Unit 9: Optional Sampling Theorem

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9.1 Optional sampling theorem

Theorem 9.1. *Let $(X_n)_{n\geq 0}$ be a uniformly integrable submartingale. Then, for any stopping time T, $\mathsf{E}X_0 \leq \mathsf{E}X_T \leq \mathsf{E}X_\infty$, where $X_\infty = \lim_{n\to\infty} X_n$.*

Proof. The optional sampling theorem for bounded stopping times shows that $\mathsf{E}X_0 \leq \mathsf{E}X_{T\wedge n} \leq \mathsf{E}X_n$, since $T \wedge n$ is a stopping time bounded by n. By Lemma 9.1 below, $(X_{T\wedge n})$ is uniformly integrable, and thus $X_{T\wedge n}$ converges to X_T a.s. and in L^1. Thus,

$$\mathsf{E}X_0 \leq \lim_n \mathsf{E}X_{T\wedge n} = \mathsf{E}X_T \leq \lim_n \mathsf{E}X_n = \mathsf{E}X_\infty.$$

The last equality follows from the uniform integrability of (X_n). □

Remark 9.1. It follows from Theorem 9.1 that if (X_n) is a uniformly integrable martingale, we have $\mathsf{E}[X_T] = \mathsf{E}[X_0]$. A more powerful version of the optional sampling theorem is given in the next theorem.

Theorem 9.2. *Let $S \leq T$ be two stopping times and (X_n) be a submartingale such that $(X_{n\wedge T})_{n\geq 0}$ is uniformly integrable. Then, $\mathsf{E}[X_T \,|\, \mathcal{F}_S] \geq X_S$, a.s.*

Proof. Let $Y_n = X_{n\wedge T}$. Since (Y_n) is uniformly integrable, by Theorem 9.1,

$$\mathsf{E}[X_0] = \mathsf{E}[Y_0] \leq \mathsf{E}[Y_S] = \mathsf{E}[X_S] \leq \mathsf{E}[Y_\infty] = \mathsf{E}[X_T].$$

In particular, we have $\mathsf{E}[X_S] \leq \mathsf{E}[X_T]$.

To prove the asserted inequality, fix an arbitrary event $A \in \mathcal{F}_S$, and define $U = S\mathbb{1}_A + T\mathbb{1}_{A^c}$. Since U is a stopping time by Exercise 9.1 and $U \leq T$, we have

$$\mathsf{E}[X_U] = \mathsf{E}[X_S\mathbb{1}_A + X_T\mathbb{1}_{A^c}] \leq \mathsf{E}[X_T].$$

It follows that

$$\mathsf{E}[X_S\mathbb{1}_A] \leq \mathsf{E}[X_T\mathbb{1}_A] = \mathsf{E}\left[\,\mathsf{E}[X_T\mathbb{1}_A \,|\, \mathcal{F}_S]\,\right] = \mathsf{E}\left[\,\mathbb{1}_A\mathsf{E}[X_T \,|\, \mathcal{F}_S]\,\right].$$

This implies that $\mathsf{E}[X_T \,|\, \mathcal{F}_S] \geq X_S$, a.s. To see this, pick

$$A = \{\mathsf{E}[X_T \,|\, \mathcal{F}_S] \leq X_S\}$$

which is in $\mathcal{F}_S$. □

Remark 9.2. Again, in Theorem 9.2, T is allowed to equal ∞, and we define $X_\infty = \lim_{n\to\infty} X_n$. The existence of X_∞ can be seen as part of the result. This comment applies to Theorem 9.3 as well.

Lemma 9.1. *Let $(X_n)_{n\geq 0}$ be a uniformly integrable submartingale. Then, for any stopping time T, $(X_{n\wedge T})_{n\geq 0}$ is uniformly integrable.*

Proof. We first show $\mathsf{E}|X_T| < \infty$, where we define $X_\infty = \lim_{n\to\infty} X_n$ (which exists due to the uniform integrability). Since $f(x) = x^+$ is convex and non-decreasing, (X_n^+) is a submartingale. Hence, treating $T\wedge n$ as a bounded stopping time, we find that $\mathsf{E}[X_{T\wedge n}^+] \leq \mathsf{E}[X_n^+]$. Clearly, the uniform integrability of (X_n) implies the uniform integrability of (X_n^+). Thus, $\sup_n \mathsf{E}[X_{T\wedge n}^+] \leq \sup_n \mathsf{E}[X_n^+] < \infty$, and $X_{T\wedge n}$ converges a.s. to some random variable Y_∞ with $\mathsf{E}|Y_\infty| < \infty$. But observe that, a.s. $Y_\infty = X_T$.

