1})^{\frac{1-k/r}{k-1}} \sup_{t,T} \|X_{t,T}\|_r^{\frac{k}{k-1}},$$

where $C_k = 3(k-1)!2^{k-1}$. Since the above bound holds for all i, multiplying the above over i gives

$$\left|\text{cum}(X_{t_1,T}, X_{t_2,T}, \ldots, X_{t_k,T})\right| \leq C_k \sup_{t,T} \|X_{t,T}\|_r^k \prod_{i=2}^{k} \alpha(t_i - t_{i-1})^{\frac{1-k/r}{k-1}}, \tag{47}$$

thus proving (i).

To prove (ii), we rewrite $\sum_{t_2,\ldots,t_k=1}^{\infty}$ as the sum of orderings, that is $\sum_{t_2,\ldots,t_k=1}^{\infty} = k! \sum_{1=t_2\leq\ldots\leq t_k}^{\infty}$. Since the number of orderings is finite, we use (i) to obtain

$$\sum_{t_2,\ldots,t_k=1}^{\infty} \left|\text{cum}(X_{t_1,T}, X_{t_2,T}, \ldots, X_{t_k,T})\right| \leq C_k \sup_{t,T} \|X_{t,T}\|_r^k \{\sum_r \alpha(r)^{\frac{(1-k/r)}{(k-1)}}\}^{k-1} < \infty,$$

which gives (ii). To prove (iii) we use a similar argument to obtain

$$\sum_{t_2,\ldots,t_k=1}^{\infty} (1+|t_j|)\left|\text{cum}(X_{t_1,T}, X_{t_2,T}, \ldots, X_{t_k,T})\right| \leq k! \sum_{1\leq t_2<\ldots<t_k<\infty} (1+|t_j|)\left|\text{cum}(X_{t_1,T}, X_{t_2,T}, \ldots, X_{t_k,T})\right|$$

substituting (47) into the above gives the result. □

Using the lemma above, we link α-mixing with the so called Brillinger-type mixing conditions.

Corollary A.1 *Suppose $\{X_{t,T}\}$ is an α-mixing time series which satisfies Assumption 2.1(i), with $\alpha(t) \leq K|t|^{-s}$.*

(i) *If there exists an* $r > 4s/(s-3)$ *such that* $\sup_{t,T} \mathbb{E}|X_{t,T}|^r < \infty$, *then* $|cov(X_{t,T}, X_{\tau,T})| \leq C|t-\tau|^{-\frac{(s+3)}{2}}$ *and*
$\sup_{t_1} \sum_{t_2,t_3,t_4=-\infty}^{\infty} |cum(X_{t_1,T}, X_{t_2,T}, X_{t_3,T}, X_{t_4,T})| < \infty$.

(ii) *If there exists an* $r > 4s/(s-6)$ *such that* $\sup_{t,T} \mathbb{E}|X_{t,T}|^r < \infty$, *then* $|cov(X_{t,T}, X_{\tau,T})| \leq C|t-\tau|^{-\frac{(s+6)}{2}}$ *and for all* $2 \leq j \leq 4$,
$\sup_{t_1} \sum_{t_2,t_3,t_4=-\infty}^{\infty} (1+|t_j|)|cum(X_{t_1,T}, X_{t_2,T}, X_{t_3,T}, X_{t_4,T})| < \infty$.

We now state a series of results which are used in Section 4. We will make heavy use of Ibragimov's inequality (see Ibragimov (1962) and Davidson (1994), Theorem 14.2 for a review). We summarize it here. Suppose that $\{X_{t,T}\}$ are alpha-mixing random variables which satisfy Assumption 2.1(i). If there exists an $r > q \geq 1$ such that $\|X_{t,T}\|_r < \infty$ and $j > 0$. Then

$$\left\|\mathbb{E}[X_{t,T}|\mathcal{F}_{t-j}] - \mathbb{E}[X_{t,T}]\right\|_q \leq 2[2^{1/q}+1]\alpha(j)^{1/q-1/r}\|X_{t,T}\|_r, \tag{48}$$

