Lecture 1

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For more details about the materials covered in this note, see Chapters 1.3 to 1.8 and 2.1 of Resnick [1].

1.1 Set-theoretic limits

Definition 1.1. Limits of sets.

(i) $\inf_{k \geq n} A_k = \bigcap_{k=n}^{\infty} A_k, \quad \sup_{k \geq n} A_k = \bigcup_{k=n}^{\infty} A_k.$

(ii) $\liminf_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k, \quad \limsup_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$

(iii) We write $\lim_{n \to \infty} A_n = A$ iff $\limsup_{n \to \infty} A_n = \liminf_{n \to \infty} A_n = A$.

Example 1.1. Let $A_n = (-1/n, 1 - 1/n]$. Then $\lim_{n \to \infty} A_n = [0, 1)$.

Example 1.2. Define a sequence of sets by $A_{2k-1} = [0, 1]$, $A_{2k} = [0, 2]$ for $k = 1, 2, \ldots$. Then $\liminf_{n \to \infty} A_n = [0, 1]$ and $\limsup_{n \to} A_n = [0, 2]$.

Proposition 1.1. *Monotone sequences of sets.*

(i) We say $\{A_n\}$ is monotone non-decreasing if $A_1 \subset A_2 \subset \cdots$. For such a sequence, $\lim_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} A_n$.

(ii) We say $\{A_n\}$ is monotone non-increasing if $A_1 \supset A_2 \supset \cdots$. For such a sequence, $\lim_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} A_n$.

Proof. Try it yourself. □

Definition 1.2. Indicator function:

$$\mathbb{1}_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \notin A. \end{cases}$$

Other common notation: $I_A(\omega)$, $\mathbb{1}\{\omega \in A\}$, $I(\omega \in A)$, etc.

Proposition 1.2. $\liminf$ *and* $\limsup$ *of sets can also be expressed by*

$$\liminf_{n \to \infty} A_n = \{\omega \in \Omega : \liminf_{n \to \infty} \mathbb{1}_{A_n}(\omega) = 1\} = \{\omega \in \Omega : \sum_{n \geq 1} \mathbb{1}_{A_n^c}(\omega) < \infty\},$$

$$\limsup_{n \to \infty} A_n = \{\omega \in \Omega : \limsup_{n \to \infty} \mathbb{1}_{A_n}(\omega) = 1\} = \{\omega \in \Omega : \sum_{n \geq 1} \mathbb{1}_{A_n}(\omega) = \infty\}.$$

Proof. Try it yourself. ☐

Proposition 1.3. *Properties of* $\liminf$ *and* $\limsup$.

(i) $\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$.

(ii) $(\liminf_{n\to\infty} A_n)^c = \limsup_{n\to\infty} A_n^c$.

(iii) $\liminf_{n\to\infty} A_n = \lim_{n\to\infty}(\inf_{k\geq n} A_k)$.

(iv) $\limsup_{n\to\infty} A_n = \lim_{n\to\infty}(\sup_{k\geq n} A_k)$.

Proof. Try it yourself. ☐

1.2 Probability triple

Definition 1.3. A probability space is a triple $(\Omega, \mathcal{F}, \mathsf{P})$ where

- Ω is the sample space, a set of all possible outcomes;
- $\mathcal{F}$ is a σ-algebra on Ω (a collection of subsets of Ω), a set of events;
- P: $\mathcal{F} \to [0,1]$ is a probability measure.

We will define "σ-algebra" and "probability measure" later, but the following two examples may be self-explanatory.

Example 1.3. Consider only one flip of a fair coin. Then

$$\Omega = \{H, T\}, \quad \mathcal{F} = \{\emptyset, \{H\}, \{T\}, \{H, T\}\},$$
$$\mathsf{P}(\{H\}) = \mathsf{P}(\{T\}) = 0.5, \quad \mathsf{P}(\emptyset) = 0, \quad \mathsf{P}(\{H, T\}) = 1.$$

Example 1.4. Consider two flips of a fair coin. Then $\Omega = \{HH, HT, TH, TT\}$. Each outcome has probability 0.25.

1.3 σ-algebra

Definition 1.4. $\mathcal{F}$ is a σ-algebra (or σ-field) on Ω if it satisfies

- $\Omega \in \mathcal{F}$;
- $\forall\, A \in \mathcal{F}$, we have $A^c \in \mathcal{F}$;

- if $A_1, A_2, \ldots, \in \mathcal{F}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Equivalently, we can say that a σ-algebra $\mathcal{F}$ is a non-empty collection of subsets of Ω closed under countable union, countable intersection and complementation.

