Unit 12: Limit Theorems for Martingales

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12.1 Weak law of large numbers

Let $X_1, X_2, \dots$ be a sequence of random variables, and define $S_n = X_1 + \cdots + X_n$ for each n. Let $(\mathcal{F}_n)_{n \geq 0}$ be a filtration such that $X_n \in \mathcal{F}_n$ for each n. We will compare the results for two settings: (i) $X_1, X_2, \dots$ are independent, (ii) $(S_n)_{n \geq 1}$ is a martingale w.r.t. $(\mathcal{F}_n)$.

Theorem 12.1. *Let $X_1, X_2, \dots$ be independent. Let $(b_n)_{n \geq 1}$ be a sequence of positive constants such that $b_n \uparrow \infty$, and define $Y_{n,k} = X_k \mathbb{1}_{\{|X_k| \leq b_n\}}$. Then, $S_n / b_n \xrightarrow{P} 0$ if and only if*

(i) $\sum_{k=1}^n \mathsf{P}(|X_k| > b_n) \to 0$;

(ii) $b_n^{-1} \sum_{k=1}^n \mathsf{E}[Y_{n,k}] \to 0$;

(iii) $b_n^{-2} \sum_{k=1}^n \left\{ \mathsf{E}[Y_{n,k}^2] - (\mathsf{E} Y_{n,k})^2 \right\} \to 0.$

Proof. See [5]. □

Theorem 12.2. *Let $(S_n)_{n \geq 1}$ be a martingale. Let $(b_n)_{n \geq 1}$ be a sequence of positive constants such that $b_n \uparrow \infty$, and define $Y_{n,k} = X_k \mathbb{1}_{\{|X_k| \leq b_n\}}$. Suppose*

(i) $\sum_{k=1}^n \mathsf{P}(|X_k| > b_n) \to 0$;

(ii) $b_n^{-1} \sum_{k=1}^n \mathsf{E}[Y_{n,k} \,|\, \mathcal{F}_{k-1}] \xrightarrow{P} 0$;

(iii) $b_n^{-2} \sum_{k=1}^n \left\{ \mathsf{E}[Y_{n,k}^2] - \mathsf{E}\left(\mathsf{E}[Y_{n,k} \,|\, \mathcal{F}_{k-1}]^2 \right) \right\} \to 0$;

Then, $S_n / b_n \xrightarrow{P} 0$.

Proof. Let $T_n = Y_{n,1} + \cdots + Y_{n,n}$. Then,

$$\mathsf{P}(S_n \neq T_n) \leq \sum_{k=1}^n \mathsf{P}(X_k \neq Y_{n,k}) = \sum_{k=1}^n \mathsf{P}(X_k > b_n) \to 0.$$

Hence, by condition (i), we only need to show $T_n/b_n \xrightarrow{\text{P}} 0$. Due to condition (ii), this is equivalent to proving that

$$\frac{1}{b_n}\sum_{k=1}^{n}\left(Y_{n,k} - \mathsf{E}[Y_{n,k} \,|\, \mathcal{F}_{k-1}]\right) \xrightarrow{\text{P}} 0.$$

Define $Z_{n,k} = Y_{n,k} - \mathsf{E}[Y_{n,k} \,|\, \mathcal{F}_{k-1}]$. Apply Markov's inequality to get

$$\mathsf{P}\left(b_n^{-1}\left|\sum_{k=1}^{n} Z_{n,k}\right| > \epsilon\right) \leq \frac{1}{\epsilon^2 b_n^2}\mathsf{E}\left[\left(\sum_{k=1}^{n} Z_{n,k}\right)^2\right].$$

For $1 \leq i < j \leq n$, we have

$$\mathsf{E}[Z_{n,i}Z_{n,j}] = \mathsf{E}[Z_{n,j}\mathsf{E}[Z_{n,i} \,|\, \mathcal{F}_j]] = 0.$$

