Unit 8: Convergence in L^1

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8.1 Review of uniform integrability (R: §6.5; D: §4.6)

Definition 8.1. Let $\{X_t : t \in T\}$ be a family of integrable random variables (i.e. $E|X_t| < \infty$ for each t). We say this family is uniformly integrable if as $M \to \infty$, we have

$$\sup_{t \in T} \mathsf{E}\left(|X_t| \mathbb{1}_{\{|X_t| > M\}}\right) = \sup_{t \in T} \int_{\{|X_t| > M\}} |X_t| \mathrm{d}\mathsf{P} \to 0.$$

Proposition 8.1. *If there exists a random variable Z such that $\mathsf{E}|Z| < \infty$ and $|X_t| \leq Z$ for every t, then $\{X_t\}$ is uniformly integrable.*

Proof. Try it yourself. □

Remark 8.1. Proposition 8.1 implies that if $\mathsf{E}(\sup_{t \in T} |X_t|) < \infty$, then $\{X_t\}_{t \in T}$ is uniformly integrable. If $\{X_t\}_{t \in T}$ is uniformly integrable, one can show that $\sup_t \mathsf{E}|X_t| < \infty$ using $\mathsf{E}|X_t| \leq \mathsf{E}\left(|X_t| \mathbb{1}_{\{|X_t| > M\}}\right) + M$.

Corollary 8.1. *If $\mathsf{E}|X_t| < \infty$ for each t and the index set T is finite, then $\{X_t\}$ is uniformly integrable.*

Proof. Try it yourself. □

Proposition 8.2. *Consider two families of integrable random variable $\{X_t : t \in T\}$ and $\{Y_t : t \in T\}$. If $|X_t| \leq |Y_t|$ for each t and $\{Y_t\}$ is uniformly integrable, then $\{X_t\}$ is uniformly integrable.*

Proof. Try it yourself. □

Example 8.1. Let $\{X_n\}$ be a sequence of random variables with $\mathsf{P}(X_n = n) = 1/n$ and $\mathsf{P}(X_n = 0) = 1 - 1/n$. Clearly, $\sup \mathsf{E}|X_n| = 1$. However, this family is not uniformly integrable because

$$\int_{\{X_n > M\}} X_n \, \mathrm{d}\mathsf{P} = \begin{cases} 1 & \text{if } M \leq n, \\ 0 & \text{if } M > n, \end{cases}$$

which yields $\sup_{n \geq 1} \int_{\{X_n > M\}} X_n \, \mathrm{d}\mathsf{P} = 1$ for every M.

Theorem 8.1. *Suppose $p \in [1, \infty)$ and $\{X_n\}$ is a sequence of random variables such that $\mathsf{E}|X_n|^p < \infty$ for all n. Then $X_n \xrightarrow{L^p} X$ if and only if (i) $\{|X_n|^p\}$ is uniformly integrable and (ii) $X_n \xrightarrow{P} X$.*

Proof. See the textbook. □

Exercise 8.1 (Crystal ball condition)**.** For any $p > 0$, the family $\{|X_t|^p\}$ is uniformly integrable if $\sup_n \mathsf{E}|X_n|^{p+\delta} < \infty$ for some $\delta > 0$.

8.2 Convergence of submartingales in L^1

Theorem 8.2. *Let (X_n) be a submartingale. The following statements are equivalent:*

(i) *(X_n) is uniformly integrable.*

(ii) *$(X_n) \xrightarrow{L^1} X_\infty$ for some random variable X_∞.*

(iii) *$(X_n) \xrightarrow{L^1} X_\infty$ and $(X_n) \xrightarrow{a.s.} X_\infty$ for some random variable X_∞.*

Proof. Clearly, (iii) implies (ii). By Theorem 8.1, (ii) implies (i). It only remains to show that (i) implies (iii). By Theorem 5.1 and Remark 8.1, uniform integrability implies that $X_n \xrightarrow{\text{a.s.}} X_\infty$, and Theorem 8.1 shows that we also have the convergence in L^1. □

Theorem 8.3. *Let (X_n) be a martingale such that $X_n \xrightarrow{L^1} X_\infty$. Then $X_n = \mathsf{E}[X_\infty \,|\, \mathcal{F}_n]$.*

Proof. According to the definition of conditional expectations, it suffices to show that for any $A \in \mathcal{F}_m$, we have $\mathsf{E}[X_\infty \mathbb{1}_A] = \mathsf{E}[X_m \mathbb{1}_A]$. Scheffe's lemma implies that $\mathsf{E}[X_n \mathbb{1}_A] \to \mathsf{E}[X_\infty \mathbb{1}_A]$ as $n \to \infty$. And the martingale property implies that for any $n > m$, a.s.,

$$\mathsf{E}[X_n \mathbb{1}_A] = \mathsf{E}[\,\mathsf{E}[X_n \mathbb{1}_A \,|\, \mathcal{F}_m]] = \mathsf{E}[X_m \mathbb{1}_A].$$

In other words, a.s., $\mathsf{E}[X_n \mathbb{1}_A]$ can only converge to $\mathsf{E}[X_m \mathbb{1}_A]$. Thus, we have $\mathsf{E}[X_m \mathbb{1}_A] = \mathsf{E}[X_\infty \mathbb{1}_A]$. □

8.3 Convergence of conditional expectations

Theorem 8.4. *Let* X *be defined on* $(\Omega, \mathcal{F}, \mathsf{P})$ *such that* $\mathsf{E}|X| < \infty$*. Then,*

$$\{\mathsf{E}[X \mid \mathcal{G}] \colon \mathcal{G} \subset \mathcal{F} \text{ is a } \sigma\text{-algebra on } \Omega\}$$

is uniformly integrable.

