Lecture 6

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For more details about the materials covered in this note, see Chapters 5.7 to 5.9 of Resnick [3] and Chapter 1.7 of Durrett [2].

6.1 Product spaces

Definition 6.1. Let $(\Omega_1, \mathcal{F}_1, \mu_1), (\Omega_2, \mathcal{F}_2, \mu_2)$ be two measure spaces.

(i) Product space: $\Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2) : \omega_i \in \Omega_i, i = 1, 2.\}$.

(ii) Product σ-algebra: $\mathcal{F}_1 \times \mathcal{F}_2 = \sigma(\{A_1 \times A_2 : A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2\})$.

(iii) Coordinate (or projection) maps: $\pi_i(\omega_1, \omega_2) = \omega_i$ for $i = 1, 2$. Note that π_i is a mapping from $\Omega_1 \times \Omega_2$ to Ω_i.

(iv) For $A \subset \Omega_1 \times \Omega_2$, the section of A at ω_1 is defined by

$$A_{\omega_1} = \{\omega_2 : (\omega_1, \omega_2) \in A\} \subset \Omega_2.$$

Similarly, we can define the section of A at ω_2.[1]

(v) For a real valued function f defined on $\Omega_1 \times \Omega_2$, the section of f at ω_1 is defined by $f_{\omega_1}(\omega_2) = f(\omega_1, \omega_2)$. So f_{ω_1} is a mapping from Ω_2 to $\mathbb{R}$.

(vi) If $A_i \subset \Omega_i$ for $i = 1, 2$, then we call $A_1 \times A_2$ a rectangle. Further, we say it is measurable if $A_i \in \mathcal{F}_i$ for $i = 1, 2$. Note that some authors use "rectangles" to refer to "measurable rectangles".

Example 6.1. When Ω_1, Ω_2 are countable and $\mathcal{F}_i = \mathcal{P}(\Omega_i)$ for $i = 1, 2$, we have $\mathcal{F}_1 \times \mathcal{F}_2 = \mathcal{P}(\Omega_1 \times \Omega_2)$. Another special case is the Borel σ-algebra on $\mathbb{R}^2$. It can be shown that $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$.

Example 6.2. Let $\Omega = [0, 1]$ and equip it with the σ-algebra generated by all one-point sets, which we denote by $\mathcal{C}$. Consider the product space $(\Omega \times \Omega, \mathcal{C} \times \mathcal{C})$. Define the diagonal set $D = \{(\omega, \omega) : \omega \in \Omega\}$. Its sections are clearly measurable with respect to $\mathcal{C}$ but it can be shown that $D \notin \mathcal{C} \times \mathcal{C}$.

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[1] This notation for sections can be confusing. We will not use it in other lectures.

Lemma 6.1. *The collection of all measurable rectangles is a semi-algebra.*

Proof. See page 144 of Resnick [3]. □

Proposition 6.1. *Properties of sections.*

(i) *If* $A \subset \Omega_1 \times \Omega_2$, *then* $(A^c)_{\omega_1} = (A_{\omega_1})^c$.

(ii) *If for a set* T, *we have* $A_t \subset \Omega_1 \times \Omega_2$ *for all* $t \in T$, *then*

$$\left(\bigcup_t A_t \right)_{\omega_1} = \bigcup_t (A_t)_{\omega_1}, \quad \left(\bigcap_t A_t \right)_{\omega_1} = \bigcap_t (A_t)_{\omega_1}.$$

(iii) *If* f, g *are functions defined on* $\Omega_1 \times \Omega_2$, *then* $(f+g)_{\omega_1} = f_{\omega_1} + g_{\omega_1}$.

(iv) *Let* f_n *be a sequence of functions defined on* $\Omega_1 \times \Omega_2$ *such that* $f_n \to f$. *Then,* $\lim_{n\to\infty} (f_n)_{\omega_1} = f_{\omega_1}$.

Proof. Try it yourself. □

Lemma 6.2. *If a set* $A \in \mathcal{F}_1 \times \mathcal{F}_2$, *then for all* $\omega_1 \in \Omega_1$, *we have* $A_{\omega_1} \in \mathcal{F}_2$.

Proof. Define $\mathcal{C}_{\omega_1} = \{A \subset \Omega_1 \times \Omega_2 \colon A_{\omega_1} \in \mathcal{F}_2\}$. If A is a measurable rectangle, we can write it as $A = A_1 \times A_2$ and thus

$$A_{\omega_1} = \{\omega_2 : (\omega_1, \omega_2) \in A_1 \times A_2\} = \begin{cases} A_2 \in \mathcal{F}_2, & \text{if } \omega_1 \in A_1, \\ \emptyset, & \text{if } \omega_1 \notin A_1. \end{cases}$$

Thus, all the measurable rectangles belong to $\mathcal{C}_{\omega_1}$.