where $\mathcal{F}_{t-j} = \sigma(X_{t-j,T}, X_{t-j-1,T}, \ldots)$. The same proof of (48) can also be used to show a simpler result

$$\left\|\mathbb{E}[Z_{t,T}|\sigma(Z_{t-j})] - \mathbb{E}[Z_{t,T}]\right\|_q \leq 2[2^{1/q}+1])\alpha[\sigma(Z_{t,T}), \sigma(Z_{t-j,T})]^{1/q-1/r}\|Z_{t,T}\|_r, \text{ for } r > q, \tag{49}$$

where $\alpha[\mathcal{F}, \mathcal{G}] = \sup_{A\in\mathcal{F}, B\in\mathcal{G}} |P(A\cap B) - P(A)P(B)|$. Using this result, for any $\tau > t$

$$\begin{aligned} & |\text{cov}(Z_{t,T}, Z_{\tau,T})| = |\mathbb{E}\left(Z_{t,T}\left\{\mathbb{E}[Z_{\tau,T}|Z_{t,T}] - \mathbb{E}[Z_{\tau,T}]\right\}\right)| \\ \leq\ & \|Z_{t,T}\|_2 \left\|\mathbb{E}[Z_{\tau,T}|Z_{t,T}] - \mathbb{E}[Z_{\tau,T}]\right\|_2 \\ \leq\ & 2[2^{1/2}+1]\|Z_{t,T}\|_2\|Z_{\tau,T}\|_r \alpha[\sigma(Z_{t,T}), \sigma(Z_{t-j,T})]^{1/2-1/r} \qquad \text{(for some } r > 2). \end{aligned} \tag{50}$$

Let $h(X_{t,T}, X_{\tau,T})$ be defined as in (3) and the weights $\{H_{t,\tau,T}\}$ be some aribtrary weights (which we specify later). Define the sum

$$Z_{t,T} = H_{t,t,T}h(X_{t,T}, X_{t,T}) + \sum_{\tau=1}^{t-1}[H_{t,\tau,T}h(X_{t,T}, X_{\tau,T}) + G_{\tau,t,T}h(X_{\tau,T}, X_{t,T})]. \tag{51}$$

In the following lemma we obtain some bounds for $Z_{t,T}$.

Lemma A.4 *Suppose* $\{X_{t,T}\}$ *is an* α*-mixing time series which satisfies Assumption 2.1(i), with* $\alpha(t) \leq K|t|^{-s}$. *Let* $Z_{t,T}$ *be defined as in (51), where* $H_{t,\tau,T}$ *is such that* $\sup_T |H_{t,\tau,T}| \leq C|t-\tau|^{-(2+\eta)}I_{|t-\tau|>0}$ *for some* $\eta > 0$. *Suppose Assumption 2.1(i) holds and there exists some* $r > q \geq 1$

such that $\sup_{t,\tau,T} \|h(X_{t,T}, X_{\tau,T})\|_r < \infty$, *then*

$$\|\mathbb{E}(Z_{t,T}|\mathcal{F}_{t-j}) - \mathbb{E}[Z_{t,T}]\|_q \leq C\left(|j|^{-(1+\eta)} + |j|^{-s(\frac{1}{q}-\frac{1}{r})}\right) \tag{52}$$

where $\mathcal{F}_j = \sigma(X_{t,T}; t \leq j)$.

PROOF. We write $\mathbb{E}(Z_{t,T}|\mathcal{F}_{t-j}) - \mathbb{E}[Z_{t,T}] = A_1 + A_2$ where

$$\begin{aligned} A_1 &= \sum_{\tau=1}^{t-1} H_{t,\tau,T}[\mathbb{E}[h(X_{t,T}, X_{\tau,T})|\mathcal{F}_{t-j}] - \mathbb{E}[h(X_{t,T}, X_{\tau,T})] \\ A_2 &= \sum_{\tau=1}^{t} H_{\tau,t,T}[\mathbb{E}[h(X_{\tau,T}, X_{t,T})|\mathcal{F}_{t-j}] - \mathbb{E}[h(X_{\tau,T}, X_{t,T})]. \end{aligned}$$

We now bound the above. As the derivation for the bounds on A_1 and A_2 are identical, we focus on A_1.