Next, we check the uniform integrability of $(X_{n\wedge T})$ by definition. Write

$$
\begin{aligned}
&\mathsf{E}\left(|X_{n\wedge T}|\mathbb{1}_{\{|X_{n\wedge T}|>M\}}\right)\\
=\ &\mathsf{E}\left(|X_n|\mathbb{1}_{\{|X_n|>M\}}\mathbb{1}_{\{n\leq T\}}\right) + \mathsf{E}\left(|X_T|\mathbb{1}_{\{|X_T|>M\}}\mathbb{1}_{\{n>T\}}\right)\\
\leq\ &\mathsf{E}\left(|X_n|\mathbb{1}_{\{|X_n|>M\}}\right) + \mathsf{E}\left(|X_T|\mathbb{1}_{\{|X_T|>M\}}\right).
\end{aligned}
$$

Pick $\epsilon > 0$. Since (X_n) is uniformly integrable, there exists $M = M_1(\epsilon)$ such that $\mathsf{E}\left(|X_n|\mathbb{1}_{\{|X_n|>M\}}\right) < \epsilon/2$ for every n. Since $\mathsf{E}|X_T| < \infty$, there exists $M = M_2(\epsilon)$ such that $\mathsf{E}\left(|X_T|\mathbb{1}_{\{|X_T|>M\}}\right) < \epsilon/2$. The uniform integrability of $(X_{n\wedge T})$ then follows. □

Exercise 9.1. Let $S \leq T$ be stopping times and $A \in \mathcal{F}_S$. Show that $U = S\mathbb{1}_A + T\mathbb{1}_{A^c}$ is also a stopping time.

9.2 Special cases of Theorem 9.2

Theorem 9.3. *Let $S \leq T$ be two stopping times and (X_n) be a submartingale. Suppose at least one of the following conditions holds:*

(i) $T \leq m$ *a.s. for some* $m < \infty$.

(ii) There exists $K < \infty$ *such that* $|X_{n\wedge T}| \leq K$ *a.s. for every* n.

(iii) $\mathsf{E}[T] < \infty$, *and there exists* $K < \infty$ *such that* $\mathsf{E}(|X_{n+1} - X_n| \mid \mathcal{F}_n) \leq K$ *a.s. on the event* $\{T > n\}$ *for every* n.

(iv) $\mathsf{E}|X_T| < \infty$ *and* $\mathsf{E}[|X_n| \mathbb{1}_{\{T>n\}}] \to 0$.

Then, $\mathsf{E}[X_T \,|\, \mathcal{F}_S] \geq X_S$*, a.s.*

Proof. We just check that each condition implies that $(X_{n\wedge T})_{n\geq 0}$ is uniformly integrable. For (i) and (ii), the proof is easy and thus omitted.

Consider (iii). Triangle inequality implies that

$$\begin{aligned} |X_{n\wedge T}| &\leq |X_0| + \sum_{j=0}^{(n\wedge T)-1} |X_{j+1} - X_j| \\ &\leq |X_0| + \sum_{j=0}^{\infty} |X_{j+1} - X_j| \mathbb{1}_{\{T>j\}} =: Z. \end{aligned}$$

Since $\mathbb{1}_{\{T>j\}} \in \mathcal{F}_j$, we have

$$\mathsf{E}(|X_{j+1} - X_j| \mathbb{1}_{\{T>j\}}) = \mathsf{E}\left(\mathbb{1}_{\{T>j\}} \mathsf{E}[|X_{j+1} - X_j| \,|\, \mathcal{F}_j] \right) \leq K\, \mathsf{P}(T > j).$$

Hence,

$$\mathsf{E}|Z| \leq \mathsf{E}|X_0| + K \sum_{j=0}^{\infty} \mathsf{P}(T > j) = \mathsf{E}|X_0| + K\mathsf{E}[T].$$

The assumption $\mathsf{E}[T] < \infty$ thus implies that $(X_{n\wedge T})_n$ is dominated by an integrable random variable Z, and thus $(X_{n\wedge T})$ is uniformly integrable.