Next, we prove $\mathcal{C}_{\omega_1}$ is a λ-system.

(1) Clearly $\Omega_1 \times \Omega_2 \in \mathcal{C}_{\omega_1}$ since $\Omega_1 \times \Omega_2$ is a measurable rectangle.

(2) If $A \in \mathcal{C}_{\omega_1}$, then $A^c \in \mathcal{C}_{\omega_1}$ since $(A^c)_{\omega_1} = (A_{\omega_1})^c$ by Proposition 6.1.

(3) Consider a sequence of disjoint sets $A_1, A_2, \dots$ such that $A_n \in \mathcal{C}_{\omega_1}$ for $n = 1, 2, \dots$. Since $(A_n)_{\omega_1} \in \mathcal{F}_2$, we have $\cup_n (A_n)_{\omega_1} \in \mathcal{F}_2$. By Proposition 6.1, this further implies $(\cup_n A_n)_{\omega_1} \in \mathcal{F}_2$ and thus $\cup_n A_n \in \mathcal{C}_{\omega_1}$.

By Lemma 6.1, the collection of all measurable rectangles is a π-system. Hence, by Dynkin's theorem, the σ-algebra generated by this π-system is contained in $\mathcal{C}_{\omega_1}$; that is, $\mathcal{F}_1 \times \mathcal{F}_2 \subset \mathcal{C}_{\omega_1}$. □

Corollary 6.1. *Let* $f \colon (\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. *Then* $f_{\omega_1} \in \mathcal{F}_2$.

Proof. Try it yourself. □

6.2 Product measures

Theorem 6.1. *Let* $(\Omega_1, \mathcal{F}_1, \mu_1), (\Omega_2, \mathcal{F}_2, \mu_2)$ *be two* σ*-finite measure spaces. Then there is a unique measure* μ *on* $(\Omega_1 \times \Omega_2,\ \mathcal{F}_1 \times \mathcal{F}_2)$ *such that*

$$\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2),$$

for any $A_1 \times A_2 \in \mathcal{F}_1 \times \mathcal{F}_2$*. We write* $\mu = \mu_1 \times \mu_2$ *and call it product measure.*

Proof. For any $A_1 \times A_2 \in \mathcal{F}_1 \times \mathcal{F}_2$, define $\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$. By the extension theorems and Lemma 6.1, we only need to prove μ is a pre-measure (i.e. a σ-additive function) on the collection of all measurable rectangles and μ is σ-finite. The latter is easy: one can show that the σ-finiteness of μ_1 and μ_2 implies that μ is σ-finite.

To prove μ is σ-additive for measurable rectangles, let $\{A_{n,1} \times A_{n,2} : n = 1, 2, \dots\}$ be a collection of disjoint measurable rectangles such that $\cup_n (A_{n,1} \times A_{n,2}) = A_1 \times A_2$ for some $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2$. We need to show that $\mu(A_1 \times A_2) = \sum_{n=1}^{\infty} \mu(A_{n,1} \times A_{n,2})$. By the definition of μ,

$$\begin{aligned} \mu(A_1 \times A_2) &= \mu_1(A_1) \times \mu_2(A_2) \\ &= \left(\int_{\Omega_1} \mathbb{1}_{A_1} d\mu_1 \right) \mu_2(A_2) \\ &= \int_{\Omega_1} \mu_2(A_2) \mathbb{1}_{A_1}(\omega_1) \mu_1(d\omega_1). \end{aligned}$$

The second line follows from the definition of the Lebesgue integral for simple functions. Observe that

$$\mu_2(A_2)\mathbb{1}_{A_1}(\omega_1) = \sum_{n=1}^{\infty} \mu_2(A_{n,2}) \mathbb{1}_{A_{n,1}}(\omega_1),$$

because $A_1 \times A_2$ is a measurable rectangle set and μ_2, as a measure, is σ-additive for disjoint subsets of Ω_2. Hence, by the monotone convergence theorem, we have