The bound is based on an interplay between the coefficients $H_{t,\tau,T}$ which decay as $|t-\tau| \to \infty$ and the mixing rate of the time series $\{X_{t,T}\}$. We define the sigma algebra $\mathcal{F}_{t-j/2}^t = \sigma(X_{k,T}; t-j/2 \leq k \leq t)$ and add and subtract $\mathbb{E}[h(X_{t,T}, X_{\tau,T})|\mathcal{F}_{t-j/2}^t]$ to A_1

$$\begin{aligned} A_1 &= \sum_{\tau=1}^{t-1} H_{t,\tau,T} \left\{\mathbb{E}[h(X_{t,T}, X_{\tau,T})|\mathcal{F}_{t-j}] - \mathbb{E}[h(X_{t,T}, X_{\tau,T})]\right\} \\ &= \sum_{\tau=1}^{t-1} H_{t,\tau,T}\mathbb{E}\left\{h(X_{t,T}, X_{\tau,T}) - \mathbb{E}[h(X_{t,T}, X_{\tau,T})|\mathcal{F}_{t-j/2}^t]\big|\mathcal{F}_{t-j}\right\} \\ &\quad + \sum_{\tau=1}^{t-1} H_{t,\tau,T}\mathbb{E}\left\{\mathbb{E}[h(X_{t,T}, X_{\tau,T})|\mathcal{F}_{t-j/2}^t] - \mathbb{E}[h(X_{t,T}, X_{\tau,T})]\big|\mathcal{F}_{t-j}\right\}. \end{aligned}$$

Thus by applying Minkowski's inequality to A_1 we have $\|A_1\|_q \leq A_{11} + A_{12}$, where

$$\begin{aligned} A_{11} &= \sum_{\tau=1}^{t-1} |H_{t,\tau,T}| \left\|\mathbb{E}\left\{h(X_{t,T}, X_{\tau,T}) - \mathbb{E}[h(X_{t,T}, X_{\tau,T})|\mathcal{F}_{t-j/2}^t]\big|\mathcal{F}_{t-j}\right\}\right\|_q \\ A_{12} &= \sum_{\tau=1}^{t-1} |H_{t,\tau,T}| \left\|\mathbb{E}\left\{\mathbb{E}[h(X_{t,T}, X_{\tau,T})|\mathcal{F}_{t-j/2}^t] - \mathbb{E}[h(X_{t,T}, X_{\tau,T})]\big|\mathcal{F}_{t-j}\right\}\right\|_q. \end{aligned}$$

The bound for A_{11} is based on the observation that for $t - j/2 < \tau$, $h(X_t, X_\tau) \in \mathcal{F}_{t-j/2}^t$, thus $\mathbb{E}[h(X_{t,T}, X_{\tau,T})|\mathcal{F}_{t-j/2}^t] = h(X_{t,T}, X_{\tau,T})$ and

$$\mathbb{E}\left\{h(X_{t,T}, X_{\tau,T}) - \mathbb{E}[h(X_{t,T}, X_{\tau,T})|\mathcal{F}_{t-j/2}^t]\big|\mathcal{F}_{t-j}\right\} = 0.$$

Therefore when we partition A_{11} into $\sum_{\tau=t-j/2}^{t}$ and $\sum_{\tau=1}^{t-j/2-1}$ the first summand is zero and A_{11} reduces to

$$A_{11} = \sum_{\tau=1}^{t-j/2} |H_{t,\tau,T}| \left\| h(X_{t,T}, X_{\tau,T}) - \mathbb{E}(h(X_{t,T}, X_{\tau,T})|\mathcal{F}_{t-j/2}^{t}) \right\|_q .$$

