Approximation of bivariate copulas by patched bivariate Fréchet copulas

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Abstract

Bivariate Fréchet (BF) copulas characterize dependence as a mixture of three simple structures: comonotonicity, independence and countermonotonicity. They are easily interpretable but have limitations when used as approximations to general dependence structures. To improve the approximation property of the BF copulas and keep the advantage of easy interpretation, we develop a new copula approximation scheme by using BF copulas locally and patching the local pieces together. Error bounds and a probabilistic interpretation of this approximation scheme are developed. The new approximation scheme is compared with several existing copula approximations, including shuffle of min, checkmin, checkerboard and Bernstein approximations and exhibits better performance, especially in characterizing the local dependence. The utility of the new approximation scheme in insurance and finance is illustrated in the computation of the rainbow option prices and stop-loss premiums.

Keywords: Bivariate Fréchet copulas; patched bivariate Fréchet copula; approximation of bivariate copulas; rainbow two-color options.

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1. Introduction

A copula is a multivariate distribution with uniform $[0,1]$ margins. For a n-dimensional distribution function F with marginal distributions $F_1, \ldots, F_n$, Sklar's Theorem (Nelsen, 2006) states that there exists a n-dimensional copula C such that for all $x_1, x_2, \cdots, x_n$,

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)).$$

If F is continuous, the copula C is uniquely defined. The copula function fully captures the dependence structure of F. For a comprehensive introduction to copulas, see Nelsen (2006). Recently, copulas are widely used for valuing risks in insurance and finance (Cherubini et al., 2004).

We focus on the bivariate copulas in this paper. Write $M(u,v) = \min\{u,v\}$, $\Pi(u,v) = uv$ and $W(u,v) = \max\{u+v-1, 0\}$ for $u, v \in [0,1]$. These functions are three important copulas, which are called respectively the Fréchet upper

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bound, the product copula and the Fréchet lower bound. It is well-known that for any bivariate copula C,

$$W(u,v) \leq C(u,v) \leq M(u,v), \qquad u, v \in [0,1].$$

The copulas $M(u,v)$, $\Pi(u,v)$ and $W(u,v)$ have simple interpretations as modeling the comonotonicity, independence and countermonotonicity. Two variables X and Y are said to be comonotonic if there exist two increasing functions f and g and one random variable Z such that $X = f(Z), Y = g(Z)$. The connecting copula for two comonotonic variables is $M(u,v)$. Two variables X and Y are countermonotonic if X and $-Y$ are comonotonic. The connecting copula for two countermonotonic variables is $W(u,v)$. The connecting copula for two independent variables is the product copula $\Pi(u,v)$. The importance of comonotonicity, independence and countermonotonicity in risk management, utility theory and actuarial pricing has been well documented in the literature; see for example Embrechets et al. (2002), Dhaene et al. (2002a), Dhaene et al. (2002b), and Yang et al. (2006).

Associated with the simplicity of interpretation of the above three copulas is the strong restriction each has in modeling the relationship of two variables. Indeed, comonotonicity, independence and countermonotonicity are extreme dependence relationships. If used alone, none of them can appropriately characterize the dependence structure presented in most real data sets. Fortunately combination of them can provide much more flexibility. The convex combination of these three copulas are called Bivariate Fréchet (BF) copulas (Nelsen, 2006). Precisely, a BF copula can be written as

$$C^F_{\alpha,\gamma}(u,v) =: \alpha M(u,v) + (1-\alpha-\gamma)\Pi(u,v) + \gamma W(u,v),$$

where $0 \leq \alpha, \gamma \leq 1$ and $\alpha + \gamma \leq 1$. A BF copula is the convex sum of three important dependence factors and can be interpreted as a mixture of three simple bivariate distributions. More discussions of BF copulas can be found in Kass et al. (2001) and Salvadori et al. (2007). The importance of the BF copula is implied by the work of Yang et al. (2006), who proved that any copula C can be decomposed uniquely as a convex combination of the Fréchet upper bound, the product copula, the Fréchet lower bound and an indecomposable copula. Yang et al. (2006) also showed that BF copulas can provide good approximations to some copulas and discussed how to obtain an optimal BF approximation for an arbitrary copula.

Of course, there are limitations in the BF copula approximations. Sometime the optimal BF copula approximation is not close enough to the original copula. The purpose of this paper is to develop an approximation scheme based on BF copulas such that the approximation error can be arbitrarily small and in the mean time preserve the ease of interpretation of BF copulas. Our idea is to first partition the unit square into small rectangles, called patches, then approximate the copula of the conditional distribution on each rectangle by a BF copula, and finally glue the BF copulas together using the law of total probability. We refer to the resulting approximating copula as a patched BF copula, or PBF copula for short. We show that this approximation scheme is consistent, that is, the uniform and L_2

norm of the approximation error go to zero when the number of patches goes to infinity.

On the other hand, several approximations of copulas have been proposed in the literatures, including shuffle of min approximation (Mikusinski et al., 1992; Nelsen, 2006; Durante et al., 2009), checkmin approximation (Mikusinski & Taylor, 2008), checkerboard approximation (Li et al., 1998; Durrleman et al., 2000) and Bernstein approximation (Scancetta & Satchell, 2004). We show that all these approximations can be interpreted as a patched copula using either comonotonicity or independence to describe the local dependence. However, sometimes using comonotonicity or independence alone can not lead to satisfying approximations of the local dependence structure. In contrast, our new PBF copula approximation uses the more flexible BF copula to characterize the local dependence and is thus more accurate than these existing approximations. We illustrate the good approximation properties of the PBF copula and compare with other approximations using various numerical examples and by applying it to compute the bivariate rainbow option prices and the stop-loss premiums.

The rest of the paper is organized as follows. Section 2 first provides a decomposition of a copula and then uses it to motivate the PBF copulas; approximation properties of the PBF copulas are also studied. Section 3 gives a probabilistic explanation of PBF copulas. Section 4 discusses the optimal PBF copula approximations of bivariate copulas and studies their properties. A detailed comparison of the PBF copula approximation and other existing approximations is given in Section 5. Section 6 illustrates the utility of the PBF copula by applying it to computation of the rainbow option prices and the stop-loss premiums. Section 7 concludes the paper. The Appendix collects some technical details.

2. Patched bivariate Fréchet copulas

2.1. A decomposition of a bivariate copula

Let C be a bivariate copula, that is, a function that has the following properties:

P1 it is defined in the unit square $I^2 = [0, 1] \times [0, 1]$ and takes values in the unit interval $I = [0, 1]$;

P2 for every pair (u, v) in I^2, $C(u, 0) = 0 = C(0, v)$, $C(u, 1) = u$, $C(1, v) = v$;

P3 for every rectangle $[u_1, u_2] \times [v_1, v_2]$ in I^2 with $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0.$$

Note that the copula C is continuous. For ease of exposition, we introduce a pair of random variables U and V whose joint distribution is the copula function C.

Let m be a positive integer. Denote $I_i = (i/m, (i + 1)/m]$ for $i = 0, \dots, m - 1$ and let I_0 be closed. Then $\{I_i, i = 0, \dots, m - 1\}$ is a partition of the unit interval into m equal-sized intervals and $\{I_i \times I_j, i, j = 0, \dots, m - 1\}$ is a partition of the unit square into m^2 equal-sized sub-squares. For $0 \le i, j \le m-1$, denote $A_{i,j} = \{(U,V) \in I_i \times I_j\}$. Then $\{A_{i,j}, 0 \le i, j \le m-1\}$ is a partition of the probability space. The law of total probability implies that

$$C(u,v) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} P(A_{i,j})P(U \le u, V \le v | A_{i,j}). \tag{2.1}$$

For each fixed pair of integers (i, j), $0 \le i, j \le m-1$, the probability of the event $A_{i,j}$ can be obtained from the copula function by the following formula

$$P(A_{i,j}) = C(\frac{i+1}{m}, \frac{j+1}{m}) + C(\frac{i}{m}, \frac{j}{m}) - C(\frac{i+1}{m}, \frac{j}{m}) - C(\frac{i}{m}, \frac{j+1}{m}). \tag{2.2}$$

Write

$$F_{i,j}(u) = P(U \le u \mid A_{i,j}), \quad G_{i,j}(v) = P(V \le v \mid A_{i,j}). \tag{2.3}$$

