Unit 14: Brownian Motion

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Continuous-time stochastic processes

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. We say $(X_t)_{0 \le t < \infty}$, a collection of random variables defined on $(\Omega, \mathcal{F}, \mathrm{P})$ indexed by $t \in [0, \infty)$, is a continuous-time stochastic process.

Sample path

For each $\omega \in \Omega$, the function $t \mapsto X_t(\omega)$ is said to be a sample path or trajectory of the process $X = (X_t)_{0 \le t < \infty}$.

Finite-dimensional distributions

The finite-dimensional distributions of X refer to the distributions of $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$ for any $n \ge 1$ and $0 \le t_1 < t_2 \le \cdots < t_n < \infty$.

Equivalence between stochastic processes

Theorem 14.1

Let X and Y be two stochastic processes. Consider the following.

1. $\mathrm{P}(X_t = Y_t \text{ for every } 0 \le t < \infty) = 1$ *(indistinguishable).*
2. $\mathrm{P}(X_t = Y_t) = 1$ *for every* $0 \le t < \infty$ *(modification).*
3. X *and* Y *have the same finite-dimensional distributions.*

Then (1) $\Rightarrow$ *(2)* $\Rightarrow$ *(3) .*

Proof.

Try it yourself. □

Continuous-time martingales

Let $(\mathcal{F}_t)_{0\le t<\infty}$ be a filtration (i.e., non-decreasing σ-algebras) such that X is adapted to $(\mathcal{F}_t)_{0\le t<\infty}$ (i.e., $X_t \in \mathcal{F}_t$ for each t). Further, assume that $\mathrm{E}|X_t| < \infty$ for each t.

Submartingales, supermartingales and martingales

- X is a submartingale if $\mathrm{E}[X_t \mid \mathcal{F}_s] \ge X_s$, a.s. for any $0 \le s < t < \infty$.
- X is a supermartingale if $-X$ is a submartingale.
- X is a martingale if it is both a supermartingale and a submartingale.

Many results for discrete-time martingales (e.g. upcrossing inequality, Doob's inequality, optional sampling theorem) continue to hold for continuous-time martingales with right-continuous paths.

Brownian motion

Brownian motion is the foundation of continuous-time martingales.

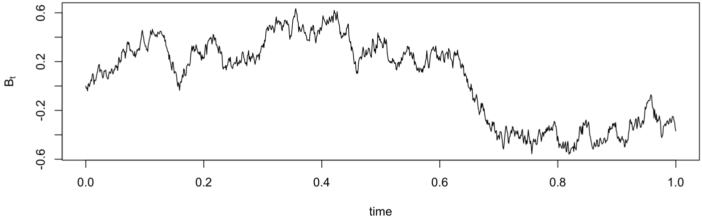
Definition 14.2

We say $B = (B_t)_{0 \le t < \infty}$[a] is a standard one-dimensional Brownian motion (i.e., Wiener process) if

1. $B_0 = 0$;
2. for any $t_0 < t_1 < \cdots < t_n$, $B(t_0), B(t_1) - B(t_0), \ldots, B(t_n) - B(t_{n-1})$ are independent;
3. for any $s, t \ge 0$, $B(s+t) - B(s) \sim N(0, t)$;
4. sample paths of B are almost surely continuous.

[a] We will sometimes write $B(t)$ instead of B_t.

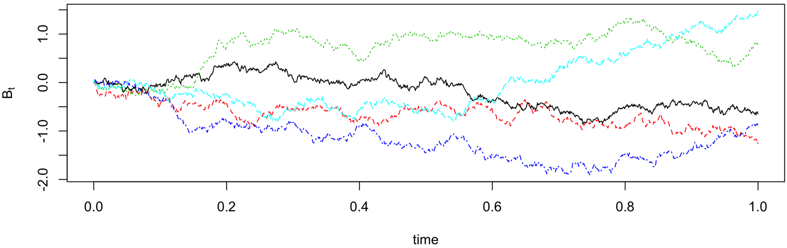
Numerical simulation of Brownian motion



Here is the R code.

```
h = 0.001
N = 1/h
Z = rnorm(N, mean=0, sd=sqrt(h))
B = c(0, cumsum(Z))
plot((0:N)/N, B, type='l', xlab='time', ylab='B_t')
```

Numerical simulation of Brownian motion



Construction of Brownian motion: Method 1

We first consider how to construct a Brownian motion on the time interval $[0, 1]$. The method we use will also justify the simulation scheme used in the previous slides and lead to the famous Donsker's theorem.

