Unit 5: Almost Sure Convergence

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5.1 Upcrossing inequality

Definition 5.1. Consider a stochastic process $(X_n)_{n\geq 0}$. Choose constants $a < b$ and we define the upcrossings of $[a, b]$ as follows. Let $T_0 = -1$, and for $k \geq 1$, let

$$\begin{aligned} T_{2k-1} &= \inf\{n > T_{2k-2} \colon X_n \leq a\}, \\ T_{2k} &= \inf\{n > T_{2k-1} \colon X_n \geq b\}. \end{aligned}$$

From time T_{2k-1} to T_{2k}, the process X crosses from below a to above b, which is called an upcrossing. Define

$$U_n^{a,b} = \sup\{k \colon T_{2k} \leq n\},$$

which gives the number of completed upcrossings up to time n.

Lemma 5.1. *Let (X_n) be a supermartingale and $U_n^{a,b}$ as defined above. Then,*

$$(b - a)\mathsf{E}[U_n^{a,b}] \leq \mathsf{E}[(X_n - a)^-], \quad \text{for any } n \geq 0.$$

Proof. Define $H_n = 1$ if $T_{2k-1} < n \leq T_{2k}$ for some $k \geq 1$, and let $H_n = 0$ otherwise. That is,

$$H_n = \sum_{k=1}^{\infty} \mathbb{1}_{[T_{2k-1}+1, T_{2k}]}(n).$$

Since T_k's are stopping times, H_n is previsible. Further, $H \cdot X$ can be seen as the return in a stock market, where we always buy some stock once its price drops below a and sell the stock once its price goes above b. Hence,

$$(H \cdot X)_n \geq (b - a)U_n^{a,b} - (X_n - a)^-,$$

where $(X_n - a)^-$ is an upper bound on the possible loss due to the last ongoing upcrossing. By Theorem 3.1,

$$\mathsf{E}[(H \cdot X)_n] \leq 0,$$

which yields the asserted inequality. $\square$

Remark 5.1. Consider a sequence of real numbers (x_n). If $\liminf x_n < \limsup x_n$, then there exist rational numbers $a < b$ such that $\liminf x_n < a < b < \limsup x_n$, which implies that the sequence (x_n) completes infinitely many upcrossings of $[a, b]$. This observation allows us to use Lemma 5.1 to prove an almost sure convergence result for supermartingales.

5.2 Almost sure convergence

Theorem 5.1. *Let (X_n) be a supermartingale such that $\sup_n \mathsf{E}[X_n^-] < \infty$. Then, $X_\infty = \lim_n X_n$ exists almost surely. Further, $\mathsf{E}|X_\infty| < \infty$.*

Proof. Define the event $A = \{\liminf X_n < \limsup X_n\}$. By Remark 5.1,

$$A = \bigcup_{a,b \in \mathbb{Q}:\, a<b} \{\liminf X_n < a < b < \limsup X_n\} \subset \bigcup_{a,b \in \mathbb{Q}:\, a<b} \{U_\infty^{a,b} = \infty\},$$

where $U_\infty^{a,b} = \lim_{n\to\infty} U_n^{a,b}$ exists by monotone convergence theorem. But Lemma 5.1 yields that

$$\mathsf{E}[U_\infty^{a,b}] = \lim_{n\to\infty} \mathsf{E}[U_n^{a,b}] \le \limsup_{n\to\infty} \frac{\mathsf{E}[(X_n - a)^-]}{b-a} < \infty,$$

where the last step follows from $(X_n - a)^- \le X_n^- + |a|$ and the assumption $\sup_n \mathsf{E}[X_n^-] < \infty$. Hence, $U_\infty^{a,b} < \infty$ a.s., from which it follows that $\mathsf{P}(A) = 0$. This proves that X_∞ exists a.s. (but it may be infinite).

Since X_n converges to X_∞ implies X_n^- converges to X_∞^-, by Fatou's lemma, we have

$$\mathsf{E}X_\infty^- \le \liminf_{n\to\infty} \mathsf{E}X_n^- < \infty.$$

It only remains to use the supermartingale property to show $\mathsf{E}X_n^+ < \infty$. By Fatou's lemma again,

$$\mathsf{E}X_\infty^+ \le \liminf_{n\to\infty} \mathsf{E}X_n^+ = \liminf_{n\to\infty} \mathsf{E}[X_n^- + X_n] \le \liminf_{n\to\infty} \mathsf{E}[X_n^-] + \mathsf{E}[X_0] < \infty.$$

Hence, $\mathsf{E}|X_\infty| < \infty$, which of course implies that X_∞ is finite, a.s. □

Corollary 5.1. *Let (X_n) be a non-negative supermartingale. Then, $X_\infty = \lim_n X_n$ exists almost surely. Further, $\mathsf{E}X_\infty \le \mathsf{E}X_0$.*

Proof. The first claim follows from Theorem 5.1. By Fatou's lemma, $\mathsf{E}X_\infty \leq \liminf_n \mathsf{E}X_n \leq \mathsf{E}X_0$. □

