Lecture 5

Instructor: Quan Zhou

For more details about the materials covered in this note, see Tao [4], Chapters 5.3 and 5.5 of Resnick [3] and Chapter 1.5 of Durrett [2].

5.1 Convergence of Lebesgue integrals

For all results below, assume the measure space $(\Omega, \mathcal{F}, \mu)$ is given, and $\{f_n\}$ is a sequence of measurable functions taking value in $(\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$. Besides, recall the following convention: if $\{a_n\}$ is a sequence such that each $a_i \in \bar{\mathbb{R}}$ and $a_1 \leq a_2 \leq \cdots$, we write $a_n \uparrow \infty$ or $a_n \to \infty$.

Theorem 5.1 (Monotone convergence theorem)**.** *If $f_n \geq 0$ and $f_n \uparrow f$ (i.e., for every $\omega \in \Omega$, $0 \leq f_1(\omega) \leq f_2(\omega) \leq \cdots$), then $\int f_n d\mu \uparrow \int f d\mu$.*

Proof. Some authors prefer to first prove Fatou's lemma and then use it to prove monotone convergence theorem (MCT). Here we prove MCT first. We assume that we have already proven Proposition 4.1 for non-negative simple functions but not for non-negative measurable functions. And we will use the following definition for $\int f d\mu$,

$$\int f d\mu = \sup \left\{ \int \varphi d\mu \colon 0 \leq \varphi \leq f,\ \varphi \text{ is a simple function} \right\}.$$

The above definition implies that

$$\text{if } 0 \leq g \leq h, \text{ then } \int g d\mu \leq \int h d\mu \tag{1}$$

since $0 \leq \varphi \leq g$ implies $0 \leq \varphi \leq h$. Besides,

$$\forall a \in \mathbb{R}, \quad a \int f d\mu = \int a f d\mu, \tag{2}$$

since $a \int \varphi d\mu = \int a \varphi d\mu$ for every non-negative simple function φ.

For a non-decreasing sequence of functions $\{f_n\}$, $\lim f_n = \sup f_n$. By (1), we have $\int f_n d\mu \leq \int f_{n+1} d\mu$ and thus $\lim \int f_n d\mu = \sup \int f_n d\mu$. Hence, to prove MCT we just need to show that

$$\int \sup_{n \geq 1} f_n d\mu = \sup_{n \geq 1} \int f_n d\mu.$$

By (1), $\int \sup_{n\geq 1} f_n d\mu \geq \int f_n d\mu$ and taking supremum on both sides we obtain that

$$\int \sup_{n\geq 1} f_n d\mu \geq \sup_{n\geq 1} \int f_n d\mu.$$

To prove the other direction, i.e. $\int \sup_{n\geq 1} f_n d\mu \leq \sup_{n\geq 1} \int f_n d\mu$, by our definition of Lebesgue integral, we need to show that for any non-negative simple function φ such that $0 \leq \varphi \leq \sup f_n$, we have

$$\int \varphi d\mu \leq \sup_{n\geq 1} \int f_n d\mu.$$

Choose an arbitrary simple function φ that satisfies $0 \leq \varphi \leq \sup f_n$. It suffices to prove that

$$(1-\epsilon) \int \varphi d\mu \leq \sup_{n\geq 1} \int f_n d\mu,$$

for every $\epsilon \in (0,1)$ since then we can take limit on both sides by letting $\epsilon \downarrow 0$. For any $\epsilon \in (0,1)$ and any $\omega \in \Omega$, there exists some $N = N(\epsilon, \omega)$ such that

$$f_N(\omega) \geq (1-\epsilon)\varphi(\omega),$$

since $\varphi(\omega) \leq \sup f_n(\omega)$. The monotonicity of $\{f_n\}$ implies that for any $n \geq N$, $f_n(\omega) \geq (1-\epsilon)\varphi(\omega)$. Define $E_n = \{\omega \in \Omega \colon f_n(\omega) \geq (1-\epsilon)\varphi(\omega)\}$. We have $E_1 \subset E_2 \subset \cdots$ (by the monotonicity of $\{f_n\}$) and $\cup_{n\geq 1} E_n = \Omega$ (since for every $\omega \in \Omega$, we can find $N(\epsilon, \omega)$.)