To bound this term we use that $\sup_T |H_{t,\tau,T}| \leq C|t-\tau|^{-(2+\eta)} I_{|t-\tau|>0}$ to give

$$A_{11} \leq 2\|h(X_{t,T}, X_{\tau,T})\|_q \sum_{\tau=1}^{t-j/2} |H_{t,\tau,T}| \leq C|j|^{-(1+\eta)}.$$

To bound A_{12} we rewrite the conditional expectation as

$$A_{12} = \sum_{\tau=1}^{t-1} |H_{t,\tau,T}| \left\| \mathbb{E}\left\{ \mathbb{E}[h(X_{t,T}, X_{\tau,T})|\mathcal{F}_{t-j/2}^{t}] \middle| \mathcal{F}_{t-j} \right\} - \mathbb{E}[h(X_{t,T}, X_{\tau,T})] \right\|_q .$$

We observe that $\mathbb{E}[h(X_{t,T}, X_{\tau,T})|\mathcal{F}_{t-j/2}^{t}] \in \sigma(X_{t,T}, \ldots, X_{t-j/2,T})$, thus it is α-mixing random variable with the same mixing rate as $\{X_t\}$. Applying (48) we have

$$\begin{aligned} & \left\| \mathbb{E}\left\{ \mathbb{E}[h(X_{t,T}, X_{\tau,T})|\mathcal{F}_{t-j/2}^{t}] \middle| \mathcal{F}_{t-j} \right\} - \mathbb{E}[h(X_{t,T}, X_{\tau,T})] \right\|_q \\ \leq \; & C \left\| \mathbb{E}[h(X_{t,T}, X_{\tau,T})|\mathcal{F}_{t-j/2}^{t}] \right\|_r \alpha(j/2)^{1/q-1/r} \leq C \left\| h(X_{t,T}, X_{\tau,T}) \right\|_r |j|^{-s(1/q-1/r)} \end{aligned}$$

to give

$$A_{12} \leq C \sup_{\tau,T} \left\| h(X_{t,T}, X_{\tau,T}) \right\|_r |j|^{-s(1/q-1/r)} \sum_{\tau=1}^{t-1} |H_{t,\tau,T}|.$$

Thus altogether we have

$$A_1 \leq C\left(\sup_{t,\tau,T} \|h(X_{t,T}, X_{\tau,T})\|_q |j|^{-(1+\eta)} + \sup_{t,\tau,T} \|h(X_{t,T}, X_{\tau,T})\|_r |j|^{-s(\frac{1}{q}-\frac{1}{r})} \right).$$

A similar bound also applies to A_2, thus

$$\|\mathbb{E}(Z_{t,T}|\mathcal{F}_{t-j}) - \mathbb{E}(Z_{t,T})\|_q \leq C\left(\sup_{t,\tau,T} \|h(X_{t,T}, X_{\tau,T})\|_q |j|^{-(1+\eta)} + \sup_{t,\tau,T} \|h(X_{t,T}, X_{\tau,T})\|_r |j|^{-s(\frac{1}{q}-\frac{1}{r})} \right).$$

Thus proving (52). □

We define the following sum

$$B^{(u)}_{T,N_T} \;=\; \sum_{t=u+1}^{N_T+u} Z_{t,T}, \tag{53}$$

where $\{Z_{t,T}\}$ is defined in (51).