Note that $F_{i,j}$ and $G_{i,j}$ are distribution functions supported on I_i and I_j, respectively. The inverse functions of $F_{i,j}$ and $G_{i,j}$ are denoted as $F_{i,j}^{-1}$ and $G_{i,j}^{-1}$, where

$$F_{i,j}^{-1}(u) = \inf\{y : F_{i,j}(y) \ge u\}, \quad G_{i,j}^{-1}(u) = \inf\{y : G_{i,j}(y) \ge u\}.$$

According to Sklar's theorem (Nelsen, 2006), there is a unique copula function, denoted as $C^{i,j}$, such that

$$P(U \le u, V \le v | A_{i,j}) = C^{i,j}(F_{i,j}(u), G_{i,j}(v)).$$

Here the uniqueness follows from the continuity of C. Plugging the above result into (2.1), we obtain

$$C(u,v) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} P(A_{i,j})C^{i,j}(F_{i,j}(u), G_{i,j}(v)). \tag{2.4}$$

Therefore the copula is decomposed as a convex sum of distributions each of which has a support in a local region of the unit square. By the definitions of the conditional marginal distributions, if $P(A_{h,k}) \ne 0$, then

$$F_{h,k}(u) = \frac{C(u, \frac{k+1}{m}) - C(u, \frac{k}{m}) + C(\frac{h}{m}, \frac{k}{m}) - C(\frac{h}{m}, \frac{k+1}{m})}{P(A_{h,k})}, \quad u \in I_h, \tag{2.5}$$

$$G_{h,k}(v) = \frac{C(\frac{h+1}{m}, v) - C(\frac{h}{m}, v) + C(\frac{h}{m}, \frac{k}{m}) - C(\frac{h+1}{m}, \frac{k}{m})}{P(A_{h,k})}, \quad v \in I_k. \tag{2.6}$$

From (2.5) and (2.6), we know that $F_{i,j}, G_{i,j}, 0 \le i, j \le m-1$ are totally determined by the values of the copula C on the grid $I \times \{i/m\}, \{i/m\} \times I$, $i = 0, \ldots, m$ of the unit square, that is, $C(u, \frac{i}{m}), C(\frac{i}{m}, v), u, v \in [0,1], 0 \le i \le m$. The decomposition (2.4) plays a crucial role in this paper to motivate our new approximation of bivariate copulas. We next show that this decomposition actually characterizes a copula.

Suppose that for each $0 \le i, j \le m-1$, $D^{i,j}$ is a copula, $H_{i,j}$ and $L_{i,j}$ are distribution functions over I_i and I_j respectively, $a_{i,j} \ge 0$, and $\sum_{i=0}^{m-1}\sum_{j=0}^{m-1} a_{i,j} = 1$. Define

$$D(u,v) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} a_{i,j}D^{i,j}(H_{i,j}(u), L_{i,j}(v)), \quad u, v \in [0,1]. \tag{2.7}$$

Theorem 2.1. *(i) The function defined in* (2.7) *is a copula; (ii) Any copula can be decomposed to a form in* (2.7) *and the decomposition is unique. Moreover, in* (2.7)*,* $a_{i,j}, D^{i,j}, H_{i,j}, L_{i,j}$*,* $0 \le i, j \le m-1$ *are uniquely determined by D, and in particular,*

$$a_{i,j} = D(\frac{i+1}{m}, \frac{j+1}{m}) + D(\frac{i}{m}, \frac{j}{m}) - D(\frac{i+1}{m}, \frac{j}{m}) - D(\frac{i}{m}, \frac{j+1}{m}). \tag{2.8}$$

Proof. It is easy to check that $D(u, v)$ is a copula. The existence of the decomposition is given in the derivation of (2.4). For $(u, v) \in I_i \times I_j$, from (2.7) we get

$$D(u, v) + D(\frac{i}{m}, \frac{j}{m}) - D(u, \frac{j}{m}) - D(\frac{i}{m}, v) = a_{i,j} D^{i,j}(H_{i,j}(u), L_{i,j}(v)).$$

Setting $u = \frac{i+1}{m}, v = \frac{j+1}{m}$, (2.8) follows. Applying the Sklar's theorem and noticing the continuity of D, we see that $D^{i,j}, H_{i,j}$ and $L_{i,j}$ are uniquely determined by D. The proof is complete. □

As illustration, in the following we give the elements in the decomposition (2.4) for some copulas, including the marginal distributions $F_{i,j}, G_{i,j}$ and the copulas $C^{i,j}$.

Example 2.1. *For BF copula* $C^F_{\alpha,\gamma}(u, v)$*, we have*

$$P(A_{i,j}) = \alpha I_{\{i=j\}} \frac{1}{m} + \beta \frac{1}{m^2} + \gamma I_{\{i+j=m-1\}} \frac{1}{m},$$

where $\beta = 1 - \alpha - \gamma$*. When* $P(A_{i,j}) \ne 0$*, it is easy to derive that*

$$F_{i,j}(u) = \min\{\max\{0, mu - i\}, 1\}, \quad G_{i,j}(v) = \min\{\max\{0, mv - j\}, 1\}$$

for $u, v \in [0, 1]$*, which are uniform distributions on* I_i *and* I_j *respectively, and*

$$C^{i,j}(x, y) = \frac{\alpha I_{\{i=j\}} M(x, y) + \beta \frac{1}{m} \Pi(x, y) + \gamma I_{\{i+j=m-1\}} W(x, y)}{\alpha I_{\{i=j\}} + \beta \frac{1}{m} + \gamma I_{\{i+j=m-1\}}},$$

which is also a BF copula, but with parameters different from those of the original copula.

Example 2.2. *Consider the Gaussian copula* $C(u, v) = \Phi_\rho(\Phi^{-1}(u), \Phi^{-1}(v))$*, where the correlation coefficient parameter* $\rho \in (-1, 1)$*,*

$$\phi_\rho(x, y) = \frac{exp\{\frac{-x^2-y^2+2\rho xy}{2(1-\rho^2)}\}}{\sqrt{2\pi(1-\rho^2)}}, \quad \Phi_\rho(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} \phi_\rho(s, t)\, dsdt$$

and Φ^{-1} *is the inverse of standard normal distribution function* Φ*. We have that for* $u, v \in [0, 1]$ *and* $s, t \in (0, 1)$*,*

$$F_{i,j}(u) = min\{max\{\frac{\int_{\Phi^{-1}(\frac{i}{m})}^{\Phi^{-1}(u)} \int_{\Phi^{-1}(\frac{j}{m})}^{\Phi^{-1}(\frac{j+1}{m})} \phi_\rho(x, y)\, dxdy}{P(A_{i,j})}, 0\}, 1\},$$

$$G_{i,j}(v) = min\{max\{\frac{\int_{\Phi^{-1}(\frac{i}{m})}^{\Phi^{-1}(\frac{i+1}{m})} \int_{\Phi^{-1}(\frac{j}{m})}^{\Phi^{-1}(v)} \phi_\rho(x, y)\, dxdy}{P(A_{i,j})}, 0\}, 1\}$$

and

$$C^{i,j}(s,t) = \frac{\int_{\Phi^{-1}(\frac{i}{m})}^{\Phi^{-1}(F_{i,j}^{-1}(s))} \int_{\Phi^{-1}(\frac{j}{m})}^{\Phi^{-1}(G_{i,j}^{-1}(t))} \phi_\rho(x,y)\, dxdy}{P(A_{i,j})},$$

where

$$P(A_{i,j}) = \int_{\Phi^{-1}(\frac{i}{m})}^{\Phi^{-1}(\frac{i+1}{m})} \int_{\Phi^{-1}(\frac{j}{m})}^{\Phi^{-1}(\frac{j+1}{m})} \phi_\rho(x,y)\, dxdy.$$