Using (1) and (2), we find that

$$(1-\epsilon) \int \varphi \cdot \mathbb{1}_{E_n} d\mu = \int (1-\epsilon)\varphi \cdot \mathbb{1}_{E_n} d\mu \leq \int f_n \cdot \mathbb{1}_{E_n} d\mu \leq \int f_n d\mu.$$

Since φ is simple, we may write $\varphi = \sum_{i=1}^k a_i \mathbb{1}_{A_i}$ for some non-negative real numbers $a_1, \ldots, a_k$ and disjoint sets $A_1, \ldots, A_k$, and

$$\int \varphi d\mu = \sum_{i=1}^k a_i \mu(A_i)$$

Observing that $\varphi \cdot \mathbb{1}_{E_n}$ is also a simple function, we obtain that

$$\int \varphi \cdot \mathbb{1}_{E_n} d\mu = \sum_{i=1}^k a_i \mu(A_i \cap E_n).$$

Taking supremum on both sides we get

$$\sup_{n\geq 1}\int \varphi\cdot \mathbb{1}_{E_n}d\mu = \sup_{n\geq 1}\sum_{i=1}^{k} a_i\mu(A_i\cap E_n) = \lim_{n\to\infty}\sum_{i=1}^{k} a_i\mu(A_i\cap E_n)$$

since $E_1 \subset E_2 \subset \cdots$. By the continuity of measures from below, as $n\to\infty$, $\mu(A_i\cap E_n)\uparrow \mu(A_i)$, which implies that

$$\sup_{n\geq 1}\int \varphi\cdot \mathbb{1}_{E_n}d\mu = \sum_{i=1}^{k} a_i\mu(A_i) = \int \varphi d\mu$$

Multiplying both sides by $(1-\epsilon)$, we get $(1-\epsilon)\int \varphi d\mu \leq \sup_{n\geq 1}\int f_n d\mu$, which concludes the proof. □

Theorem 5.2 (Fatou's lemma)**.** *If $f_n \geq 0$, then*

$$\int (\liminf_{n\to\infty} f_n)d\mu \leq \liminf_{n\to\infty}\int f_n d\mu.$$

Proof. Note that $\liminf_{n\to} f_n = \lim_{n\to\infty}(\inf_{k\geq n} f_k)$, and $\{\inf_{k\geq n} f_k : n = 1,2,\ldots\}$ is a non-negative non-decreasing sequence. So by MCT,

$$\int \liminf_{n\to\infty} f_n d\mu = \int \lim_{n\to\infty}(\inf_{k\geq n} f_k)d\mu = \lim_{n\to\infty}\int (\inf_{k\geq n} f_k)d\mu.$$

By the monotonicity of Lebesgue integrals, $\int(\inf_{k\geq n} f_k)d\mu \leq \inf_{k\geq n}\int f_k d\mu$, from which the result follows. □

Theorem 5.3 (Dominated convergence theorem)**.** *If $f_n \to f$ and there exists an integrable function g such that $|f_n|\leq g$, then $\int f_n d\mu \to \int f d\mu$ and $\int |f_n - f|d\mu \to 0$.*

Proof. By assumption, we have $-g \leq f_n \leq g$. By Fatou's lemma,

$$\liminf_{n\to\infty}\int (f_n+g)d\mu \geq \int \liminf_{n\to\infty}(f_n+g)d\mu = \int (f+g)d\mu,$$
$$\liminf_{n\to\infty}\int (g-f_n)d\mu \geq \int \liminf_{n\to\infty}(g-f_n)d\mu = \int (g-f)d\mu.$$