Lemma A.5 *Suppose $\{X_{t,T}\}$ is an α-mixing time series which satisfies Assumption 2.1(i), with $\alpha(t) \leq K|t|^{-s}$. Let $Z_{t,T}$ and $B^{(u)}_{T,N_T}$ be defined as in (51) and (53) respectively, where $\sup_T |H_{t,\tau,T}| \leq C|t-\tau|^{-(2+\eta)} I_{|t-\tau|>0}$ for some $\eta > 0$. If for some $r > qs/(s-2)$ we have $\sup_{t,\tau,T} \|h(X_{t,T}, X_{\tau,T})\|_r < \infty$. Then*

$$\sup_u \big\| B^{(u)}_{T,N_T} - \mathbb{E}[B^{(u)}_{T,N_T}] \big\|_q \leq C N_T^{1/2} \left[\Gamma(1+\eta) + \Gamma\left(\frac{s}{q} - \frac{s}{r}\right) \right], \tag{54}$$

where C is a finite constant that is independent of N_T and T.

PROOF. Let

$$N_{t,T}(j) = \mathbb{E}(Z_{t,T}|\mathcal{F}_{t-j}) - \mathbb{E}(Z_{t,T}|\mathcal{F}_{t-j-1}).$$

By Lemma A.4 we have

$$\|N_{t,T}(j)\|_q \leq C\left(\sup_{t,\tau,T} \|h(X_{t,T}, X_{\tau,T})\|_q |j|^{-(1+\eta)} + \sup_{t,\tau,T} \|h(X_{t,T}, X_{\tau,T})\|_r |j|^{-s(\frac{1}{q}-\frac{1}{r})} \right).$$

This allows us to use the representation $Z_{t,T} - \mathbb{E}(Z_{t,T}) = \sum_{j=0}^{\infty} N_{t,T}(j)$ to prove the result. Substituting the above into $\|B^{(u)}_{T,N_T}\|_q$ and using the Burkholder inequality gives

$$\begin{aligned} \left\| B^{(u)}_{T,N_T} - \mathbb{E}[B^{(u)}_{T,N_T}] \right\|_q &= \big\| \sum_{t=u}^{N_T+u} \sum_{j=0}^{\infty} N_{t,T}(j) \big\|_q \leq \sum_{j=0}^{\infty} \big\| \sum_{t=u}^{N_T+u} N_{t,T}(j) \big\|_q \leq \sum_{j=0}^{\infty} \left(\sum_{t=u}^{N_T+u} \|N_{t,T}(j)\|_q^2 \right)^{1/2} \\ &\leq C N_T^{1/2} \sup_{t,\tau,T} \|h(X_{t,T}, X_{\tau,T})\|_r \sum_{j=1}^{\infty} \left(|j|^{-(1+\eta)} + |j|^{-s(\frac{1}{q}-\frac{1}{r})} \right) \end{aligned}$$

as required. □

Lemma A.6 *Suppose $\{X_{t,T}\}$ satisfies Assumption 2.2(ii). Let $Z_{t,T}$ and $B^{(u)}_{T,N_T}$ be defined as in (51) and (53) respectively, where $|H_{t,\tau,T}| = 0$ if $|t-\tau| > M$. Then*

$$\sup_u \big\| B^{(u)}_{T,N_T} - \mathbb{E}[B^{(u)}_{T,N_T}] \big\|_4 \leq C M^{1/2} N_T^{1/2} \tag{55}$$

where C is a finite constant that is independent of N_T and T.

PROOF. Expanding

$$\left\| B^{(u)}_{T,N_T} - \mathbb{E}[B^{(u)}_{T,N_T}] \right\|_4^4 = 3\mathrm{var}[B^{(u)}_{T,N_T}]^2 + \mathrm{cum}_4[B^{(u)}_{T,N_T}].$$

Using indecomposable partitions and Assumption 2.2(i) it is straightforward to show that $\mathrm{var}[B^{(u)}_{T,N_T}] = O(N_T M)$ and $\mathrm{cum}_4[B^{(u)}_{T,N_T}] = O(N_T^2 M^2)$. Thus we obtain the desired result. □