Example 5.1. Consider the branching process example given in Unit 1. Let $\{Z_{n,i} \colon n \in \mathbb{N}_0,\, i \in \mathbb{N}\}$ be a collection of i.i.d. random variables taking values in $\{0, 1, 2, \dots\}$. Let $X_0 = 1$, and for each $n \geq 1$,

$$X_n = \sum_{i=1}^{X_{n-1}} Z_{n-1,i}.$$

Define $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$, and $W_n = X_n / \mu^n$ where $\mu = \mathsf{E}[Z_{0,1}]$. Then, W_n is a martingale, since

$$\mathsf{E}[X_{n+1} \,|\, \mathcal{F}_n] = \mathsf{E}[Z_{n,1} + \cdots + Z_{n,X_n} \,|\, \mathcal{F}_n] = \mu X_n.$$

Since a martingale is also a supermartingale, $W_\infty = \lim W_n$ exists a.s.. However, we may not have the convergence in L^1; see Exercise 5.1 below.

Example 5.2. We now give a numerical simulation of the branching process. For simplicity, we let $X_0 = 1$ and generate X_n by sampling it from $\text{Pois}(\mu X_{n-1})$; that is, we assume each $Z_{n,i}$ follows a Poisson distribution with rate μ. Results for $\mu = 1.2$ and $\mu = 0.95$ are shown in Figure 1. We will prove later that whether (X_n) converges in L^1 only depends on whether $\mu > 1$.

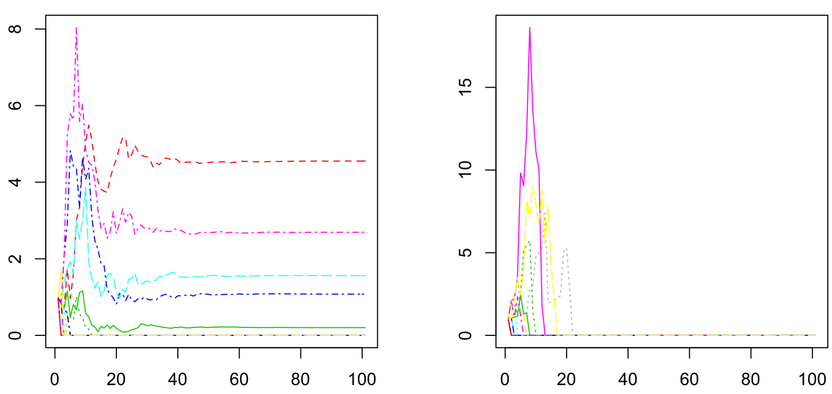

Figure 1: 15 simulated trajectories of (W_n) in Example 5.2, with $\mu = 1.2$ in the left panel and $\mu = 0.95$ in the right.

Example 5.3. We give a counterexample which shows that the condition of Theorem 5.1 cannot be replaced by $\sup_n |X_n| < \infty$, a.s. Let $(Z_n)_{n\geq 1}$ be i.i.d. with $Z_1 \sim \mathrm{Unif}(0,1)$. Let $X_0 = 0$, and for each $n \geq 1$, let

$$X_n = \begin{cases} 1, & \text{if } X_{n-1} = 0, U_n \geq 1/2, \\ -1, & \text{if } X_{n-1} = 0, U_n < 1/2, \\ 0, & \text{if } X_{n-1} \neq 0, U_n \geq n^{-2}, \\ n^2 X_{n-1}, & \text{if } X_{n-1} \neq 0, U_n < n^{-2}. \end{cases}$$

It is easy to show that (X_n) is a martingale. Borel-Cantelli lemma implies that (X_n) is bounded a.s.; i.e., $\sup_n |X_n| < \infty$ a.s. However, a.s., (X_n) is not convergent with $\limsup_n X_n = 1$ and $\liminf_n X_n = -1$.

Exercise 5.1. Consider Example 5.1. Suppose $\mu \leq 1$ and $\mathsf{P}(Z_{0,1} = 1) < 1$. Show that $X_n \overset{\text{a.s.}}{\to} 0$.

Exercise 5.2. Let $Z_1, Z_2, \dots$ be i.i.d. such that $\mathsf{P}(Z_1 = 1) = \mathsf{P}(Z_1 = -1) = 1/2$. Define $\mathcal{F}_n = \sigma(Z_1, Z_2, \dots, Z_n)$, and $X_n = Z_1 + \cdots + Z_n$ (we set $X_0 = 0$). Let our betting strategy be $H_1 = 1$, and

$$H_n = 2^{n-1} \mathbb{1}_{\{Z_1 = \cdots = Z_{n-1} = -1\}}, \qquad \text{for each } n \geq 2.$$

Define $S_n = (H \cdot X)_n = \sum_{i=1}^n H_i Z_i$. Does S_n converge a.s.? If S_n converges a.s., find the limit.

Exercise 5.3. Let $Y_1, Y_2, \dots$ be i.i.d. non-negative random variables such that $\mathsf{E} Y_1 = 1$ and $\mathsf{P}(Y_1 = 1) < 1$. Let $X_n = \prod_{i=1}^n Y_i$. Prove that $X_n \overset{\text{a.s.}}{\to} 0$.

References

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