Definition 1.5. Let $\mathcal{C}$ be a collection of subsets of Ω. The σ-algebra generated by $\mathcal{C}$, denoted by $\sigma(\mathcal{C})$, is a σ-algebra that satisfies

- $\mathcal{C} \subset \sigma(\mathcal{C})$;
- if $\mathcal{G}$ is another σ-algebra such that $\mathcal{C} \subset \mathcal{G}$, then $\sigma(\mathcal{C}) \subset \mathcal{G}$.

Theorem 1.1. *Let $\mathcal{C}$ be an arbitrary collection of subsets of Ω. There is always a unique minimal σ-algebra that contains $\mathcal{C}$, i.e. $\sigma(\mathcal{C})$ is well defined.*

Proof. First, there always exists some σ-algebra that contains $\mathcal{C}$ since $\mathcal{C} \subset \mathcal{P}(\Omega)$ (see Example 1.5). Complete the proof yourself by taking intersection of all σ-algebras that contain $\mathcal{C}$. □

Example 1.5. Examples of σ-algebra.

(i) $\mathcal{P}(\Omega)$: The power set of Ω; that is, the set of all subsets of Ω.

(ii) $\{\Omega, \emptyset\}$ is the trivial σ-algebra.

(iii) For any $A \subset \Omega$ such that $A \neq \emptyset$ and $A \neq \Omega$, $\{\Omega, A, A^c, \emptyset\}$ is a σ-algebra.

(iv) $\mathcal{B}(\Omega)$: The Borel σ-algebra is the σ-algebra generated by all the open subsets (or equivalently, all the closed subsets) of Ω.

1.4 Borel sets on $\mathbb{R}$

Using the notation defined in Example 1.5, we let $\mathcal{B}(\mathbb{R}) = \sigma(\text{open subsets of } \mathbb{R})$.

Theorem 1.2. *The Borel sets on $\mathbb{R}$ can be equivalently defined as*

$$
\begin{aligned}
\mathcal{B}(\mathbb{R}) &= \sigma\left(\{(a,b)\colon -\infty \leq a \leq b \leq \infty\}\right) \\
&= \sigma\left(\{[a,b)\colon -\infty < a \leq b \leq \infty\}\right) \\
&= \sigma\left(\{[a,b]\colon -\infty < a \leq b < \infty\}\right) \\
&= \sigma\left(\{(-\infty,b]\colon -\infty < b < \infty\}\right).
\end{aligned}
$$

Proof. See the textbook. □

Theorem 1.3. *The Borel σ-algebra on the interval* $(0,1]$ *satisfies* $\mathcal{B}((0,1]) = \mathcal{B}(\mathbb{R}) \cap (0,1]$, *i.e.* $\mathcal{B}((0,1]) = \{A \cap (0,1] \colon A \in \mathcal{B}(\mathbb{R})\}$.

Proof. It can be shown that $\mathcal{B}((0,1]) = \sigma(\{(a,b] \colon 0 \le a \le b \le 1\})$. Then, the result follows from Theorem 1.4 by letting $\Omega = \mathbb{R}$, $\Lambda = (0,1]$ and $\mathcal{S}$ be the collection of all the intervals of the form $(a,b]$ for $-\infty \le a \le b < \infty$. □

Theorem 1.4. *Let $\mathcal{S}$ be a collection of subsets of Ω, and $\Lambda \subset \Omega$. Define $\mathcal{S} \cap \Lambda = \{A \cap \Lambda \colon A \in \mathcal{S}\}$. Then,*

$$\sigma(\mathcal{S} \cap \Lambda) = \sigma(\mathcal{S}) \cap \Lambda = \{A \cap \Lambda \colon A \in \sigma(\mathcal{S})\},$$

where $\sigma(\mathcal{S} \cap \Lambda)$ is understood as a σ-algebra on Λ and $\sigma(\mathcal{S})$ is a σ-algebra on Ω.

Proof. Note that $\mathcal{S} \cap \Lambda$, $\sigma(\mathcal{S}) \cap \Lambda$ are collections of sets. The use of the intersection symbol $\cap$ here is not most standard, but it is convenient. Since $\sigma(\mathcal{S} \cap \Lambda)$ and $\sigma(\mathcal{S})$ are σ-algebras defined on different spaces, we will write $\Omega \setminus A$ (or $\Lambda \setminus A$) instead of A^c (which could be ambiguous).