Example 2.3. *For the Farlie-Gumbel-Morgenstern copula with parameter $\theta \in [-1, 1]$,*

$$C(u,v) = uv + \theta uv(1-u)(1-v),$$

we have that for $u, v \in [0, 1]$,

$$\begin{aligned} F_{i,j}(u) &= min\{max\{\frac{\frac{1}{m}(u-\frac{i}{m})+\frac{\theta}{m}(1-\frac{2j+1}{m})(u-\frac{i}{m})(1-u-\frac{i}{m})}{P(A_{i,j})}, 0\}, 1\}, \\ G_{i,j}(v) &= min\{max\{\frac{\frac{1}{m}(v-\frac{j}{m})+\frac{\theta}{m}(1-\frac{2i+1}{m})(v-\frac{j}{m})(1-v-\frac{j}{m})}{P(A_{i,j})}, 0\}, 1\} \end{aligned}$$

and for $s, t \in (0, 1)$,

$$C^{i,j}(s,t) = \frac{(F_{i,j}^{-1}(s)-\frac{i}{m})(G_{i,j}^{-1}(t)-\frac{j}{m}) + \theta(F_{i,j}^{-1}(s)-\frac{i}{m})(G_{i,j}^{-1}(t)-\frac{j}{m})(1-G_{i,j}^{-1}(t)-\frac{j}{m})(1-F_{i,j}^{-1}(s)-\frac{i}{m})}{P(A_{i,j})},$$

where

$$P(A_{i,j}) = \frac{1}{m^2} + \frac{\theta}{m^2}(1-\frac{2j+1}{m})(1-\frac{2i+1}{m}).$$

2.2. Patched bivariate Fréchet copulas

Yang et al. (2006) studied how to use a bivariate Fréchet copula to approximate an arbitrary bivariate copula. Motivated by their work, we consider replacing each of the copulas $D^{i,j}(s,t)$ in (2.7) by a BF copula $C^F_{\alpha_{i,j},\gamma_{i,j}}(s,t)$. As a result, we obtain the following function

$$D_F(u,v) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} a_{i,j} C^F_{\alpha_{i,j},\gamma_{i,j}}(H_{i,j}(u), L_{i,j}(v)), \tag{2.9}$$

which is itself a copula according to part (i) of Theorem 2.1. We call this copula a patched bivariate Fréchet (PBF) copula, because it is a mixture of distributions each of which is generated by a bivariate Fréchet copula and supported on a local subregion of the unit square.

Given a bivariate copula C, define $P(A_{i,j})$, $F_{i,j}(u)$ and $G_{i,j}(v)$ as in (2.2) and (2.3). Letting $a_{i,j} = P(A_{i,j})$, $H_{i,j}(u) = F_{i,j}(u)$ and $L_{i,j}(v) = G_{i,j}(v)$ in (2.9), we obtain the following copula

$$C_F(u,v) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} P(A_{i,j}) C^F_{\alpha_{i,j},\gamma_{i,j}}(F_{i,j}(u), G_{i,j}(v)), \tag{2.10}$$

which is called a PBF copula generated by the copula C. Varying $\alpha_{i,j}$ and $\gamma_{i,j}$ over their permissible range, we obtain the following C-generated PBF family

$$\mathcal{F}_C^{(m)} = \{C_F(u,v), u, v \in [0,1] | 0 \leq \alpha_{i,j}, \gamma_{i,j} \leq 1, \alpha_{i,j} + \gamma_{i,j} \leq 1 \; for \; 0 \leq i, j \leq m-1\},$$

where the superscript (m) in $\mathcal{F}_C^{(m)}$ denotes the dependence on the number of subregions.

It follows from (2.2), (2.5) and (2.6) that, the member of $\mathcal{F}_C^{(m)}$ for a fixed pair of $\alpha_{i,j}$ and $\gamma_{i,j}$ is totally determined by the values of C on the grid $I \times \{i/m\}, \{i/m\} \times I$, $i = 0, \ldots, m$ of the unit square. The following theorem shows that a much stronger result holds, that is, all members of $\mathcal{F}_C^{(m)}$ are identical to the generating copula C on this grid.

Theorem 2.2. *For each* $0 \leq i, j \leq m$ *and* $u, v \in I$,

$$C_F(\frac{i}{m}, v) = C(\frac{i}{m}, v), \quad C_F(u, \frac{j}{m}) = C(u, \frac{j}{m}). \tag{2.11}$$

Proof. Given $u, v \in [0,1]$, we can find $i, j \leq m-1$ such that $(u,v) \in I_i \times I_j$. Note that $F_{h,k}(u) = 0$ if $h > i$, $F_{h,k}(u) = 1$ if $h < i$, $G_{h,k}(v) = 0$ if $k > j$ and $G_{h,k}(v) = 1$ if $k < j$. Using these facts and the properties **P1**–**P2** of the copula in an application of (2.4) yields

$$C(u,v) = \sum_{k=0}^{i-1} P(A_{k,j}) G_{k,j}(v) + \sum_{h=0}^{j-1} P(A_{i,h}) F_{i,h}(u) + C(\frac{i}{m}, \frac{j}{m}) + P(A_{i,j}) C^{i,j}(F_{i,j}(u), G_{i,j}(v)) \tag{2.12}$$

and

$$C_F(u,v) = \sum_{k=0}^{i-1} P(A_{k,j}) G_{k,j}(v) + \sum_{h=0}^{j-1} P(A_{i,h}) F_{i,h}(u) + C(\frac{i}{m}, \frac{j}{m}) + P(A_{i,j}) C^F_{\alpha_{i,j},\gamma_{i,j}}(F_{i,j}(u), G_{i,j}(v)). \tag{2.13}$$

Compare (2.13) with (2.12) to yield

$$C(u,v) - C_F(u,v) = P(A_{i,j}) \{C^{i,j}(F_{i,j}(u), G_{i,j}(v)) - C^F_{\alpha_{i,j},\gamma_{i,j}}(F_{i,j}(u), G_{i,j}(v))\}. \tag{2.14}$$

Noticing that for $0 \leq k, l \leq m-1$, $F_{k,l}((k+1)/m) = G_{k,l}((l+1)/m) = 1$, and using the properties of copulas, we see that the quantities in the curly brackets of (2.14) are identically zero if $u = (i+1)/m$ or $v = (j+1)/m$, $0 \leq i, j \leq m-1$. The desired results then follows. □

The next theorem gives bounds of the approximation errors of using members of a PBF copula family to approximate the generating copula. Denote the C-volume of a rectangular region $[u_1, u_2] \times [v_1, v_2]$ of I^2 as

$$V_C([u_1, u_2] \times [v_1, v_2]) = C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1).$$

Theorem 2.3. *(i) (Uniform norm)*

$$\sup_{u,v \in I^2} |C(u,v) - C_F(u,v)| \leq \max_{0 \leq i,j \leq n-1} V_C(I_i \times I_j) \leq \frac{1}{m}; \tag{2.15}$$

(ii) (L_2 norm)

$$\int_0^1 \int_0^1 [C(u,v) - C_F(u,v)]^2 \, dudv \leq \frac{1}{m^3}.$$

Proof. (i) Note that in (2.14) both $C^{i,j}(F_{i,j}(u), G_{i,j}(v))$ and $C^F_{\alpha_{i,j},\gamma_{i,j}}(F_{i,j}(u), G_{i,j}(v))$ are between 0 and 1. The first inequality in (2.15) thus follows from (2.14) since $P(A_{i,j}) = V_C(I_i \times I_j)$. Because the marginal distribution induced by the copula C is the uniform distribution, the second inequality in (2.15) follows from $V_C(I_i \times I_j) \leq V_C(I_i \times I) = 1/m$.