Lemma A.7 *Suppose $\{X_{t,T}\}$ is an α-mixing time series which satisfies Assumption 2.1(i). Let $Z_{t,T}$ and $B^{(u)}_{T,N_T}$ be defined as in (51) and (53) respectively, where $H_{t,\tau,T} = 0$ for $|t-\tau| > M$. If $M < N_T$, then*

$$\left| cov\left(B^{(u)}_{T,N_T}, B^{(u+jN_T)}_{T,N_T} \right) \right| \;\leq\; C\alpha\left(jN_T - M\right)^{1-2/(2+\delta)} \left\| B^{(u)}_{T,N_T} \right\|_2 \left\| B^{(u+jN_T)}_{T,N_T} \right\|_{2+\delta}, \qquad (56)$$

for some $\delta > 0$, and

$$\left| cov\left[\exp\left(ixB^{(u)}_{T,N_T} \right), \exp\left(ix \prod_{j=1}^{m} B^{(u+jN_T)}_{T,N_T} \right) \right] \right| \;\leq\; C\alpha\left(jN_T - M\right). \qquad (57)$$

PROOF. To prove (56) we use Ibragimov's inequality stated in (50) to bound the covariance

$$\left| \mathrm{cov}\left(B^{(u)}_{T,N_T}, B^{(u+jN_T)}_{T,N(T)} \right) \right| \;\leq\; C\alpha[\sigma(B^{(u)}_{T,N_T}), \sigma(B^{(u+jN_T)}_{T,N(T)})]^{1-\frac{2}{2+\delta}} \left\| B^{(u)}_{T,N(T)} \right\|_2 \left\| B^{(u+jN_T)}_{T,N(T)} \right\|_{2+\delta}.$$

By definition of $B^{(u)}_{T,N_T}$ and $B^{(u+jN_T)}_{T,N(T)}$ we have $\sigma(B^{(u)}_{T,N_T}) \subseteq \sigma(Y_{k,T}; k \leq u) \subseteq \mathcal{F}^{u}_{-\infty}$ and $\sigma(B^{(u+jN_T)}_{T,N_T}) \subseteq \sigma(Y_{k,T}; k \geq u + jN_T) \subseteq \mathcal{F}^{\infty}_{u+jN_T-M}$. We observe from the definition of $\{Y_{t,T}\}$ that $\sigma(Y_{k,T}; k \geq u + jN_T) \subseteq \sigma(X_{k,T}; k \geq u + jN_T - M)$ and $\sigma(Y_{k,T}; k \leq u) \subseteq \sigma(X_{k,T}; k \leq u)$. Thus

$$\begin{aligned}
&\left| \mathrm{cov}\left(B^{(u)}_{T,N_T}, B^{(u+jN_T)}_{T,N(T)} \right) \right| \\
\leq\;& C\alpha[\sigma(X_{k,T}; k \leq u), \sigma(X_{k,T}; k \geq u + jN_T - M)]^{1-\frac{2}{2+\delta}} \left\| B^{(u)}_{T,N(T)} \right\|_2 \left\| B^{(u+jN_T)}_{T,N(T)} \right\|_{2+\delta} \\
\leq\;& C\alpha\left(jN_T - M\right)^{1-2/(2+\delta)} \left\| B^{(u)}_{T,N(T)} \right\|_2 \left\| B^{(u+jN_T)}_{T,N(T)} \right\|_{2+\delta}
\end{aligned}$$

for some $\delta > 0$.

The proof of (57) is identical and we omit the details. □

B Proof of Theorem 2.2

The proof is very similar to the proof of Theorem 2.1, however, there is no need to truncate the weight function $G_{t,\tau,M}$, as by definition $|G_{t,\tau,M}| = 0$ for $|t-\tau| > M$. Let

$$Y_{t,T} = G_{t,t,M}h(X_{t,T}, X_{t,T}) + \sum_{\tau=1}^{t-1}[G_{t,\tau,M}h(X_{t,T}, X_{\tau,T}) + G_{\tau,t,M}h(X_{\tau,T}, X_{t,T})].$$