Step 1. It can be shown that $\sigma(\mathcal{S}) \cap \Lambda$ is a σ-algebra on Λ (see below). Further, because $\mathcal{S} \subset \sigma(\mathcal{S})$, we have $\mathcal{S} \cap \Lambda \subset \sigma(\mathcal{S}) \cap \Lambda$. Hence, by the minimality of $\sigma(\mathcal{S} \cap \Lambda)$, we have $\sigma(\mathcal{S} \cap \Lambda) \subset \sigma(\mathcal{S}) \cap \Lambda$.

Step 2. Define $\mathcal{G} = \{A \subset \Omega \colon A \cap \Lambda \in \sigma(S \cap \Lambda)\}$. Again, we can show $\mathcal{G}$ is a σ-algebra (see below). For any set $B \in \mathcal{S}$, $B \cap \Lambda \in \mathcal{S} \cap \Lambda$ (by definition) and thus $B \cap \Lambda \in \sigma(\mathcal{S} \cap \Lambda)$. Hence, $\mathcal{S} \cap \Lambda \subset \sigma(\mathcal{S} \cap \Lambda)$ and $\mathcal{S} \subset \mathcal{G}$. By the minimality of $\sigma(\mathcal{S})$, $\sigma(\mathcal{S}) \subset \mathcal{G}$. By the definition of $\mathcal{G}$, $\sigma(\mathcal{S}) \cap \Lambda \subset \sigma(\mathcal{S} \cap \Lambda)$.

Step 3. Since $\sigma(\mathcal{S} \cap \Lambda) \subset \sigma(\mathcal{S}) \cap \Lambda \subset \sigma(\mathcal{S} \cap \Lambda)$, we conclude that $\sigma(\mathcal{S}) \cap \Lambda = \sigma(\mathcal{S} \cap \Lambda)$.

To prove $\sigma(\mathcal{S}) \cap \Lambda$ is a σ-algebra on Λ, we check the three postulates in the definition. First, $\Lambda = \Omega \cap \Lambda \in \sigma(\mathcal{S}) \cap \Lambda$ since $\Omega \in \sigma(\mathcal{S})$. Second, if $B \in \sigma(\mathcal{S}) \cap \Lambda$, we can write $B = A \cap \Lambda$ for some $A \in \sigma(\mathcal{S})$. Then $\Lambda \setminus B = \Lambda \setminus (A \cap \Lambda) = (\Omega \setminus A) \cap \Lambda \in \sigma(\mathcal{S}) \cap \Lambda$ since $\Omega \setminus A \in \sigma(\mathcal{S})$. Lastly, if $B_1, B_2, \dots \in \sigma(\mathcal{S}) \cap \Lambda$, we can write $B_i = A_i \cap \Lambda$ for some $A_i \in \sigma(\mathcal{S})$. Then, by the distributive law of sets, $\cup B_i = \cup(A_i \cap \Lambda) = (\cup A_i) \cap \Lambda \in \sigma(\mathcal{S}) \cap \Lambda$ since $\cup A_i \in \sigma(\mathcal{S})$.

Finally, we prove $\mathcal{G}$ is a σ-algebra on Ω. To simplify the notation, let $\mathcal{H} = \sigma(\mathcal{S} \cap \Lambda)$. First, $\Omega \in \mathcal{G}$ since $\Omega \cap \Lambda = \Lambda \in \mathcal{H}$ (recall $\mathcal{H}$ is a σ-algebra on Λ.) Second, if $A \in \mathcal{G}$, $A \cap \Lambda \in \mathcal{H}$. Then, $(\Omega \setminus A) \cap \Lambda = \Lambda \setminus (A \cap \Lambda) \in \mathcal{H}$. By the definition of $\mathcal{G}$, this means $\Omega \setminus A \in \mathcal{G}$. Lastly, if $A_1, A_2, \dots \in \mathcal{G}$, then $A_i \cap \Lambda \in \mathcal{H}$. This implies that $(\cup_i A_i) \cap \Lambda = \cup_i (A_i \cap \Lambda) \in \mathcal{H}$; that is, $\cup_i A_i \in \mathcal{G}$. □

References

[1] Sidney Resnick. *A Probability Path*. Springer, 2019.