Since $\mathsf{E}\left[Z_{n,k}^2\right] = \mathsf{E}[Y_{n,k}^2] - \mathsf{E}\left(\mathsf{E}[Y_{n,k} \,|\, \mathcal{F}_{k-1}]^2\right)$, by condition (iii),

$$\mathsf{P}\left(b_n^{-1}\left|\sum_{k=1}^{n} Z_{n,k}\right| > \epsilon\right) \leq \frac{1}{\epsilon^2 b_n^2}\sum_{k=1}^{n}\mathsf{E}\left[Z_{n,k}^2\right] \to 0,$$

which completes the proof. □

Remark 12.1. The three conditions in Theorem 12.2 are sufficient but not necessary. See [4] for counterexamples where the weak law of large numbers holds but condition (i) is violated.

12.2 Almost sure convergence of random series

We still use the same notation as in Section 12.1: let $X_1, X_2, \dots$ be random variables and $S_n = X_1 + \cdots + X_n$.

Theorem 12.3 (Kolmogorov convergence criterion)**.** *Let $X_1, X_2, \dots$ be independent with $\mathsf{E}[X_n] = 0$ for each n. If $\sum_{n=1}^{\infty}\mathsf{E}[X_n^2] < \infty$, then, almost surely, S_n converges to a finite limit.*

Theorem 12.4. *Let $(S_n)_{n\geq 1}$ be a square integrable martingale. Then S_n converges (to a finite limit) a.s. on the event $\{\sum_{n=1}^{\infty}\mathsf{E}[X_n^2 \,|\, \mathcal{F}_{n-1}] < \infty\}$.*

Proof. This is essentially Theorem 7.3. The square variation process of S is

$$\langle S \rangle_n = \sum_{k=1}^{n} \mathsf{E}[(S_k - S_{k-1})^2 \mid \mathcal{F}_{k-1}] = \sum_{k=1}^{n} \mathsf{E}[X_k^2 \mid \mathcal{F}_{k-1}].$$

By Theorem 7.3, S_n converges a.s. on the event $\{\langle S \rangle_\infty < \infty\}$. □

Theorem 12.5 (Chow, 1965)**.** *Let $(S_n)_{n \geq 1}$ be a martingale and $1 \leq p \leq 2$. Then S_n converges (to a finite limit) a.s. on the event*

$$\left\{ \sum_{n=1}^{\infty} \mathsf{E}[|X_n|^p \mid \mathcal{F}_{n-1}] < \infty \right\}.$$

Proof. See [2]. □

Theorem 12.6 (Strong Law of Large Numbers)**.** *Let $(S_n)_{n \geq 1}$ be a martingale, and $(W_n)_{n \geq 1}$ be non-decreasing, positive and previsible. Then, S_n/W_n converges to zero a.s. on the event*

$$\left\{ W_n \to \infty, \ \text{and} \ \sum_{n=1}^{\infty} W_n^{-p} \, \mathsf{E}[|X_n|^p \mid \mathcal{F}_{n-1}] < \infty \right\} \tag{1}$$

where $1 \leq p \leq 2$.

Proof. Since W_n is previsible, by Theorem 12.5, $\sum_{k=1}^{n} (X_k/W_k)$ converges to a finite limit a.s. on the given event. By Kronecker's lemma, we further have $S_n/W_n \to 0$ a.s. on the given event. □

12.3 Central limit theorems

We consider triangular arrays in this section. Let $\{X_{n,k} \colon n \geq 1, \ 1 \leq k \leq n\}$ be a triangular array of random variables. For each (n, k), let

$$S_{n,k} = X_{n,1} + \cdots X_{n,k},$$

and $S_n = S_{n,n}$. For each n, let $(\mathcal{F}_{n,k})_{0 \leq k \leq n}$ be a filtration such that $X_{n,k} \in \mathcal{F}_{n,k}$ for each k.

