Lecture 0

Instructor: Quan Zhou

We will not go through the materials covered in this note in class, since you should have already learned them from an introductory real analysis course. A good understanding of the following basic concepts and results is necessary to the successful completion of the course.

0.1 Basic set theory

Definition 0.1. Cardinality of a set.

(i) If there exists a bijection from the set A to $\{1, 2, \ldots, n\}$ for some natural number n, we say A is finite.

(ii) If there exists a bijection from the set A to all natural numbers $\mathbb{N} = \{0, 1, 2, \ldots, \}$, we say A is a countably infinite.

(iii) We say A is countable if it is finite or countably infinite.

(iv) We say A is uncountable if it is not countable.

Example 0.1. The set of all real numbers, $\mathbb{R}$, is uncountable. The set of all rational numbers, $\mathbb{Q}$, is countable.

Definition 0.2. Some basic definitions in set theory.

(i) If ω is an element in A, we write $\omega \in A$. If A is a subset of B (i.e. every element in A is also in B), we write $A \subset B$ (note that A may equal B.)

(ii) Complement of A: $A^c = \{\omega \in \Omega \colon \omega \notin A\}$.

(iii) Set difference: $A \setminus B = \{\omega \colon \omega \in A \text{ and } \omega \notin B\}$.

(iv) Intersection: $\bigcap_{t \in T} A_t = \{\omega \in \Omega \colon \omega \in A_t, \ \forall\, t \in T\}$.

(v) Union: $\bigcup_{t \in T} A_t = \{\omega \in \Omega \colon \exists\, t \in T, \text{ s.t. } \omega \in A_t\}$.

(vi) $\{A_t \colon \ t \in T\}$ is *pairwise disjoint* (or *mutually disjoint*) if whenever $t, t' \in T$ and $t \neq t'$, we have $A_t \cap A_{t'} = \emptyset$.

Theorem 0.1. *An arbitrary (finite, countable or uncountable) union of open sets is open, and the intersection of a finite number of open sets is open.*

Example 0.2. The intersection of an infinite number of open sets can be closed, e.g. $\bigcap_{n=1}^{\infty}(-1/n,\, 1/n) = \{0\}$. Similarly, the union of an infinite number of closed sets can be open, e.g. $\bigcup_{n=2}^{\infty}[1/n,\, 1-1/n] = (0,1)$.

Remark 0.1. In this course, when we use the terms "open sets" and "closed sets", usually we are referring to open and closed intervals of $\mathbb{R}$. So we do not need the definition of "open sets" in more general contexts.

Proposition 0.1. *Algebra of sets.*

(i) *Associative law:* $(A \cup B) \cup C = A \cup (B \cup C)$.

(ii) *Distributive law:* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(iii) *De Morgan's laws:* $(\cup A_t)^c = \cap(A_t^c)$ *and* $(\cap A_t)^c = \cup(A_t^c)$.

0.2 Sequences and their limits

Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be sequences of real numbers; i.e., $a_n \in \mathbb{R}$ and $b_n \in \mathbb{R}$ for each n.

Definition 0.3. We say the limit of $(a_n)_{n\geq 1}$ exists and write $\lim_{n\to\infty} a_n = a$ if $a \in \mathbb{R}$ and for any $\epsilon > 0$, there exists some $N_\epsilon < \infty$ such that $|a_n - a| < \epsilon$ for all $n \geq N_\epsilon$.

Proposition 0.2 (Limit laws)**.** *Suppose* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$. *Then the following formulae hold.*

(i) $\lim_{n\to\infty}(a_n + b_n) = a + b$.

(ii) $\lim_{n\to\infty}(a_n b_n) = ab$.

(iii) $\lim_{n\to\infty}(c\, a_n) = ac$ *for any* $c \in \mathbb{R}$.

(iv) *If* $a_n > 0$ *for each* n *and* $a > 0$, $\lim_{n\to\infty} a_n^{-1} = a^{-1}$.

(v) $\lim_{n\to\infty} \max(a_n, b_n) = \max(a, b)$.

(vi) $\lim_{n\to\infty} \min(a_n, b_n) = \min(a, b)$.

Definition 0.4. For any $A \subset \mathbb{R}$, we define $\sup(A)$ as follows.

- If A is empty, let $\sup(A) = -\infty$.
- If A is non-empty and has no upper bound, let $\sup(A) = \infty$.
- If A is non-empty and has an upper bound, let $\sup(A)$ be the least upper bound of A.

Define the infimum of A by $\inf(A) = -\sup\{x\colon -x \in A\}$. For the real sequence $(a_n)_{n\geq 1}$, define its supremum by $\sup_{n\geq 1} a_n = \sup\{a_n\colon n \geq 1\}$.

Definition 0.5. Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ denote the extended real number system. For any $A \subset \bar{\mathbb{R}}$, we define its supremum by

$$\sup A = \begin{cases} \infty, & \text{if } \infty \in A, \\ \sup(A \setminus \{-\infty\}), & \text{if } \infty \notin A. \end{cases}$$

Theorem 0.2 (Monotone convergence theorem)**.** *Let $(a_n)_{n\geq 1}$ be a monotone increasing sequence, i.e., $a_{n+1} \geq a_n$ for each n. If $\sup_{n\geq 1} a_n < \infty$, then $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} a_n = \sup_{n\geq 1} a_n$.*

Remark 0.2. For a monotone increasing sequence $(a_n)_{n\geq 1}$ with $\sup_{n\geq 1} a_n = \infty$, we adopt the convention that $\lim_{n\to\infty} a_n$ is defined to be ∞, though we still say a_n diverges in this case.