(ii) Note that

$$\int_0^1 \int_0^1 [C(u,v) - C_F(u,v)]^2 \, dudv = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \int_{\frac{i}{m}}^{\frac{i+1}{m}} \int_{\frac{j}{m}}^{\frac{j+1}{m}} [C(u,v) - C_F(u,v)]^2 \, dudv.$$

Using (2.14) we obtain that

$$\int_{\frac{i}{m}}^{\frac{i+1}{m}} \int_{\frac{j}{m}}^{\frac{j+1}{m}} [C(u,v) - C_F(u,v)]^2 \, dudv \leq \frac{\{V_C(I_i \times I_j)\}^2}{m^2} \leq \frac{V_C(I_i \times I_j)}{m^3}.$$

Since $\sum_{i=0}^{m-1} \sum_{j=0}^{m-1} V_C(I_i \times I_j) = 1$, the desired result follows. □

For a fixed copula C, the C-generated PBF copula family $\mathcal{F}_C^{(m)}$ has two special members corresponding to the lower and upper bounds of the family. In particular, for $0 \leq i, j \leq m-1$,

$$C^F_{0,1}(F_{i,j}(u), G_{i,j}(v)) \leq C^F_{\alpha_{i,j},\gamma_{i,j}}(F_{i,j}(u), G_{i,j}(v)) \leq C^F_{1,0}(F_{i,j}(u), G_{i,j}(v)). \tag{2.16}$$

Putting all pieces together, we obtain that for each $C_F \in \mathcal{F}_C^{(m)}$,

$$C_F^{lo}(u,v) \leq C_F(u,v) \leq C_F^{up}(u,v),$$

where

$$C_F^{lo}(u,v) = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} P(A_{i,j}) C^F_{0,1}(F_{i,j}(u), G_{i,j}(v))$$

and

$$C_F^{up}(u,v) = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} P(A_{i,j}) C^F_{1,0}(F_{i,j}(u), G_{i,j}(v)).$$

The distance between the upper bound and the lower bound can be measured. In fact, for any $(u,v) \in I_i \times I_j$, using (2.13),

$$\begin{aligned} C_F^{up}(u,v) - C_F^{lo}(u,v) &= P(A_{i,j})\{C^F_{1,0}(F_{i,j}(u), G_{i,j}(v)) - C^F_{0,1}(F_{i,j}(u), G_{i,j}(v))\} \\ &= P(A_{i,j})[\min\{F_{i,j}(u), G_{i,j}(v)\} - \max\{F_{i,j}(u) + G_{i,j}(v) - 1, 0\}] \leq P(A_{i,j}) \leq \frac{1}{m}. \end{aligned}$$

The bounds obtained here can be viewed as an extension of the Fréchet–Hoeffding copula boundaries.

3. A probabilistic interpretation of patched bivariate Fréchet copulas

By extending results of Yang et al. (2006), this section gives an alternative representation of patched bivariate Fréchet copulas in terms of random vectors. The probability space is decomposed into pieces on each of which the random variables have a simple relationship. This probabilistic representation gives an intuitive description of the dependence structure of the PBF copulas and also provides an convenient tool for computations involving PBF copulas.

As before we suppose the joint distribution of the random variables U and V is the copula C. We want to find a probabilistic representation of the C-generated PBF copulas. Let Z and W be two independent random variables each of which has a uniform $[0, 1]$ distribution. Moreover, it is assumed that (Z, W) is independent of (U, V). For fixed $0 \le i, j \le m-1$, consider the disjoint random sets $B^+_{i,j}, B^\perp_{i,j}, B^-_{i,j}$. Suppose that $B^+_{i,j}$, $B^\perp_{i,j}$, $B^-_{i,j}$ are independent of the random variables U, V, Z and W, and satisfy that

$$P(B^+_{i,j}) = \alpha_{i,j}, \quad P(B^\perp_{i,j}) = 1 - \alpha_{i,j} - \gamma_{i,j}, \quad P(B^-_{i,j}) = \gamma_{i,j}.$$

Recall that $A_{i,j} = \{U \in I_i, V \in I_j\}$. Write

$$A^+_{i,j} = B^+_{i,j} \bigcap A_{i,j}, \quad A^\perp_{i,j} = B^\perp_{i,j} \bigcap A_{i,j}, \quad A^-_{i,j} = B^-_{i,j} \bigcap A_{i,j}.$$

Then $A^+_{i,j}, A^\perp_{i,j}, A^-_{i,j}$ is a partition of $A_{i,j}$ for each (i, j), and $A_{i,j}$, $0 \le i, j \le m-1$ is a partition of the whole probability space.

Define

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \left\{ \begin{bmatrix} \xi \\ \eta \end{bmatrix} I_{A_{i,j}} \right\} = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \begin{bmatrix} \xi_{i,j} \\ \eta_{i,j} \end{bmatrix}, \tag{3.1}$$

where

$$\begin{bmatrix} \xi_{i,j} \\ \eta_{i,j} \end{bmatrix} = \begin{bmatrix} F^{-1}_{i,j}(W) \\ G^{-1}_{i,j}(W) \end{bmatrix} I_{A^+_{i,j}} + \begin{bmatrix} F^{-1}_{i,j}(W) \\ G^{-1}_{i,j}(Z) \end{bmatrix} I_{A^\perp_{i,j}} + \begin{bmatrix} F^{-1}_{i,j}(W) \\ G^{-1}_{i,j}(1-W) \end{bmatrix} I_{A^-_{i,j}}. \tag{3.2}$$

Theorem 3.1. *(i) The distribution function of the random vector $(\xi, \eta)^T$ is exactly the PBF copula given in* (2.10)*;*

(ii) The random vector $([m\xi], [m\eta])$ *has the same distribution as* $([mU], [mV])$*, where* $[m\xi], [m\eta], [mU]$ *and* $[mV]$ *denote the integer parts of $m\xi, m\eta, mU$ and mV respectively.*

The first part of this theorem is a simple consequence of the assumed independence of various quantities in (3.1), and the following equivalent representation of (3.2):

$$\begin{bmatrix} \xi_{i,j} \\ \eta_{i,j} \end{bmatrix} = I_{A_{i,j}} \left\{ \begin{bmatrix} F^{-1}_{i,j}(W) \\ G^{-1}_{i,j}(W) \end{bmatrix} I_{B^+_{i,j}} + \begin{bmatrix} F^{-1}_{i,j}(W) \\ G^{-1}_{i,j}(Z) \end{bmatrix} I_{B^\perp_{i,j}} + \begin{bmatrix} F^{-1}_{i,j}(W) \\ G^{-1}_{i,j}(1-W) \end{bmatrix} I_{B^-_{i,j}} \right\}. \tag{3.3}$$

Since the support of the distribution of $(\xi_{i,j}, \eta_{i,j})$ is the local square $I_i \times I_j$, (3.1) gives an intuitive explanation about what "patching" means. On each "patch", the meaning of BF copula is clearly revealed in (3.2) through random variables. Obviously, (W, W) are comonotonic, (W, Z) are independent, $(W, 1-W)$ are countermonotonic, and the inverse probability transformations $F_{i,j}^{-1}$ and $G_{i,j}^{-1}$ are applied in order to match the marginal distributions. As a consequence, $(\xi_{i,j}, \eta_{i,j})$ are comonotonic on $A_{i,j}^{+}$, independent on $A_{i,j}^{\perp}$ and countermonotonic on $A_{i,j}^{-}$.

The second part of Theorem 3.1 is a rephrasing of Theorem 2.2 in terms of random variables. It indicates that the PBF copulas are identical to the generating copula when rounding downwards the random variables to multiples of $1/m$.

Next we use the probability representation given in part (i) of Theorem 3.1 to derive an explicit formula for calculating integrals with respect to a PBF copula. We present the formula in a general form that allows for arbitrary marginal distributions.

For two random variables X and Y with a copula C and marginal distributions F and G, we want to compute an integral with respect to their joint distribution. Write $X = F^{-1}(U)$ and $Y = G^{-1}(V)$ where the inverse function $F^{-1}(x) = \inf\{s : F(s) \geq x\}$ and G^{-1} is similarly defined, then the joint distribution of U and V is the copula C. Define $X^* = F^{-1}(\xi)$, $Y^* = G^{-1}(\eta)$, where ξ and η are given in (3.1). Then the marginal distributions of X^* and Y^* are F and G respectively, which are the same as those of X and Y, and the copula of (ξ, η) is a PBF copula approximation to the copula C.