$$
\begin{aligned}
\mu(A_1 \times A_2) &= \int_{\Omega_1} \left(\sum_{n=1}^{\infty} \mu_2(A_{n,2}) \mathbb{1}_{A_{n,1}}(\omega_1) \right) \mu_1(d\omega_1) \\
&= \sum_{n=1}^{\infty} \int_{\Omega_1} \mu_2(A_{n,2}) \mathbb{1}_{A_{n,1}}(\omega_1) \mu_1(d\omega_1) \\
&= \sum_{n=1}^{\infty} \mu_2(A_{n,2}) \int_{\Omega_1} \mathbb{1}_{A_{n,1}}(\omega_1) \mu_1(d\omega_1) \\
&= \sum_{n=1}^{\infty} \mu_1(A_{n,1}) \mu_2(A_{n,2}) \\
&= \sum_{n=1}^{\infty} \mu(A_{n,1} \times A_{n,2}),
\end{aligned}
$$

which completes the proof. □

Example 6.3. We can construct independent random variables (to be defined in the next lecture) using product measure. Assume μ_1, μ_2 are probability measures. Let $X_i \colon (\Omega_i, \mathcal{F}_i, \mu_i) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and consider the product space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mu_1 \times \mu_2)$. Define

$$
X_1^*(\omega_1, \omega_2) = X_1(\omega_1), \qquad X_2^*(\omega_1, \omega_2) = X_2(\omega_2).
$$

They are both random variables on the product space. Now using the definition of rectangle sets and product measure, we can find that

$$
\begin{aligned}
&\mu_1 \times \mu_2(\{(\omega_1, \omega_2) \colon X_1^* \le x_1, X_2^* \le x_2\}) \\
&= \mu_1(\{\omega_1 \colon X_1(\omega_1) \le x_1\})\, \mu_2(\{\omega_2 \colon X_2(\omega_2) \le x_2\}) \\
&= \mu_1 \times \mu_2(\{(\omega_1, \omega_2) \colon X_1^* \le x_1\}) \mu_1 \times \mu_2(\{(\omega_1, \omega_2) \colon X_2^* \le x_2\}).
\end{aligned}
$$

In the next lecture, we will see that this implies X_1^* and X_2^* are independent under the measure $\mu_1 \times \mu_2$.

6.3 Fubini's theorem

Theorem 6.2 (Fubini's theorem)**.** *Let $(\Omega_1, \mathcal{F}_1, \mu_1), (\Omega_2, \mathcal{F}_2, \mu_2)$ be two σ-finite measure spaces and let $f \colon (\Omega_1 \times \Omega_2,\ \mathcal{F}_1 \times \mathcal{F}_2) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$. Then if* *either $f \geq 0$ or f is integrable (with respect to $\mu_1 \times \mu_2$), we have*

$$\int_{\Omega_1 \times \Omega_2} f d(\mu_1 \times \mu_2) = \int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1, \omega_2) \mu_2(d\omega_2) \right) \mu_1(d\omega_1)$$
$$= \int_{\Omega_2} \left(\int_{\Omega_1} f(\omega_1, \omega_2) \mu_1(d\omega_1) \right) \mu_2(d\omega_2).$$

When $f \geq 0$, this result is also known as Tonelli's theorem.

Proof. See the textbook. □

Example 6.4. Consider two measure spaces $(\Omega_1 = [0,1], \mathcal{B}([0,1]), m)$ and $(\Omega_2 = [0,1], \mathcal{P}([0,1]), \#)$ where m denotes the Lebesgue measure and $\#$ is the counting measure. On the product space we define a measurable function $f(\omega_1, \omega_2) = \mathbb{1}(\omega_1 = \omega_2)$. However,

$$\int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1, \omega_2) \#(d\omega_2) \right) m(d\omega_1) = \int_{\Omega_1} \left(\int_{\Omega_2} \mathbb{1}_{\{\omega_1\}}(\omega_2) \#(d\omega_2) \right) m(d\omega_1) = 1.$$
$$\int_{\Omega_2} \left(\int_{\Omega_1} f(\omega_1, \omega_2) m(d\omega_1) \right) \#(d\omega_2) = \int_{\Omega_2} \left(\int_{\Omega_1} \mathbb{1}_{\{\omega_2\}}(\omega_1) m(d\omega_1) \right) \#(d\omega_2) = 0.$$

Note that Fubini's theorem cannot be applied since $([0,1], \mathcal{P}([0,1]), \#)$ is not σ-finite.

References

[1] Dennis D. Cox. *The Theory of Statistics and Its Applications.* Unpublished.

[2] Rick Durrett. *Probability: Theory and Examples*, volume 49. Cambridge university press, 2019.

[3] Sidney Resnick. *A Probability Path.* Springer, 2019.