Canceling the constant $\int g d\mu$ on both sides, we get

$$\liminf_{n\to\infty}\int f_n d\mu \geq \int f d\mu \geq \limsup_{n\to\infty}\int f_n d\mu.$$

Since lim sup is always greater than or equal to lim inf, we conclude that $\int f_n d\mu \to \int f d\mu$. To prove that f_n converges to f in L^1, note that $|f_n - f| \leq 2g$ since $|f_n|$ is bounded by g. So, by the first conclusion of DCT, $\int |f_n - f| d\mu \to 0$. □

Theorem 5.4 (Bounded convergence theorem)**.** *If $f_n \to f$, $\mu(\Omega) < \infty$ and there exists some constant $M < \infty$ such that $f_n(\omega) \leq M$ for every $n \in \mathbb{N}$ and $\omega \in \Omega$, then $\int f_n d\mu \to \int f d\mu$.*

Proof. This follows from DCT. We can simply let $g(\omega) = M$ for every $\omega \in \Omega$. Then g is integrable since $\mu(\Omega) < \infty$. □

Example 5.1. Consider the probability space $([0,1], \mathcal{B}([0,1]), m)$ where m denotes the Lebesgue measure (note that $m([0,1]) = 1$). Define $X_n(\omega) = n$ if $\omega \in (0, 1/n)$ and $X_n(\omega) = 0$ otherwise. Hence $\mathsf{P}(X_n = n) = 1/n$ and $\mathsf{P}(X_n = 0) = 1 - 1/n$. Then $E(X_n) = 1$, but $X_n \to 0$ pointwise. That is, $\lim_{n\to\infty} \int X_n \, d\mathsf{P} = 1$ and $\int \lim_{n\to\infty} X_n \, d\mathsf{P} = 0$.

5.2 Change of variables and densities

Theorem 5.5 (Change of variables)**.** *Let $f\colon (\Omega, \mathcal{F}, \mu) \to (\Lambda, \mathcal{G})$ and $g\colon (\Lambda, \mathcal{G}) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$. Then,*

$$\int_\Omega g(f(\omega))\mu(d\omega) = \int_\Lambda g(\lambda)(\mu \circ f^{-1})(d\lambda),$$

provided that either integral is defined.

Proof. See Durrett [2, Theorem 5.5.1]. □

Example 5.2. Consider a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and a random variable X. Recall that the distribution (or the law) of X is given by the push-forward measure $\mathsf{P} \circ X^{-1}$. Hence, using the change-of-variable formula, we obtain

$$E(X) = \int_\Omega X(\omega)\, \mathsf{P}(d\omega) = \int_{\mathbb{R}} x\, (\mathsf{P} \circ X^{-1})(dx).$$

More generally, for a function $g : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$, we have

$$E[g(X)] = \int_\Omega g(X(\omega))\, \mathsf{P}(d\omega) = \int_{\mathbb{R}} g(x)\, (\mathsf{P} \circ X^{-1})(dx).$$

The change-of-variable formula is also known as transformation theorem (see Resnick's book). A more interesting name is "the law of unconscious statistician". This is because in practice, the expectation of a random variable is almost always computed by evaluating an integral on the real line, though expectation is defined as a Lebesgue integral on the sample space Ω.

Theorem 5.6. *Let* $f : (\Omega, \mathcal{F}, \mu) \to ([0, \infty], \mathcal{B}([0, \infty]))$. *Define*

$$\nu(A) = \int_A f d\mu$$

for any $A \in \mathcal{F}$. *Then* ν *is a measure on* $(\Omega, \mathcal{F})$, *and* f *is called the density of* ν *(with respect to* μ*).*

Proof. Try it yourself. □

References

[1] Dennis D. Cox. *The Theory of Statistics and Its Applications*. Unpublished.

[2] Rick Durrett. *Probability: Theory and Examples*, volume 49. Cambridge university press, 2019.

[3] Sidney Resnick. *A Probability Path*. Springer, 2019.

[4] Terence Tao. *Analysis*, volume 1. Springer, 2006.