Let

$$X_{i,j}^{+} = F^{-1} \circ F_{i,j}^{-1}(W), \quad Y_{i,j}^{+} = G^{-1} \circ G_{i,j}^{-1}(W),$$
$$X_{i,j}^{\perp} = F^{-1} \circ F_{i,j}^{-1}(W), \quad Y_{i,j}^{\perp} = G^{-1} \circ G_{i,j}^{-1}(Z),$$
$$X_{i,j}^{-} = F^{-1} \circ F_{i,j}^{-1}(W), \quad Y_{i,j}^{-} = G^{-1} \circ G_{i,j}^{-1}(1-W).$$

Applying F^{-1} to ξ and G^{-1} to η in (3.1) and (3.2), we obtain

$$\begin{bmatrix} X^* \\ Y^* \end{bmatrix} = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \left\{ \begin{bmatrix} F^{-1}(\xi) \\ G^{-1}(\eta) \end{bmatrix} I_{A_{i,j}} \right\} = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \begin{bmatrix} F^{-1}(\xi_{i,j}) \\ G^{-1}(\eta_{i,j}) \end{bmatrix}, \tag{3.4}$$

and using (3.3), we have

$$\begin{bmatrix} F^{-1}(\xi_{i,j}) \\ G^{-1}(\eta_{i,j}) \end{bmatrix} = \begin{bmatrix} X_{i,j}^{+} \\ Y_{i,j}^{+} \end{bmatrix} I_{A_{i,j}^{+}} + \begin{bmatrix} X_{i,j}^{\perp} \\ Y_{i,j}^{\perp} \end{bmatrix} I_{A_{i,j}^{\perp}} + \begin{bmatrix} X_{i,j}^{+} \\ Y_{i,j}^{+} \end{bmatrix} I_{A_{i,j}^{-}}. \tag{3.5}$$

By part (i) of Theorem 3.1, we know the copula of X^* and Y^* is a PBF copula generated by C, the copula of (X, Y). The following theorem immediately follows from (3.4) and (3.5).

Theorem 3.2. *For any measurable function f on R^2,*

$$E\{f(X^*, Y^*)\} = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} P(A_{i,j})[\alpha_{i,j} E\{f(X_{i,j}^{+}, Y_{i,j}^{+})\} + \beta_{i,j} E\{f(X_{i,j}^{\perp}, Y_{i,j}^{\perp})\} + \gamma_{i,j} E\{f(X_{i,j}^{-}, Y_{i,j}^{-})\}],$$

where $\beta_{i,j} = 1 - \alpha_{i,j} - \gamma_{i,j}$.

Since the marginal distributions of (X^*, Y^*) are the same as those of (X, Y) and the copula of the former is a PBF copula approximation of the later, we can apply Theorem 3.2 to approximate integrals with respect to the joint distribution of (X, Y). The next result shows that the approximation error can be bounded.

Theorem 3.3. *Suppose that for the measurable function g defined on I^2, the derivative $\frac{\partial^2 g(u,v)}{\partial u \partial v}$ exists and is continuous. Then*

$$|E\{g(U, V)\} - E\{g(\xi, \eta)\}| \leq \frac{1}{m^{3/2}}\{\int_0^1 \int_0^1 |\frac{\partial^2 g(u, v)}{\partial u \partial v}|^2 \, dudv\}^{1/2}.$$

Proof. Note that

$$g(u, v) = \int_0^u \int_0^v \frac{\partial^2 g(s, t)}{\partial s \partial t} \, dsdt + g(0, v) + g(u, 0) - g(0, 0).$$

Denote $T = \int_0^1 g(0, v)dv + \int_0^1 g(u, 0)du - g(0, 0)$ and $\overline{C}(s, t) = P(U > s, V > t)$. We have that

$$\begin{aligned}
E[g(U, V)] &= \int_0^1 \int_0^1 g(u, v)C(du, dv) \\
&= \int_0^1 \int_0^1 \int_0^u \int_0^v \frac{\partial^2 g(s, t)}{\partial s \partial t} \, dsdt \, C(du, dv) + T \\
&= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{\partial^2 g(s, t)}{\partial s \partial t} I_{\{u \geq s, v \geq t\}} \, C(du, dv) \, dsdt + T \\
&= \int_0^1 \int_0^1 \frac{\partial^2 g(s, t)}{\partial s \partial t} \overline{C}(s, t) dsdt + T.
\end{aligned}$$

Similarly,

$$E[g(\xi, \eta)] = \int_0^1 \int_0^1 \frac{\partial^2 g(s, t)}{\partial s \partial t} \overline{C^F}(s, t) \, dsdt + T.$$

Then, applying Theorem 2.1 and the Cauchy–Schwarz inequality, we have that

$$|E[g(U, V)] - E[g(\xi, \eta)]| \leq \int_0^1 \int_0^1 |\frac{\partial^2 g(s, t)}{\partial s \partial t}| \times |\overline{C}(s, t) - \overline{C^F}(s, t)| \, dsdt \leq \frac{1}{m^{3/2}}\{\int_0^1 \int_0^1 |\frac{\partial^2 g(s, t)}{\partial s \partial t}|^2 \, dsdt\}^{1/2}.$$

The proof is complete. □

Taking $g(u, v) = f(F^{-1}(u), G^{-1}(v))$ in Theorem 3.3, we obtain a bound for the error of using $E\{f(X^*, Y^*)\}$ to approximate $E\{f(X, Y)\}$.

Remark 3.1. *Application of the above theorem requires existence of the partial derivative of g. Error bounds can be obtained in some special cases when the partial derivative does not exist. For example, when $g(u, v) = \sum_{i=1}^{l} a_i I_{\{u \leq u_i, v \leq v_i\}}$, one can find that*

$$|E[g(U, V)] - E[g(\xi, \eta)]| \leq \frac{1}{m} \sum_{i=1}^{l} |a_i|.$$

4. The optimal PBF approximation

According to Section 2.2, a family of PBF copula $\mathcal{F}_C^{(m)}$ is generated from a given copula C. In this section we discuss how to get an optimal member from this family that minimizes the mean squared approximation error.

4.1. Computation of the optimal approximation

Given $C_F \in \mathcal{F}_C^{(m)}$, the mean squared approximation error is

$$\int_0^1 \int_0^1 [C(u,v) - C_F(u,v)]^2\, dudv = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} P(A_{i,j})^2 S_{i,j}(\alpha_{i,j}, \gamma_{i,j}),$$

where

$$S_{i,j}(\alpha_{i,j}, \gamma_{i,j}) = \int_{\frac{i}{m}}^{\frac{i+1}{m}} \int_{\frac{j}{m}}^{\frac{j+1}{m}} \{C^{i,j}(F_{i,j}(u), G_{i,j}(v)) - C^F_{\alpha_{i,j},\gamma_{i,j}}(F_{i,j}(u), G_{i,j}(v))\}^2\, dudv.$$

We want to find the optimal coefficients $\alpha_{i,j}, \gamma_{i,j},\ i, j \le m-1$, that minimizes the mean squared approximation error. If $P(A_{i,j}) \neq 0$ and $(\alpha^*_{i,j}, \gamma^*_{i,j})$ solve the problem

$$\min_{0 \le \alpha_{i,j}, \gamma_{i,j} \le 1, \alpha_{i,j}+\gamma_{i,j} \le 1} S_{i,j}(\alpha_{i,j}, \gamma_{i,j}), \tag{4.1}$$

then the optimal approximation of C is

$$C^*(u,v) = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} P(A_{i,j}) C^F_{\alpha^*_{i,j},\gamma^*_{i,j}}(F_{i,j}(u), G_{i,j}(v)). \tag{4.2}$$

The optimization problem (4.1) can be solved by applying the method of Yang et al. (2006). First we solve the unconstrained optimization problem

$$S_{i,j}(a^*_{i,j}, b^*_{i,j}) = \min_{a_{i,j}, b_{i,j}} S_{i,j}(a_{i,j}, b_{i,j}). \tag{4.3}$$

