Unit 3: Previsible Processes and Fair Games

Instructor: Quan Zhou

3.1 Previsible processes

From now on, we always assume the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\geq 0}, \mathsf{P})$ is given.

Definition 3.1. We say $(H_n)_{n\geq 1}$ is previsible (or predictable) if $H_n \in \mathcal{F}_{n-1}$ for each $n \geq 1$.

Definition 3.2. Given an adapted stochastic process $(X_n)_{n\geq 0}$ and a previsible process $(H_n)_{n\geq 1}$, we define

$$(H \cdot X)_n = \sum_{k=1}^{n} H_k (X_k - X_{k-1}), \qquad \text{for } n \geq 1.$$

Define $(H \cdot X)_0 = 0$.

Remark 3.1. Suppose (X_n) is a martingale. Define $Z_n = X_n - X_{n-1}$, and observe that $\mathsf{E}[Z_n \,|\, \mathcal{F}_{n-1}] = 0$. Martingales are can be interpreted as fair games, where Z_n is the net profit per unit stake at game n. The random variable (H_n) can be thought of as the money you bet at game n. We require it to be previsible, which means that H_n is determined once we know the outcomes of the first $n-1$ games. This is a natural constraint: you cannot decide how much you bet after you see the outcome of the game. The random variable $(H \cdot X)_n$ is your net profit after n games.

Remark 3.2. When (X_n) is a martingale, $(H \cdot X)_n$ is called the martingale transform of (X_n) by (H_n). It can also be thought of as the integral of (H_n) with respect to (X_n). In continuous-time, we can replace (X_n) by, e.g., a Brownian motion $(B_t)_{t\geq 0}$ and define the stochastic integral $\int H \mathrm{d}B_t$.

Exercise 3.1. Let $(X_n)_{n\geq 1}$ be a previsible martingale w.r.t. $(\mathcal{F}_n)_{n\geq 0}$. Prove that for each n, $X_n = X_0$, a.s.

3.2 Fair games

Definition 3.3. For an adapted stochastic process $(X_n)_{n\geq 0}$ with $\mathsf{E}|X_n| < \infty$ for each n, we say

(i) (X_n) is a supermartingale if for each n, $\mathsf{E}[X_{n+1} \,|\, \mathcal{F}_n] \leq X_n$, a.s.,

(ii) (X_n) is a submartingale if for each n, $\mathsf{E}[X_{n+1} \,|\, \mathcal{F}_n] \geq X_n$, a.s.

Theorem 3.1. *Let (X_n) be a supermartingale. If (H_n) is a non-negative previsible process and each H_n is bounded, then $H \cdot X$ is a supermartingale.*

Proof. Clearly, $(H \cdot X)_n \in \mathcal{F}_n$, since (X_n) is adapted and (H_n) is previsible. Triangle inequality and the local boundedness of (H_n) imply $\mathsf{E}|(H \cdot X)_n| < \infty$. To prove it is a supermartingale, note that for $n \geq 1$,

$$\begin{aligned}\mathsf{E}[(H \cdot X)_n \,|\, \mathcal{F}_{n-1}] &= \mathsf{E}[(H \cdot X)_{n-1} + H_n(X_n - X_{n-1}) \,|\, \mathcal{F}_{n-1}] \\ &= (H \cdot X)_{n-1} + \mathsf{E}[H_n(X_n - X_{n-1}) \,|\, \mathcal{F}_{n-1}] \\ &= (H \cdot X)_{n-1} + H_n \, \mathsf{E}[X_n - X_{n-1} \,|\, \mathcal{F}_{n-1}] \\ &\leq (H \cdot X)_{n-1},\end{aligned}$$

since $H_n \geq 0$ and (X_n) is a supermartingale. This completes the proof. □

Remark 3.3. Theorem 3.1 holds true if we replace "supermartingale" with "submartingale". Since a martingale is both a supermartingale and submartingale, we can also replace "supermartingale" with "martingale" in Theorem 3.1. The following corollary shows that in the martingale case, the assumption $H_n \geq 0$ can be dropped.

Corollary 3.1. *Let (X_n) be a martingale. If (H_n) is a previsible process and each H_n is bounded, then $H \cdot X$ is a martingale.*

Proof. The proof is analogous to that of Theorem 3.1. □

Theorem 3.2. *Let $(X_n)_{n \geq 0}$ be adapted with $\mathsf{E}|X_0| < \infty$. Then (X_n) is a martingale if and only if, for any locally bounded previsible process H, $H \cdot X$ is a martingale.*

Proof. The "only if" part follows from Theorem 3.1. To prove the "if" part, pick a positive integer m and let $H_n = \mathbb{1}_{\{n=m\}}$. Then, $(H \cdot X)_{m-1} = 0$, and the martingale property of $H \cdot X$ implies that

$$\begin{aligned}0 &= (H \cdot X)_{m-1} \\ &= \mathsf{E}[(H \cdot X)_m \,|\, \mathcal{F}_{m-1}] \\ &= \mathsf{E}[X_m - X_{m-1} \,|\, \mathcal{F}_{m-1}].\end{aligned}$$

Since m is arbitrary, this shows that (X_n) is a martingale. □

References

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[3] David Williams. *Probability with martingales*. Cambridge university press, 1991.