Construction of Brownian motion: Method 1

Let $Z_1, Z_2, \ldots$ be i.i.d. random variables with mean zero and variance $\sigma^2 \in (0, \infty)$. Define $S_n = Z_1 + \cdots + Z_n$ and set $S_0 = 0$. Define $(X_t^n)_{0 \le t \le 1}$ as the scaled linear interpolation of $(S_j)_{1 \le j \le n}$:

$$
\begin{aligned}
X_t^n &= \frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) \frac{1}{\sigma\sqrt{n}} Z_{\lfloor nt \rfloor + 1} \\
&=: \frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor} + E_{n,t}.
\end{aligned}
$$

Note that $E_{n,t} \xrightarrow{\mathrm{P}} 0$ as $n \to \infty$. Since $\lfloor nt \rfloor / n \to t$, $S_{\lfloor nt \rfloor} / \sigma\sqrt{n}$ (and thus X_t^n) converges in distribution to $\sqrt{t} N(0, 1)$ by CLT.

Construction of Brownian motion: Method 1

Similarly, an application of the multivariate CLT yields that, for any $0 \leq t_0 < t_1 < \cdots < t_m \leq 1$, $(X^n(t_0), X^n(t_1), \ldots, X^n(t_m))$ converges in distribution to the finite-dimensional distribution specified in Definition 14.2 (i.e., a multivariate normal distribution with independent increments).

To show that this implies the existence of Brownian motion, we need

- Prohorov's theorem,
- Kolmogorov-Chentsov continuity theorem (see Method 2),
- Arzelà-Ascoli theorem (see [4]).

Prohorov's theorem

Let (S, d) be a metric space and $\mathcal{B}(S)$ denote the Borel σ-algebra (i.e., the σ-algebra generated by all open sets w.r.t. the metric d).

Relatively compactness and tightness

Let Π be a collection of probability measures on $(S, \mathcal{B}(S))$. We say Π is relatively compact if every sequence of probability measures in Π contains a weakly convergence subsequence (with limit being another probability measure on $(S, \mathcal{B}(S))$). We say Π is tight if for every $\epsilon > 0$, there is a compact set $K \subset S$ such that $\inf_{\mathrm{P} \in \Pi} \mathrm{P}(K) \geq 1 - \epsilon$

Theorem 14.3

Suppose (S, d) is complete and separable. Then Π is relatively compact if and only if it is tight.

Construction of Brownian motion: Method 1

Space $\mathcal{C}([0,1])$

Let $\mathcal{C}([0,1])$ be the space of continuous functions on $[0,1]$ endowed with the metric

$$d(\omega_1, \omega_2) = \sup_{0 \le t \le 1} |\omega_1(t) - \omega_2(t)|.$$

It can be shown that $(\mathcal{C}([0,1]), d)$ is complete and separable.

By construction each X^n takes values in $\mathcal{C}([0,1])$. Let P^n denote the distribution of X^n. If we can show $(\mathrm{P}_n)_{n \ge 1}$ is tight, then Prohorov's theorem implies that $(\mathrm{P}_n)_{n \ge 1}$ has a subsequence converging weakly to some probability measure W. In particular, the process X defined on $(\mathcal{C}([0,1]), \mathcal{B}(\mathcal{C}([0,1])), W)$ by $X_t(\omega) = \omega(t)$ is a Brownian motion. W is known as Wiener measure.

Construction of Brownian motion: Method 1

This argument can be extended to $\mathcal{C}([0,\infty))$, the space of all continuous functions on $[0,\infty)$, with metric

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{0 \le t \le n} \left(|x(t) - y(t)| \wedge 1 \right).$$

$(\mathcal{C}([0,\infty)), d)$ is complete and separable. Further, $\mathcal{B}(\mathcal{C}([0,\infty))$ coincides with the σ-algebra generated by the collection of sets

$$\{\omega \in \mathcal{C}([0,\infty)) \colon (\omega(t_1), \omega(t_2), \dots, \omega(t_n)) \in A\}, \quad n \ge 1, A \in \mathcal{B}(\mathbb{R}^n).$$

We omit the proof of the tightness of $(\mathrm{P}_n)_{n \ge 1}$, which requires Kolmogorov-Chentsov continuity theorem and Arzelà-Ascoli theorem.