The optimal $a^*_{i,j}$ and $b^*_{i,j}$ have closed-form expressions. Define

$$\mathcal{A}(i,j) = \int_{\frac{i}{m}}^{\frac{i+1}{m}} \int_{\frac{j}{m}}^{\frac{j+1}{m}} \{M(F_{i,j}(u), G_{i,j}(v)) - \Pi(F_{i,j}(u), G_{i,j}(v))\}^2\, dudv,$$

$$\mathcal{B}(i,j) = \int_{\frac{i}{m}}^{\frac{i+1}{m}} \int_{\frac{j}{m}}^{\frac{j+1}{m}} \{M(F_{i,j}(u), G_{i,j}(v)) - \Pi(F_{i,j}(u), G_{i,j}(v))\}\{W(F_{i,j}(u), G_{i,j}(v)) - \Pi(F_{i,j}(u), G_{i,j}(v))\}\, dudv,$$

$$\mathcal{C}(i,j) = \int_{\frac{i}{m}}^{\frac{i+1}{m}} \int_{\frac{j}{m}}^{\frac{j+1}{m}} \{W(F_{i,j}(u), G_{i,j}(v)) - \Pi(F_{i,j}(u), G_{i,j}(v))\}^2\, dudv,$$

$$\mathcal{D}(i,j) = \int_{\frac{i}{m}}^{\frac{i+1}{m}} \int_{\frac{j}{m}}^{\frac{j+1}{m}} \{C^{i,j}(F_{i,j}(u), G_{i,j}(v)) - \Pi(F_{i,j}(u), G_{i,j}(v))\}\{M(F_{i,j}(u), G_{i,j}(v)) - \Pi(F_{i,j}(u), G_{i,j}(v))\}\, dudv,$$

$$\mathcal{E}(i,j) = \int_{\frac{i}{m}}^{\frac{i+1}{m}} \int_{\frac{j}{m}}^{\frac{j+1}{m}} \{C^{i,j}(F_{i,j}(u), G_{i,j}(v)) - \Pi(F_{i,j}(u), G_{i,j}(v))\}\{W(F_{i,j}(u), G_{i,j}(v)) - \Pi(F_{i,j}(u), G_{i,j}(v))\}\, dudv.$$

We have

$$a^*_{i,j} = \frac{\mathcal{D}(i,j)\mathcal{C}(i,j) - \mathcal{B}(i,j)\mathcal{E}(i,j)}{\mathcal{A}(i,j)\mathcal{C}(i,j) - \mathcal{B}(i,j)^2}, \quad b^*_{i,j} = \frac{\mathcal{A}(i,j)\mathcal{E}(i,j) - \mathcal{B}(i,j)\mathcal{D}(i,j)}{\mathcal{A}(i,j)\mathcal{C}(i,j) - \mathcal{B}(i,j)^2}.$$

When $0 \le a^*_{i,j}, b^*_{i,j} \le 1, a^*_{i,j} + b^*_{i,j} \le 1$, the optimal value $S_{i,j}(\alpha^*_{i,j}, \gamma^*_{i,j})$ is obtained at points $\alpha^*_{i,j} = a^*_{i,j}, \gamma^*_{i,j} = b^*_{i,j}$. Otherwise, $S_{i,j}(\alpha^*_{i,j}, \gamma^*_{i,j})$ must be achieved at the boundary of the admissible region of $(\alpha_{i,j}, \gamma_{i,j})$. In this case, let $c^*_{i,j}, d^*_{i,j}, f^*_{i,j}$ denote the solutions of the following unconstrained optimization problems, that is,

$$S_{i,j}(c^*_{i,j}, 0) = \min_{c_{i,j}} S_{i,j}(c_{i,j}, 0), \quad S_{i,j}(0, d^*_{i,j}) = \min_{d_{i,j}} S_{i,j}(0, d_{i,j})$$

and

$$S_{i,j}(f^*_{i,j}, 1 - f^*_{i,j}) = \min_{f_{i,j}} S_{i,j}(f_{i,j}, 1 - f_{i,j}).$$

Then

$$c^*_{i,j} = \frac{\mathcal{D}(i,j)}{\mathcal{A}(i,j)}, \quad d^*_{i,j} = \frac{\mathcal{E}(i,j)}{\mathcal{C}(i,j)}, \quad f^*_{i,j} = \frac{\mathcal{G}(i,j)}{\mathcal{H}(i,j)},$$

where

$$\mathcal{G}(i,j) = \int_{\frac{i}{m}}^{\frac{i+1}{m}} \int_{\frac{j}{m}}^{\frac{j+1}{m}} \{C^{i,j}(F_{i,j}(u), G_{i,j}(v)) - W(F_{i,j}(u), G_{i,j}(v))\}\{M(F_{i,j}(u), G_{i,j}(v)) - W(F_{i,j}(u), G_{i,j}(v))\}\, dudv,$$

$$\mathcal{H}(i,j) = \int_{\frac{i}{m}}^{\frac{i+1}{m}} \int_{\frac{j}{m}}^{\frac{j+1}{m}} \{M(F_{i,j}(u), G_{i,j}(v)) - W(F_{i,j}(u), G_{i,j}(v)\}^2\, dudv.$$

The optimal value of the problem (4.3) is equal to the minimum of $S_{i,j}(c^*_{i,j}, 0)$, $S_{i,j}(0, d^*_{i,j})$, $S_{i,j}(f^*_{i,j}, 1 - f^*_{i,j})$, $S_{i,j}(1, 0)$, $S_{i,j}(0, 1)$ and $S_{i,j}(0, 0)$.

4.2. Properties of the optimal approximation

We first show that the optimal approximation of a BF copula is itself. This property is special to our PBF copula approximation and is not shared by any other existing copula approximations to be discussed in Section 5.

Theorem 4.1. *For each BF copula C and $m \ge 2$, its PBF optimal approximation equals to the BF copula C.*

Proof. Following Example 2.1, $C^{i,j}$ is a BF copula, and thus the optimal approximation of $C^{i,j}$ under the squared error equals $C^{i,j}$. Therefore, $C^* = C$. □

Next, we show that coefficients associated with the optimal approximation inherent some symmetry properties of the original copula. Let $\beta^*_{i,j} = 1 - \alpha^*_{i,j} - \gamma^*_{i,j}$, and denote $\mathcal{R} = (\alpha^*_{i,j})_{m\times m}$, $\mathcal{S} = (\beta^*_{i,j})_{m\times m}$ and $\mathcal{T} = (\gamma^*_{i,j})_{m\times m}$ as the $m \times m$ optimal coefficient matrices. A copula C is called symmetric if and only if $C(u,v) = C(v,u)$ for all $(u,v) \in I^2$ (Nelsen, 2006). A copula C is called radial symmetric if and only if $C(u,v) = \overline{C}(u,v)$ for all $(u,v) \in I^2$ (Georges et al., 2001), where $\overline{C}$ is C's survival copula defined as $\overline{C}(u,v) = u + v - 1 + C(1-u, 1-v)$. A matrix $\mathcal{P}$ is called centrosymmetric if and only if $\mathcal{Q}\mathcal{P}\mathcal{Q}^T = \mathcal{P}$ (Weaver, 1985), where

$$Q = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Theorem 4.2. *(i) If the copula $C(u, v)$ is symmetric, the optimal coefficient matrices $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$ are symmetric;*
(ii) If the copula $C(u, v)$ is radially symmetric, then the optimal coefficient matrices $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$ are centrosymmetric.

Proof. (i) For the symmetric copula C, we have $P(A_{i,j}) = P(A_{j,i})$ and $F_{i,j}(u) = P(U \leq u|A_{i,j}) = P(V \leq u|A_{j,i}) = G_{j,i}(u)$. From $P(U \leq u, V \leq v|A_{i,j}) = P(V \leq u, U \leq v|A_{j,i})$, we see that $C^{i,j}(F_{i,j}(u), G_{i,j}(v)) = C^{j,i}(F_{j,i}(v), G_{j,i}(u))$. Thus $C^{i,j}(F_{i,j}(u), G_{i,j}(v)) = C^{j,i}(G_{i,j}(v), F_{i,j}(u))$ follows. By the continuity of $G_{i,j}, F_{i,j}$, we see that $C^{i,j}(s, t) = C^{j,i}(t, s)$. Since M, W and Π are symmetric copulas, from our discussion of the forms of the optimal approximation we see that $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{T}$ are symmetric matrices.