Proof. Pick any σ-algebra $\mathcal{G} \subset \mathcal{F}$ and let $Y = \mathsf{E}[X \mid \mathcal{G}]$. Jensen's inequality implies that $|Y| \leq Z = \mathsf{E}[|X| \mid \mathcal{G}]$. Further, for any $A \in \mathcal{G}$,

$$\begin{aligned} \mathsf{E}[|Y|\mathbb{1}_A] &\leq \mathsf{E}[Z\mathbb{1}_A] = \mathsf{E}\left[\,\mathsf{E}[|X| \mid \mathcal{G}]\mathbb{1}_A\,\right] \\ &= \mathsf{E}\left[\,\mathsf{E}[|X|\mathbb{1}_A \mid \mathcal{G}]\,\right] = \mathsf{E}[|X|\mathbb{1}_A]. \end{aligned}$$

In particular, this holds for $A = \{|Y| > M\}$ where M can be any constant.

Now pick arbitrarily $\epsilon > 0$ and let $\delta = \delta(\epsilon)$ be as given in Exercise 8.2. Since $\mathsf{E}|X| < \infty$ and

$$\mathsf{P}(|Y| > M) \leq \mathsf{P}(Z > M) \leq M^{-1}\mathsf{E}[Z] = M^{-1}\mathsf{E}|X|,$$

we can pick sufficiently large $M = M(\epsilon)$ such that $\mathsf{P}(|Y| > M) \leq \delta$, and Exercise 8.2 implies that

$$\epsilon \geq \mathsf{E}\left[|X|\mathbb{1}_{\{|Y|>M\}}\right] \geq \mathsf{E}\left[|Y|\mathbb{1}_{\{|Y|>M\}}\right].$$

Since both ϵ and Y are arbitrary, this proves the result. □

Theorem 8.5. *Let* $\mathcal{F}_\infty = \sigma(\cup_{n\geq 0}\mathcal{F}_n)$ *and* X *be integrable. Then,* $\mathsf{E}[X \mid \mathcal{F}_n]$ *converges to* $\mathsf{E}[X \mid \mathcal{F}_\infty]$ *a.s. and in* L^1*.*

Proof. Recall that $Y_n = \mathsf{E}[X \mid \mathcal{F}_n]$ is a martingale. By Theorem 8.4, Y_n converges to some limiting random variable $Y_\infty \in \mathcal{F}_\infty$ both a.s. and in L^1. Further, Theorem 8.3 implies that

$$\mathsf{E}[Y_\infty \mid \mathcal{F}_n] = Y_n = \mathsf{E}[X \mid \mathcal{F}_n].$$

That is, $\mathsf{E}[Y_\infty\mathbb{1}_A] = \mathsf{E}[X\mathbb{1}_A]$ for any $A \in \cup_{n\geq 1}\mathcal{F}_n$. Dynkin's $\pi - \lambda$ theorem then shows that this holds for any $A \in \sigma(\cup_{n\geq 1}\mathcal{F}_n) = \mathcal{F}_\infty$; that is, $Y_\infty = \mathsf{E}[X \mid \mathcal{F}_\infty]$. □

Exercise 8.2. Let X be defined on $(\Omega, \mathcal{F}, \mathsf{P})$ such that $\mathsf{E}|X| < \infty$. For any $\epsilon > 0$, there exists some $\delta = \delta(\epsilon) > 0$ such that

$$\sup_{A\in\mathcal{F}\colon \mathsf{P}(A)\leq\delta} \mathsf{E}[|X|\mathbb{1}_A] \leq \epsilon.$$

Exercise 8.3. Let $\mathcal{F}_\infty = \sigma(\cup_{n\geq 0}\mathcal{F}_n)$, and suppose $X_n \stackrel{L^1}{\to} X_\infty$ where each X_n is integrable. We do not assume (X_n) is adapted to $(\mathcal{F}_n)$. Show that $\mathsf{E}[X_n \,|\, \mathcal{F}_n] \stackrel{L^1}{\to} \mathsf{E}[X_\infty \,|\, \mathcal{F}_\infty]$.

8.4 Levy's zero-one law

Theorem 8.6. *Let* $\mathcal{F}_\infty = \sigma(\cup_{n\geq 0}\mathcal{F}_n)$ *and* $A \in \mathcal{F}_\infty$. *Then* $\mathsf{E}[\mathbb{1}_A \,|\, \mathcal{F}_n] \stackrel{a.s.}{\to} \mathbb{1}_A$.

Proof. This follows from Theorem 8.5 and the assumption $A \in \mathcal{F}_\infty$. □

Example 8.2. Let $X_1, X_2, \dots$ be independent and $A \in \mathcal{T}$, where we recall the tail σ-field $\mathcal{T}$ is defined by

$$\mathcal{T} = \bigcap_{n\geq 1} \sigma(X_n, X_{n+1}, \dots).$$

For each n, let $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$, which is independent of A. Hence, $\mathsf{E}[\mathbb{1}_A \,|\, \mathcal{F}_n] = \mathsf{P}(A)$, but Theorem 8.6 implies that $\mathsf{E}[\mathbb{1}_A \,|\, \mathcal{F}_n] \stackrel{\text{a.s.}}{\to} \mathbb{1}_A$, which means that $\mathsf{P}(A)$ has to be zero or one. This shows that Theorem 8.6 generalizes Kolmogorov's zero-one law.

References

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