Consider (iv). We write

$$\mathsf{E}|X_{n\wedge T}| = \mathsf{E}\left(|X_n| \mathbb{1}_{\{T>n\}}\right) + \mathsf{E}\left(|X_T| \mathbb{1}_{\{T\leq n\}}\right).$$

Hence,

$$\mathsf{E}\left[|X_{n\wedge T}| \mathbb{1}_{\{|X_{n\wedge T}|>M\}}\right] \leq \mathsf{E}\left(|X_n| \mathbb{1}_{\{T>n\}} \mathbb{1}_{\{|X_{n\wedge T}|>M\}}\right) + \mathsf{E}\left(|X_T| \mathbb{1}_{\{|X_{n\wedge T}|>M\}}\right).$$

Given any $\epsilon > 0$, we can pick sufficiently large $N = N(\epsilon)$ such that for all $n > N$, $\mathsf{E}\left(|X_n| \mathbb{1}_{\{T>n\}}\right) \leq \epsilon/2$, and then sufficiently large $M = M(N, \epsilon)$ such that $\sup_{1\leq n\leq N} \mathsf{E}\left(|X_n| \mathbb{1}_{\{|X_{n\wedge T}|>M\}}\right) \leq \epsilon/2$ and $\mathsf{E}\left(|X_T| \mathbb{1}_{\{|X_{n\wedge T}|>M\}}\right) < \epsilon/2$. Hence, $(X_{n\wedge T})$ is uniformly integrable. □

Example 9.1. Consider condition (iv) of Theorem 9.3. We give an example which shows that the first condition cannot be dropped. Let $(Z_n)_{n\geq 0}$ be independent random variables such that $Z_n \sim N(0, \sigma_n^2)$ for each n. Let $\sigma_0^2 = 1$, and for each $n \geq 1$, let

$$\sigma_n^2 = a_n^2 - a_{n-1}^2, \quad \text{where } a_n = \frac{n^2}{\log(n+2)}.$$

Define $X_n = Z_0 + Z_1 + \cdots + Z_n$, which is clearly a martingale with respect to $(\mathcal{F}_n)$, where $\mathcal{F}_n = \sigma(Z_0, \ldots, Z_n)$. Let $\chi^{-1}(q)$ denote the q-th quantile of the standard normal distribution, i.e., $\mathsf{P}(Z_0 \leq \chi^{-1}(q)) = q$. Define

$$T = \sum_{n=1}^{\infty} (n+1) \mathbb{1}_{[(n+1)^{-2}, n^{-2})} \chi^{-1}(Z_0),$$

which is a stopping time since $T \in \mathcal{F}_0$. We now prove that

$$\mathsf{E}[|X_n| \mathbb{1}_{\{T > n\}}] \to 0, \quad \text{but } \mathsf{E}|X_T| = \infty,$$

and thus optional sampling theorem does not hold. Let $Y_n = X_n - Z_0$. Using the independence among (Z_n), we find that

$$\mathsf{E}[|X_n| \mathbb{1}_{\{T > n\}}] \leq \mathsf{E}[|Z_0| \mathbb{1}_{\{T > n\}}] + \mathsf{P}(T > n) \mathsf{E}|Y_n|.$$

Observe that $Y_n \sim N(0, a_n^2)$, which yields $\mathsf{E}|Y_n| = C a_n$ for some universal constant C. Since

$$\mathsf{P}(T > n) = \mathsf{P}(\chi^{-1}(Z_0) < n^{-2}) = n^{-2},$$

we have $\mathsf{P}(T > n) \mathsf{E}|Y_n| \to 0$, which further implies $\mathsf{E}[|X_n| \mathbb{1}_{\{T > n\}}] \to 0$. However, $\mathsf{E}|X_T| \geq \mathsf{E}|Y_T| = \sum_{n=1}^{\infty} C a_n \mathsf{P}(T = n) = \infty$.