(ii) Using $C(u, v) = u + v - 1 + C(1 - u, 1 - v)$, it is easy to check that $P(A_{i,j}) = P(A_{m-1-i,m-1-j})$ and

$$F_{i,j}(u) + F_{m-1-i,m-1-j}(1 - u) = 1, G_{i,j}(v) + G_{m-1-i,m-1-j}(1 - v) = 1. \tag{4.4}$$

Since $P(U \leq u, V \leq v|A_{i,j}) = P(U \geq 1 - u, V \geq 1 - v|A_{m-1-i,m-1-j})$, we have

$$C^{i,j}(F_{i,j}(u), G_{i,j}(v)) = \overline{C}^{m-1-i,m-1-j}(1 - F_{m-1-i,m-1-j}(1 - u), 1 - G_{m-1-i,m-1-j}(1 - v)). \tag{4.5}$$

Combining (4.4) and (4.5), we obtain that $C^{i,j}(F_{i,j}(u), G_{i,j}(v)) = \overline{C}^{m-1-i,m-1-j}(F_{i,j}(u), G_{i,j}(v))$, which in turn implies $C^{i,j}(s, t) = \overline{C}^{m-1-i,m-1-j}(s, t)$. For each copula $D \in \{M, W, \Pi\}$, $D(u, v) = u + v - 1 + D(1 - u, 1 - v)$. Taking $\mathcal{D}(i, j)$ for example, by change of variables and using (4.4) and (4.5), we have that

$$\begin{aligned}
&\mathcal{D}(i, j) \\
&= \int_{\frac{i}{m}}^{\frac{i+1}{m}} \int_{\frac{j}{m}}^{\frac{j+1}{m}} \{C^{m-1-i,m-1-j}(F_{m-1-i,m-1-j}(1 - u), G_{m-1-i,m-1-j}(1 - v)) - \Pi(F_{m-1-i,m-1-j}(1 - u), G_{m-1-i,m-1-j}(1 - v))\} \\
&\qquad \times \{M(F_{m-1-i,m-1-j}(1 - u), G_{m-1-i,m-1-j}(1 - v)) - \Pi(F_{m-1-i,m-1-j}(1 - u), G_{m-1-i,m-1-j}(1 - v))\}\, dudv \\
&= \int_{1-\frac{i+1}{m}}^{1-\frac{i}{m}} \int_{1-\frac{j+1}{m}}^{1-\frac{j}{m}} \{C^{m-1-i,m-1-j}(F_{m-1-i,m-1-j}(s), G_{m-1-i,m-1-j}(t)) - \Pi(F_{m-1-i,m-1-j}(s), G_{m-1-i,m-1-j}(t))\} \\
&\qquad \times \{M(F_{m-1-i,m-1-j}(s), G_{m-1-i,m-1-j}(t)) - \Pi(F_{m-1-i,m-1-j}(s), G_{m-1-i,m-1-j}(t))\}\, dsdt \\
&= \mathcal{D}(m - 1 - i, m - 1 - j).
\end{aligned}$$

Similar equations hold for the functions $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{E}, \mathcal{H}, \mathcal{G}$. The desired result then follows from our discussion of the forms of the optimal approximation. □

5. Comparing the PBF copula approximation with other approximations

5.1. Review of existing copula approximations

To give the probabilistic interpretations of various copula approximations, we introduce several random variables. As in previous sections, the random vector (U, V) has the copula C as its joint distribution. Let W, Z, $\{X_{1,r}, r = 1, \cdots, m\}$ and $\{X_{2,r}, r = 1, \cdots, m\}$ be mutually independent uniform $[0, 1]$ random variables, and be independent of (U, V).

Straight shuffle of min approximation. Following Mikusinski et al. (1992), a straight shuffle of min approximation of the copula C can be represented probabilistically as the connecting copula C_{SM} of the random vector

$$\begin{bmatrix} U_{SM} \\ V_{SM} \end{bmatrix} = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1}\left\{\begin{bmatrix} s_{i,j} + P(A_{i,j})W \\ t_{i,j} + P(A_{i,j})W \end{bmatrix}\right\} I_{A_{i,j}}, \tag{5.1}$$

where $A_{i,j} = \{\frac{i}{m} < U \le \frac{i+1}{m}, \frac{j}{m} < V \le \frac{j+1}{m}\}$, $s_{i,j} = \frac{i}{m} + P(\frac{i}{m} \le U \le \frac{i+1}{m}, 0 < V \le \frac{j}{m})$ and $t_{i,j} = \frac{j}{m} + P(0 < U \le \frac{i}{m}, \frac{j}{m} \le V \le \frac{j+1}{m})$. To compare with PBF copula, we rewrite the copula C_{SM} in the form of patched decomposition (2.4). Note that, conditional on $A_{i,j}$, the marginal distributions are

$$F_{i,j}^{SM}(u) = P(U_{SM} \le u | A_{i,j}) = \max\{\min\{\frac{u - s_{i,j}}{P(A_{i,j})}, 1\}, 0\}$$

and

$$G_{i,j}^{SM}(u) = P(V_{SM} \le v | A_{i,j}) = \max\{\min\{\frac{v - t_{i,j}}{P(A_{i,j})}, 1\}, 0\},$$

which are uniform distributions with supports $[s_{i,j}, s_{i,j+1}]$ and $[t_{i,j}, t_{i+1,j}]$ respectively. Thus, the copula C_{SM} can be written in the form (2.4) as

$$C_{SM}(u, v) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} P(A_{i,j}) M(F_{i,j}^{SM}(u), G_{i,j}^{SM}(v)), \tag{5.2}$$

where the local dependence structure is comonotonic.

Checkmin approximation. According to Mikusinski & Taylor (2008), the checkmin approximation of a copula C is the connecting copula C_{CM} of the random vector

$$\begin{bmatrix} U_{CM} \\ V_{CM} \end{bmatrix} = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} \begin{bmatrix} \frac{W+i}{m} \\ \frac{W+j}{m} \end{bmatrix} I_{A_{i,j}}. \tag{5.3}$$

Conditional on $A_{i,j}$, the conditional marginal distributions of (U_{CM}, V_{CM}) are

$$F_{i,j}^{CM}(u) = P(U_{CM} \le u | A_{i,j}) = \max\{\min\{mu - i, 1\}, 0\}$$

and

$$G_{i,j}^{CM}(u) = P(V_{CM} \le v | A_{i,j}) = \max\{\min\{mv - j, 1\}, 0\},$$

which are uniform distributions on $[\frac{i}{m}, \frac{i+1}{m}]$ and $[\frac{j}{m}, \frac{j+1}{m}]$ respectively. The checkmin copula can be written in the form of patched decomposition as

$$C_{CM}(u, v) = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} P(A_{i,j}) M(F_{i,j}^{CM}(u), G_{i,j}^{CM}(v)), \tag{5.4}$$

where the local dependence structure is comonotonic.

Checkerboard approximation. Checkerboard approximation provides an copula approximation that converges to the original copula in a sense that is stronger than the uniform convergence Li et al. (1998); Durrleman et al. (2000). According to Mikusinski & Taylor (2008), the checkerboard approximation can be represented as the connecting copula C_{CB} of the random vector

$$\begin{bmatrix} U_{CB} \\ V_{CB} \end{bmatrix} = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \begin{bmatrix} \frac{W+i}{m} \\ \frac{Z+j}{m} \end{bmatrix} I_{A_{i,j}}. \tag{5.5}$$

The conditional marginal distributions are

$$F_{i,j}^{CB}(u) = P(U_{CB} \leq u | A_{i,j}) = \max\{\min\{mu - i, 1\}, 0\}$$

and

$$G_{i,j}^{CB}(u) = P(V_{CB} \leq v | A_{i,j}) = \max\{\min\{mv - j, 1\}, 0\},$$

which are uniform distributions on $[\frac{i}{m}, \frac{i+1}{m}]$ and $[\frac{j}{m}, \frac{j+1}{m}]$ respectively. The checkerboard copula C_{CB} can be written in the form of patched decomposition as

$$C_{CB}(u, v) = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} P(A_{i,j}) \Pi(F_{i,j}^{CB}(u), G_{i,j}^{CB}(v)), \tag{5.6}$$

where the local dependent structure is independence.