Donsker's theorem

With some extra work, we obtain the following functional CLT.

Theorem 14.4

Let $Z_1, Z_2, \ldots$ be i.i.d. random variables with mean zero and variance $\sigma^2 \in (0, \infty)$. Define $S_n = Z_1 + \cdots + Z_n$ and set $S_0 = 0$. Define $X^n = (X^n_t)_{0 \le t < \infty}$ by

$$X^n_t = \frac{1}{\sigma\sqrt{n}} S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) \frac{1}{\sigma\sqrt{n}} Z_{\lfloor nt \rfloor + 1}.$$

The distribution of X^n converges weakly to the Wiener measure as $n \to \infty$.

Let $\mathbb{R}^{[0,\infty)}$ denote the set of all real-valued functions on $[0,\infty)$ and $\mathcal{B}(\mathbb{R}^{[0,\infty)})$ be the σ-algebra generated by the collection of sets

$$\{\omega \in \mathbb{R}^{[0,\infty)} \colon (\omega(t_1), \omega(t_2), \ldots, \omega(t_n)) \in A\}, \quad n \geq 1, A \in \mathcal{B}(\mathbb{R}^n).$$

The second method for constructing Brownian motion directly finds a stochastic process X on $\mathbb{R}^{[0,\infty)}$ that is distributed as a Brownian motion.

Construction of Brownian motion: Method 2

A standard application of Kolmogorov extension theorem yields the following result.

Theorem 14.5

For every $x \in \mathbb{R}$, there exists a probability measure P_x on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$ such that $\mathrm{P}_x\{\omega \colon \omega(0) = x\} = 1$ and for any $0 = t_0 < t_1 < \cdots < t_n$ and Borel sets $A_1, \ldots, A_n$, we have

$$\mathrm{P}_x(\{\omega \colon \omega(t_i) \in A_i\}) = \int_{A_1} \cdots \int_{A_n} \prod_{k=1}^{n} p_{t_k - t_{k-1}}(x_{k-1}, x_k) \, \mathrm{d}x_n \cdots \mathrm{d}x_1,$$

where $x_0 = 0$ and

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t}.$$

Construction of Brownian motion: Method 2

Choosing $x = 0$ in the previous theorem, we get a probability measure that satisfies conditions (1), (2), (3) in Definition 14.2. But condition (4) is hard to satisfy. Indeed, we have the following result:

Lemma 14.6

$$\{A \subset \mathcal{C}([0,\infty)) \colon A \in \mathcal{B}(\mathbb{R}^{[0,\infty)})\} = \{\emptyset\}.$$

For example, $\mathcal{C}([0,\infty))$ is not measurable.

Construction of Brownian motion: Method 2

Main idea: construct the discrete-time version of Brownian motion at $t \in \mathbb{Q}$ and then extend it to $\mathbb{R}$.

Question: Does a continuous function $f : \mathbb{Q} \to \mathbb{R}$ always have a continuous extension to $\mathbb{R}$?

Consider the function $f : \mathbb{Q} \to \mathbb{R}$ defined by $f(x) = 0$ if $x < \sqrt{2}$ and $f(x) = 1$ if $x > \sqrt{2}$.

Hölder continuity

Let $f\colon S \to \mathbb{R}$ for some $S \subset \mathbb{R}$. We say f is Hölder continuous of order $\gamma > 0$ (or γ-Hölder continuous) at x if there exist $\epsilon > 0, C < \infty$ such that for any $y \in (x - \epsilon, x + \epsilon)$,

$$|f(y) - f(x)| \le C|y - x|^{\gamma}. \tag{1}$$

If (1) holds for any x, y and some fixed $C < \infty$, then we say f is Hölder continuous of order γ.

When $\gamma = 1$, this is known as Lipschitz continuity.

Continuity of functions

Assume $\gamma \in (0, 1]$.

Pointwise:
Differentiable at $x \Rightarrow$ Lipschitz continuous at $x \Rightarrow \gamma$-Hölder continuous at $x \Rightarrow$ continuous at x.

Global:
Continuously differentiable $\Rightarrow$ Lipschitz continuous $\Rightarrow \gamma$-Hölder continuous $\Rightarrow$ uniformly continuous $\Rightarrow$ continuous.