Definition 0.6. The limit supremum and limit infimum of a_n are defined by

$$\limsup_{n\to\infty} a_n = \inf_{n\geq 1} \sup_{m\geq n} a_m, \qquad \liminf_{n\to\infty} a_n = \sup_{n\geq 1} \inf_{m\geq n} a_m.$$

Remark 0.3. Note that $b_n = \inf_{m\geq n} a_m$ is a monotone increasing sequence, and thus the limit of b_n exists by the monotone convergence theorem (the limit may be ∞). So the names "limit infimum" and "limit supremum" are justified.

Proposition 0.3. *We always have $\liminf a_n \leq \limsup a_n$. Further, a_n converges to some real number c if and only if $\limsup a_n = \liminf a_n = c$.*

Proposition 0.4 (Comparison principle)**.** *Suppose $a_n \leq b_n$ for each n. Then, we have*

$$\sup a_n \leq \sup b_n, \quad \inf a_n \leq \inf b_n,$$
$$\limsup a_n \leq \limsup b_n, \quad \liminf a_n \leq \liminf b_n.$$

Remark 0.4. The squeeze theorem (i.e., sandwich theorem) can be seen as a corollary of the comparison principle.

Proposition 0.5. *Let $a \in \mathbb{R}$. Then, a_n converges to a if and only if every subsequence of a_n converges to a.*

0.3 Series

Given a sequence a_n of real numbers, we can use the expression $\sum_{n=1}^{\infty} a_n$ to denote a series, which is a sum of (countably) infinitely many terms.

Definition 0.7. Let $S_n = \sum_{i=1}^{n} a_i$ for each n. If $\lim_{n\to\infty} S_n = A \in \mathbb{R}$, we say the series $\sum_{i=1}^{\infty} a_i$ is convergent and write $\sum_{i=1}^{\infty} a_i = A$. We say $\sum_{i=1}^{\infty} a_i$ is absolutely convergent if the series $\sum_{i=1}^{\infty} |a_i|$ is convergent.

Remark 0.5. By Theorem 0.2 and Remark 0.2, if $a_n \geq 0$ for each n, then $\sum_{i=1}^{\infty} a_n$ is always defined and may equal ∞.

Proposition 0.6. *Let p denote a real number. The series $\sum_{n=1}^{\infty} n^{-p}$ is convergent if and only if $p > 1$.*

Proposition 0.7 (Series laws)**.** *Suppose $\sum_{i=1}^{\infty} a_i = A$ and $\sum_{i=1}^{\infty} b_i = B$ for some $A, B \in \mathbb{R}$. Then the following statements hold.*

(i) $\sum_{i=1}^{\infty}(a_i + b_i) = A + B$.

(ii) $\sum_{i=1}^{\infty} c\, a_i = cA$ *for any* $c \in \mathbb{R}$.

(iii) $A = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{\infty} a_i$ *for any* $k \in \mathbb{N}$. *(Note that the convergence of the series* $\sum_{i=k+1}^{\infty} a_i$ *is part of the conclusion.)*

Proposition 0.8. *If the series $\sum_{i=1}^{\infty} a_i$ is absolutely convergent, then its value does not depend on the order of the summation; any rearrangement of the terms $a_1, a_2, \ldots$ still yields an absolutely convergent series.*

Example 0.3. Consider the series $1/3 - 1/4 + 1/5 - 1/6 + 1/7 - 1/8 + \ldots$. It is convergent but not absolutely convergent. The current arrangement yields a positive sum. But a rearrangement of the series, $1/3 - 1/4 - 1/6 + 1/5 - 1/8 - 1/10 + 1/7 - 1/12 - 1/14 + \ldots$, yields a negative sum.

0.4 Functions and their limits

Consider a function $f\colon \mathbb{R} \to \mathbb{R}$.

Definition 0.8. We write $\lim_{x \downarrow x_0} f(x) = y$ if for any $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that $|f(x) - y| < \epsilon$ for all $x \in (x_0, x_0 + \delta_\epsilon)$. We call $\lim_{x \downarrow x_0} f(x)$ the right limit of f at x_0 and denote it by $f(x_0+)$. The left limit $f(x_0-)$ is defined similarly. When $f(x_0+), f(x_0-)$ both exist and are equal to y, we write $\lim_{x \to x_0} f(x) = y$.

Definition 0.9. We have the following definitions.

(i) We say f is continuous at $x_0 \in \mathbb{R}$ if $\lim_{x \to x_0} f(x) = f(x_0)$.

(ii) We say f is continuous if f is continuous at every $x \in \mathbb{R}$.

(iii) We say f is uniformly continuous if for every $\epsilon > 0$, there exists some $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$.

Proposition 0.9. *Let $f\colon [a, b] \to \mathbb{R}$ for some real numbers $a < b$. If f is continuous on $[a, b]$, then f is also bounded on $[a, b]$.*

Definition 0.10. Consider a sequence of real-valued functions $(f_n)_{n \geq 1}$.

(i) We say f_n converges pointwise to some function f if $\lim_{n \to \infty} f_n(x) = f(x)$ for each x.

(ii) We say f_n converges uniformly to some function f if $\lim_{n \to \infty} \sup_x |f_n(x) - f(x)| = 0$; that is, for every $\epsilon > 0$, there exists some $N_\epsilon \in \mathbb{N}$ such that for each $n \geq N_\epsilon$ and each x, we have $|f_n(x) - f(x)| < \epsilon$.

Remark 0.6. In both "uniformly continuous" and "uniformly convergent", the word "uniformly" essentially means that we can find a bound for the convergence rate that is independent of x. Of course, uniform convergence implies pointwise convergence, which can be proved by checking the definitions.