Example 9.2. Consider Example 1.1, the expected number of flips until we see HHHH. To solve this problem, let $Z_1, Z_2, \ldots$ be i.i.d. random variables such that $\mathsf{P}(Z_1 = 1) = \mathsf{P}(Z_1 = 0) = 1/2$; we interpret $Z_n = 1$ as the n-th flip landing on heads and $Z_n = 0$ as the n-th flip landing on tails. Let $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$. My betting strategy is as follows: at each flip, I bet all my remaining balance (all the money I have from previous bets) plus one additional dollar on heads. Hence, my balance can be represented by $(X_n)_{n \geq 0}$, where $X_0 = 0$ and, for $n \geq 1$,

$$X_n = 2 Z_n (1 + X_{n-1}).$$

Let $Y_n = X_n - n$ denote the net profit after n flips. $(Y_n)_{n\geq 0}$ is a martingale, since $\mathsf{E}[X_n \,|\, \mathcal{F}_{n-1}] = X_{n-1} + 1$. Let

$$T = \min\{n \geq 4 \colon Z_{n-3} = Z_{n-2} = Z_{n-1} = Z_n = 1\}.$$

Clearly, T is a stopping time. Further, we can prove that $\mathsf{E}T < \infty$; see Exercise 9.2. Before time T, X_n is bounded, since we can only have at most 3 consecutive heads and $X_n = 0$ whenever $Z_n = 0$. Hence, condition (iii) of Theorem 9.3 is satisfied, and we have $\mathsf{E}[Y_T] = \mathsf{E}[X_T] - T = 0$. A direct calculation gives $X_T = 30$.

Alternative construction. Here is a more general construction of the betting strategy. At each flip, I bet one dollar on the next four flips being HHHH. That is, for $n \geq 4$, my balance X_n is given by

$$\begin{aligned} X_n = X_{n-4} &+ 2^4 \cdot \mathbb{1}_{\{Z_{n-3}=Z_{n-2}=Z_{n-1}=Z_n=1\}} + 2^3 \cdot \mathbb{1}_{\{Z_{n-2}=Z_{n-1}=Z_n=1\}} \\ &+ 2^2 \cdot \mathbb{1}_{\{Z_{n-1}=Z_n=1\}} + 2 \cdot \mathbb{1}_{\{Z_n=1\}}. \end{aligned}$$

The same argument yields that $\mathsf{E}[Y_T] = \mathsf{E}[X_T] - T = 0$, where $Y_n = X_n - n$. It is clear that $X_T = 2^4 + 2^3 + 2^2 + 2 = 30$. This method can be used to quickly find answers for other patterns; e.g., the number of expected flips needed to get HTHH is $2^4 + 2 = 18$.

Example 9.3. Three people play the following game. At each round, two of them are randomly selected, and first one gives 1 dollar to the other. When one player has no remaining dollars, this player leaves the game, and the other two continue. Let x, y, z (integers) denote the initial number of dollars of the three players. What is the expected number of rounds until one of them has all $x + y + z$ dollars?

To find the answer, let X_n, Y_n, Z_n denote the number of dollars of the 3 players after n rounds; set $X_0 = x, Y_0 = y$ and $Z_0 = z$. It is easy to verify that $W_n = X_nY_n + X_nZ_n + Y_nZ_n + n$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$ where $\mathcal{F}_n = \sigma((X_k, Y_k, Z_k)_{0\leq k\leq n})$. Let $T = \min\{n \geq 0 \colon \max(X_n, Y_n, Z_n) = x + y + z\}$ denote the time that the game ends. As in the last example, one can show that $\mathsf{E}T < \infty$. Further, the increment $|W_{n+1} - W_n|$ is clearly bounded before the game ends. Hence, by Theorem 9.3, $\mathsf{E}[W_T] = \mathsf{E}[W_0]$. Since $W_T = T$, and $\mathsf{E}[W_0] = xy + xz + yz$, we find that $T = xy + xz + yz$.

Exercise 9.2. Consider Example 9.2. Prove that $\mathsf{E}T < \infty$.

Exercise 9.3. Consider n people, each with a unique hat. They now mix the n hats together and play the following game. At each round, each remaining person randomly selects one hat. Those selecting their own hats leave the game, and the others mix their hats together again. Let R denote the number of rounds needed until everyone has her own hat. Find $\mathsf{E}[R]$.

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