We observe $Q_{T,M} = \frac{1}{T}\sum_{t=1}^{T} Y_{t,T}$. Define the partial block sums

$$U_{j,T} = \sum_{t=jr_T+1}^{jr_T+p_T} [Y_{t,T} - \mathbb{E}(Y_{t,T})] \quad \text{and} \quad V_{j,T} = \sum_{t=jr_T+p_T+1}^{(j+1)r_T} [Y_{t,T} - \mathbb{E}(Y_{t,T})], \tag{58}$$

so

$$Q_{T,M} - \mathbb{E}[Q_{T,M}] \;=\; \frac{1}{T}\sum_{j=1}^{k_T} (U_{j,T} + V_{j,T}), \tag{59}$$

where $k_T = T/(p_T + q_T)$.

The following four lemmas are analogous to Lemmas 4.1-4.4.

Lemma B.1 *Suppose Assumptions 2.1(i) (where $s > 1$) and 2.2 hold. Let $Q_{T,M}$ and $\{U_{j,T}\}$ be defined as in (5) and (58) respectively. Then*

$$\sqrt{\frac{T}{M}}\,(Q_{T,M} - \mathbb{E}[Q_{T,M}]) = \frac{1}{\sqrt{TM}}\sum_{j=1}^{k_T} U_{j,T} + F_1, \tag{60}$$

where $\|F_1\|_2 = O\left((\frac{q_T}{p_T+q_T})^{1/2}\right)$

PROOF. It is clear from (59) that $F_1 = \frac{1}{\sqrt{TM}}\sum_{j=1}^{k_T} V_{j,T}$. Since $\mathbb{E}[V_{j,T}] = 0$ the above is simply the variance

$$\left\|\frac{1}{\sqrt{MT}}\sum_{j=1}^{k_T} V_{j,T}\right\|_2^2 = \frac{1}{MT}\sum_{j_1,j_2=1}^{k_T} \mathrm{cov}\,(V_{j_1,T}, V_{j_2,T})\,.$$

By using Lemmas A.6 and A.7 we can bound the above with

$$\begin{aligned}\left\|\frac{1}{\sqrt{MT}}\sum_{j=1}^{k_T}V_{j,T}\right\|_2^2 &\leq \frac{Ck_T}{MT}\|V_{j,T}\|_4^2\left(1+\sum_{j=1}^{k_T}\alpha(jp_T-M)^{1/2}\right)\\ &\leq \frac{Ck_Tq_TM}{TM}\Gamma\left(\frac{s}{2}\right)=\left(\frac{q_T}{p_T+q_T}\right).\end{aligned}$$

□

Lemma B.2 *Suppose Assumptions 2.1(i) (where $s>1$) and 2.2 hold. Let $Q_{T,M}$ and $\{U_{j,T}\}$ be defined as in (5) and (58) respectively. Then*

$$var\left[\sqrt{\frac{T}{M}}Q_T\right]=\frac{1}{TM}\sum_{j=1}^{k_T}var[U_{j,T}]+O\left(\frac{q_T}{p_T+q_T}+\frac{1}{(q_T-M)^{s/2}}\right).$$

PROOF. By using Lemma B.1 we have

$$\text{var}\left[\sqrt{\frac{T}{M}}Q_T\right]=\text{var}\left(\frac{1}{\sqrt{TM}}\sum_{j=1}^{k_T}U_{j,T}\right)+O\left(\frac{q_T}{p_T+q_T}\right).$$

Therefore similar to the proof of Lemma B.2 we need only to take differences

$$I=\text{var}\left(\frac{1}{\sqrt{TM}}\sum_{j=1}^{k_T}U_{j,T}\right)-\frac{1}{TM}\sum_{j=1}^{k_T}\text{var}[U_{j,T}]=\frac{1}{TM}\sum_{j_1\neq j_2}^{k_T}\text{cov}\left(U_{j_1,T},U_{j_2,T}\right).$$