If $f : \mathbb{Q} \to \mathbb{R}$ is uniformly continuous, it has a unique continuous extension from $\mathbb{Q}$ to $\mathbb{R}$.

Kolmogorov-Chentsov theorem

For simplicity, we consider the time interval $[0, 1]$ first.

Theorem 14.7

Let $(X_t)_{0 \le t \le 1}$ be real-valued. Suppose there exist $\alpha, \beta > 0, C < \infty$ s.t.

$$\mathrm{E}(|X_t - X_s|^{\alpha}) \le C|t - s|^{1+\beta}, \qquad \textit{for all } s, t \in [0, 1].$$

Then, for any $\gamma < \beta/\alpha$, for almost every ω there exists $C(\omega)$ s.t.

$$|X_t(\omega) - X_s(\omega)| \le C(\omega)|t - s|^{\gamma}, \qquad \textit{for all } s, t \in \mathbb{Q}_2 \cap [0, 1],$$

where $\mathbb{Q}_2 = \{k2^{-n} \colon n, k \ge 0\}$ denotes the dyadic rationals. Further, there is a modification (unique up to indistinguishability) $\tilde{X} = (\tilde{X}_t)_{0 \le t \le 1}$ of X whose paths are a.s. Hölder continuous of order γ.

Construction of Brownian motion: Method 2

On $(\mathbb{R}^{[0,1]}, \mathcal{B}(\mathbb{R}^{[0,1]}))$, we have shown that there is a probability measure P under which the process X defined by $X_t(\omega) = \omega(t)$ has stationary, independent, and normally distributed increments, and $\mathrm{P}(X_0 = 0) = 1$.

By Kolmogorov-Chentsov theorem, there is a modification B of X such that B is a.s. Hölder continuous of order $\gamma \in (0, 1/2)$, since

$$\mathrm{E}(|X_t - X_s|^{2k}) = C_k |t - s|^k, \quad \text{for } k \geq 1 \text{ and some } C_k < \infty.$$

Because B has the same finite-dimensional distributions as X, B is a Brownian motion.

Construction of Brownian motion: Method 2

Using a limiting argument, we can extend the construction of B_t to $t \in [0, \infty)$. The paths of $(B_t)_{0 \le t < \infty}$ are a.s. *locally* Hölder continuous of order $\gamma \in (0, 1/2)$ (see [4] for the definition).

How about $\gamma \ge 1/2$?

Theorem 14.8

For any $\gamma > 1/2$, the paths of Brownian motion are a.s. nowhere γ-Hölder continuous (i.e., not γ-Hölder continuous at any t).

This implies that paths of Brownian motion are a.s. nowhere differentiable.

Construction of Brownian motion: Method 3

The last method constructs Brownian motion as an L^2-limit.

Consider the time interval $[0, 1]$ equipped with the Lebesgue measure λ. Let $L^2([0, 1])$ be the Hilbert space of square integrable (w.r.t. λ) functions with inner product

$$\langle f, g\rangle = \int_{[0,1]} f(x)g(x)\lambda(\mathrm{d}x).$$

Let $(b_n)_{n\geq 1}$ be an orthonormal basis; that is, $\langle b_n, b_m\rangle = \mathbb{1}_{\{n=m\}}$, and

$$\lim_{n\to\infty} \left\| f - \sum_{k=1}^{n} \langle f, b_k\rangle b_k \right\| = 0, \quad \forall f \in L^2([0, 1]).$$

Construction of Brownian motion: Method 3

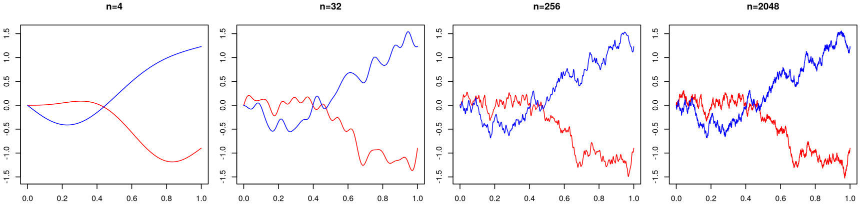
Let $Z_1, Z_2, \dots$ be i.i.d. $N(0,1)$ random variables defined on some probability space $(\Omega, \mathcal{F}, \mathrm{P})$. For each $n \geq 1$ and $t \in [0,1]$, define

$$
\begin{aligned}
X_t^n &= \sum_{i=1}^{n} \langle \mathbb{1}_{[0,t]},\, b_i \rangle Z_i \\
&= \int_{[0,t]} \left(\sum_{i=1}^{n} Z_i b_i(s) \right) \lambda(\mathrm{d}s).
\end{aligned}
$$

It can be shown that $(X_t^n)_{n \geq 1}$ converges in L^2 to some random variable X_t. Further, the process $(X_t)_{0 \leq t \leq 1}$ has the same finite-dimensional distributions as Brownian motion.