Theorem 12.7 (Lindeberg-Feller CLT)**.** *For each n, let $X_{n,1}, \ldots, X_{n,n}$ be independent with mean zero and finite variance. Suppose* *(i)* $\sum_{k=1}^{n} \mathsf{E} X_{n,k}^2 \to \sigma^2 \in (0, \infty)$,

(ii) for all $\epsilon > 0$, $\lim_{n\to\infty} \sum_{k=1}^{n} \mathsf{E}\left[|X_{n,k}|^2 \mathbb{1}_{\{|X_{n,k}|>\epsilon\}}\right] = 0$.

Then $S_n/\sigma \xrightarrow{D} N(0,1)$.

Theorem 12.8. *For each* n*, let* $(S_{n,k})_{1\le k\le n}$ *be a square integrable martingale with* $\mathsf{E}[S_{n,k}] = 0$*. Let* V *be a random variable s.t.* $\mathsf{P}(|V| < \infty) = 1$*. Suppose*

(i) $\sum_{k=1}^{n} \mathsf{E}[X_{n,k}^2 \,|\, \mathcal{F}_{n,k-1}] \xrightarrow{P} V$*, as* $n \to \infty$*;*

(ii) for all $\epsilon > 0$, $\sum_{k=1}^{n} \mathsf{E}\left[|X_{n,k}|^2 \mathbb{1}_{\{|X_{n,k}|>\epsilon\}} \,|\, \mathcal{F}_{n,k-1}\right] \xrightarrow{P} 0$*, as* $n \to \infty$*.*

Further, suppose at least one of the following two conditions holds.

(iii) $V \in \mathcal{F}_0$ *where* $\mathcal{F}_0 = \cap_{n\ge 1} \mathcal{F}_{n,0}$*.*

(iv) $\mathcal{F}_{n,k} \subset \mathcal{F}_{n+1,k}$ *for each* $n \ge 1$ *and* $1 \le k \le n$*.*

Then $S_n \xrightarrow{D} Z$ *where the random variable* Z *has characteristic function*

$$\mathsf{E}[e^{itZ}] = \mathsf{E}[e^{-t^2 V/2}].$$

Further, if $V > 0$ *a.s., then* $S_n/\sqrt{U_n} \xrightarrow{D} N(0,1)$ *where* $U_n = \sum_{k=1}^{n} X_{n,k}^2$*.*

Proof. See [1, 3, 4]. □

Example 12.1. Let $Z_1, Z_2, \dots$ be i.i.d. such that $\mathsf{P}(Z_1 = 1) = \mathsf{P}(Z_1 = -1) = 1/2$. Define $Y_1 = Z_1$, and for $n \ge 2$,

$$Y_n = Z_n \sum_{i=1}^{n-1} \frac{Z_i}{i}.$$

By Theorem 12.3, we can define $\sqrt{V} = \sum_{i=1}^{\infty} (Z_i/i)$, which exists and is finite a.s. And since $\mathsf{E}[Y_k \,|\, Z_1, \dots, Z_{k-1}] = 0$, $(Y_n)_{n\ge1}$ is a martingale difference sequence. One can use Theorem 12.8 to show that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i \xrightarrow{\mathrm{D}} W, \quad \frac{\sum_{i=1}^{n} Y_i}{\sqrt{\sum_{i=1}^{n} Y_i^2}} \xrightarrow{\mathrm{D}} N(0,1),$$

where W has characteristic function $\mathsf{E}[e^{-t^2 V/2}]$.

Exercise 12.1. Fill in the details of Example 12.1. That is, construct the triangular array $(X_{n,k})$ and verify the conditions of Theorem 12.8.

References

[1] Bruce M Brown. Martingale central limit theorems. *The Annals of Mathematical Statistics*, pages 59–66, 1971.

[2] Yuan Shih Chow. Local convergence of martingales and the law of large numbers. *The Annals of Mathematical Statistics*, 36(2):552–558, 1965.

[3] GK Eagleson. Martingale convergence to mixtures of infinitely divisible laws. *The Annals of Probability*, pages 557–562, 1975.

[4] Peter Hall and Christopher C Heyde. *Martingale limit theory and its application*. Academic press, 2014.

[5] Michel Loève. *Probability Theory I*. Springer, 1977.