Following the same steps as in the proof of Lemma B.1

$$\begin{aligned}I &\leq \frac{Ck_T}{TM}\|V_{j,T}\|_4^2\left(\sum_{j=1}^{k_T}\alpha(jq_T-M)^{1/2}\right)\\ &\leq \frac{Ck_TMp_T}{TM(q_T-M)^{s/2}}\Gamma(s^{1/2})\leq C\frac{p_T}{p_T+q_T}[(q_T-M)]^{-s/2}\leq\frac{C}{(q_T-M)^{s/2}},\end{aligned}$$

thus we obtain the desired bound. □

Let

$$\sigma_T^2=\text{var}\left(\sqrt{\frac{T}{M}}Q_T\right)=\text{ and }\sigma_{U,T}^2=\frac{1}{TM}\sum_{j=1}^{k_T}\text{var}[U_{j,T}]. \tag{61}$$

Then assuming that $q_T/(p_T+q_T) \to 0$ and $q_T - M \to \infty$ as $T \to \infty$, Lemmas 4.1 and 4.2 imply

$$\sqrt{\frac{T}{M\sigma_T^2}}\left(Q_{T,M} - \mathbb{E}(Q_{T,M})\right) = \sqrt{\frac{T}{M\sigma_{U,T}^2}} S_{k_T} + o_p(1), \text{ where } S_{k_T} = \frac{1}{T}\sum_{j=1}^{k_T} U_{j,T}. \quad (62)$$

To show normality of $\sqrt{T}S_{k_T}$ we define an "unobserved" sum $\widetilde{S}_{k_T}$ where

$$\widetilde{S}_{k_T} = \frac{1}{T}\sum_{j=1}^{k_T} \widetilde{U}_{j,T}, \quad (63)$$

and $\widetilde{U}_{j,T}$ and $U_{j,T}$ have identical distributions, but $\{\widetilde{U}_{j,T}\}$ are independent random variables.

Lemma B.3 *Suppose Assumptions 2.1(i) (where $s > 1$) and 2.2 are satisfied. Let $U_{j,T}$ and $\widetilde{U}_{j,T}$ be as defined in (58) and (63) respectively. If $\frac{T}{(q_T-M)^{s-1}} \to 0$, as $q_T \to \infty$ and $T \to \infty$ where s denotes the mixing rate of $\{X_{t,T}\}$ defined in Assumption 2.1(i), then the distribution function of $\sqrt{T}S_{k_T}$ converges to the distribution function of $\sqrt{T}\widehat{S}_{k_T}$.*

PROOF. The proof is identical to the proof of Lemma B.3. □

Lemma B.4 *Suppose Assumptions 2.1(i) (where $s > 1$) and 2.2 holds. Let $\widetilde{S}_{k_T}$ and $\widehat{\sigma}_{u,T}^2$ be defined as in (61) and (63) respectively. Then*

$$\sqrt{\frac{T}{M\sigma_{U,T}^2}}\widetilde{S}_{k_T} \xrightarrow{D} \mathcal{N}(0,1),$$

with $p_T/T \to 0$ as $T \to \infty$.

PROOF. The proof is identical to the proof of Lemma 4.4. □

PROOF of Theorem 2.2 By using Lemmas B.1-B.4, if q_T and p_T are such that

$$q_T/(p_T+q_T) \to 0, \quad (q_T - M) \to \infty, \quad T/(q_T-M)^{s-1} \to 0, \text{ and } p_T/T \to 0, \quad (64)$$

as $T \to \infty$. Then

$$\sqrt{\frac{T}{M\sigma_T^2}}\left(Q_T - \mathbb{E}(Q_T)\right) \xrightarrow{D} \mathcal{N}(0,1)$$

Set $M = T^{\delta}$, $p_T = T^{\delta_1}$ and $q_T = T^{\delta_2}$, then condition (64) are satisfied when $\delta < \delta_2 < \delta_1 < 1$, $\delta_2 > 1/(s-1)$. Thus proving the result. □

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