For example, let $b_1 = 1$ and $b_n(x) = \sqrt{2}\cos((n-1)\pi x)$ for $n \geq 2$. Then,

$$X_t^n = Z_1 t + \sum_{k=1}^{n-1} \frac{\sqrt{2}\sin(k\pi t)}{k\pi} Z_{k+1}.$$



Simulation of two sample paths for difference choices of n.

Construction of Brownian motion: Method 3

We can still use Kolmogorov-Chentsov continuity theorem to show that X has a continuous modification. But now we have a shortcut. By choosing a proper orthonormal basis of $L^2([0,1])$, we can have $\|X^n - X\|_\infty \xrightarrow{\text{a.s.}} 0$. Since a uniform limit of continuous functions is again continuous, this would guarantee that X is continuous a.s. and thus X is a Brownian motion. See [4, Chap. 21.5].

This is also known as Lévy's construction of Brownian motion.

Application to stochastic integrals

Let $(b_n)_{n\geq 1}$ be an orthonormal basis of $L^2([0,1])$ such that

$$B_t = \sum_{i=1}^{\infty} \langle \mathbb{1}_{[0,t]},\, b_i \rangle Z_i$$

is a Brownian motion. Given $f \in L^2([0,1])$ and $t \in [0,1]$, we define

$$\begin{aligned} \int_0^t f(s)\mathrm{d}B_s &= \int_{[0,t]} f(s) \left(\sum_{i=1}^{\infty} Z_i b_i(s) \right) \lambda(\mathrm{d}s) \\ &= \sum_{i=1}^{\infty} \langle f \mathbb{1}_{[0,t]},\, b_i \rangle Z_i. \end{aligned} \tag{2}$$

This is called the stochastic integral of f w.r.t. B.

Another characterization of Brownian motion

Theorem 14.9

Equivalently, $B = (B_t)_{0 \le t < \infty}$ is a standard one-dimensional Brownian motion if

1. *B is a Gaussian process; that is, all finite dimensional distributions are multivariate normal;*
2. $\mathsf{E}[B_t] = 0$ *and* $\mathrm{Cov}(B_s, B_t) = s \wedge t$ *for any* $s, t \ge 0$*;*
3. *sample paths of B are almost surely continuous.*

Proof.

Try it yourself. □

Exercises

Let $B = (B_t)_{0 \le t < \infty}$ be a Brownian motion.

Exercise 14.1

Prove Theorem 14.9.

Exercise 14.2

Let $c > 0$. Show that $X = (X_t)_{0 \le t < \infty}$ is also a Brownian motion where

$$X_t = c^{-1/2} B_{ct}.$$

Exercise 14.3

Let $Y = \int_0^1 B_s \mathrm{d}s$. Find $\mathrm{E}[Y]$ and $\mathrm{E}[Y^2]$.

Exercises

Exercise 14.4

Let $a > 0$ and $\mathcal{F}_t = \sigma(B_t)$. Show that $(G_t)_{0 \le t < \infty}$ is a martingale where

$$G_t = \exp\left(aB_t - \frac{1}{2}a^2 t\right).$$

Exercise 14.5

For $t \in [0, 1]$, let $X_t = \int_0^t f(s)\mathrm{d}B_s$ denote the stochastic integral defined in (2). Show that for any $s, t \ge 0$, $\mathrm{Cov}(X_s, X_t) = \int_0^{s \wedge t} f^2(u)\mathrm{d}u$.

References

[1] Patrick Billingsley. *Convergence of probability measures*. John Wiley & Sons, 2013.

[2] Rick Durrett. *Probability: theory and examples*, volume 49. Cambridge university press, 2019.

[3] Ioannis Karatzas and Steven Shreve. *Brownian motion and stochastic calculus*, volume 113. Springer Science & Business Media, 2012.

[4] Achim Klenke. *Probability theory: a comprehensive course*. Springer Science & Business Media, 2013.