Bernstein approximation. Bernstein approximation of a copula is defined as a linear combination of Bernstein polynomials with coefficients being the values of the original copula; see Scancetta & Satchell (2004) and the reference therein. Different from the three approximations discussed above, the Bernstein approximation has a continuous density. Here, in order to compare the Bernstein approximation with our PBF approximation, we rewrite it in the form of patched decomposition (2.4). For a given copula C, the Bernstein approximation is the connecting copula $C_{BS}(u, v)$ of the random vector

$$\begin{bmatrix} U_{BS} \\ V_{BS} \end{bmatrix} = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} \begin{bmatrix} X_1^{(i+1)} \\ X_2^{(j+1)} \end{bmatrix} I_{A_{i,j}}, \tag{5.7}$$

where $X_k^{(1)} \leq X_k^{(2)} \leq \cdots \leq X_k^{(m)}$ are the order statistics of $\{X_{k,r}, r = 1, \cdots, m\}$ for $k = 1, 2$ (Mikusinski & Taylor, 2008). The conditional marginal distributions are

$$F_{i,j}^{BS}(u) = P(U_{BS} \leq u | A_{i,j}) = \sum_{k=i+1}^{m} \binom{m}{k} u^k (1-u)^{m-k}$$

and

$$G^{BS}_{i,j}(u) = P(V_{BS} \leq v | A_{i,j}) = \sum_{l=j+1}^{m} \binom{m}{l} v^l (1-v)^{m-l},$$

which, unlike in the previous three approximations, are not uniform distributions. They are also clearly different from and even unrelated to the original marginal distributions $F_{i,j}$ and $G_{i,j}$. The Bernstein approximation $C_{BS}(u, v)$ can be written as

$$C_{BS}(u, v) = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} P(A_{i,j}) \Pi(F^{BS}_{i,j}(u), G^{BS}_{i,j}(v)), \tag{5.8}$$

where the local dependent structure is independence.

We are now ready to compare the probabilistic structure of the optimal PBF copula approximation with those of the above four copula approximations. All the approximations have a representation in the form of patched decomposition (2.4). From this representation, we observe that the approximations differ in two aspects: First, conditional on each subspace $A_{i,j}$, the conditional marginal distributions are different. The PBF copula approximation has the same conditional marginal distributions as the original copula on the rectangles, while checkmin, checkerboard and Bernstein approximations have conditional marginal distributions that are unrelated to the original copula. Second, the local dependence characterization is different. The optimal PBF copula distributes the probability of the cell $A_{i,j}$ along three parts, the comonotonic part, the countermonotonic part and the independent part, thus by adjusting the weights of each part, it has the flexibility to capture well the true local dependence structure. In contrast, other approximations use locally either the comonotonicity or independence, which are usually too restrictive to capture the true local dependence structure.

5.2. Numerical comparison between the optimal PBF approximation and other approximations

In this subsection we compare through examples the optimal PBF copula approximation with the straight shuffle of min, checkmin, checkerboard, and Bernstein approximations in terms of contours and approximation errors. Recall that the s-contour of copula $C(u, v)$ is defined as

$$L_s = \{(u, v) \in [0, 1]^2 | C(u, v) = s\}, \qquad s \in [0, 1].$$

Archimedean copulas can be expressed as

$$C(u, v) = \phi^{[-1]}(\phi(u) + \phi(v)),$$

where $\phi^{[-1]}$ denotes the pseudo-inverse function of the generator ϕ. The Gumbel and Clayton copulas to be discussed below are special cases of the Archimedean copulas. An archimedean copula and its generator are said to be strict if $\phi(0) = \infty$ and non-strict otherwise.

We consider six different copula functions:

1. Gaussian Copula discussed in Example 2.2 with $\rho = -0.8$;
2. Student t copula $C(u, v) = t_{\rho,\nu}(t_\nu^{-1}(u), t_\nu^{-1}(v))$ with parameters $\rho = 0.3, \nu = 3$;
3. Farlie-Gumbel-Morgenstern (FGM) copula discussed in Example 2.3 with $\theta = 1/2$.
4. Gumbel copula with a strict generator $\phi(t) = (-\ln t)^{\theta_1}$, where $\theta_1 = 2$. Note that this Gumbel copula expresses upper tail dependence and can be written as

$$C(u, v) = \exp\{-[(-ln(u))^2 + (-ln(v))^2]^{\frac{1}{2}}\}.$$

5. Clayton copula with a non-strict generator $\phi(t) = \frac{t^{-\theta_2}-1}{\theta_2}$, where $\theta_2 = -0.5$. This copula can be expressed as

$$C(u, v) = \{\max(\sqrt{u} + \sqrt{v} - 1, 0)\}^2.$$

   It is non-strict since the generator satisfies $\phi(0) = 2 < \infty$. The copula has a zero set, i.e., $C(\frac{1}{4}, \frac{1}{4}) = 0$.
6. A mixture of copula

$$C(u, v) = \lambda C_1(u, v) + (1 - \lambda)C_2(u, v),$$

   where C_1 is the Gumbel copula with $\theta_1 = 2$, C_2 is the Clayton copula with $\theta_2 = 1/3$, and the mixture parameter $\lambda = 1/3$. This copula is a convex combination of two Archimedean copulas, one with upper tail dependence and the other with lower tail dependence.

Table 5.1 gives the coefficients of the optimal BPF approximation of the above six copulas. For the Gaussian copula, the positive $\gamma^*_{i,j}$ coefficients reflect well the negative correlation. The optimal coefficient matrices for the Gumbel and the Clayton copulas are symmetric, as expected by Theorem 4.2, since the Archimedean copulas are symmetric. We also observed from Table 5.1 that the optimal PBF approximation captures very well the upper tail dependence of the Gumbel copula: The comonotonic weighting $\alpha^*_{2,2}$ is much larger than other $\alpha^*_{i,j}$'s.

Contours of various approximations of the Clayton copula are shown in Figure 5.1 and indicate that the PBF approximation is closer to the original copula than other approximations. Contours of approximations of other five copulas give the same message but graphs are not shown to save space. Figure 5.2 presents the L_2 errors of various approximations of the above six copulas for varying m. It is clear from this figure that the PBF approximation performs uniformly better than the other approximations.

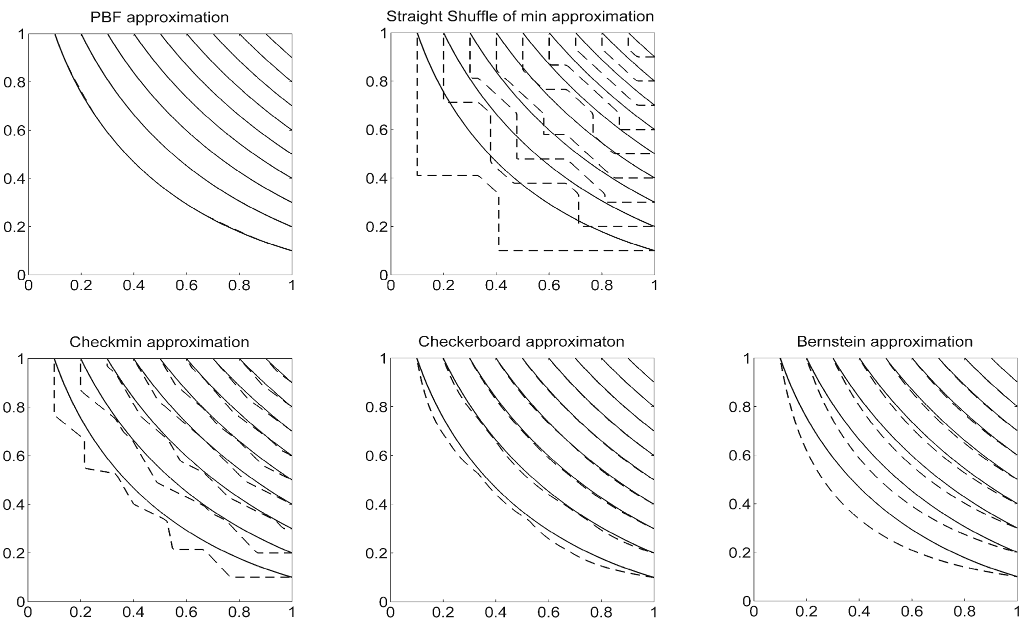
6. Applications in Insurance and Finance

We apply the PBF approximation for computing the rainbow option price and the stop-loss premium and compare the results with those obtained from straight shuffle of min, checkmin, checkerboard and Bernstein approximations.

Table 5.1: The coefficients of the optimal PBF approximation of six copulas for $m = 3$. The parameters of each copula are given in the text.

Gaussian	$(\alpha^*_{i,j}, \gamma^*_{i,j})$	$i = 0$	$i = 1$	$i = 2$
	$j = 2$	$(0, 0.4033)$	$(0, 0.1295)$	$(0, 0.0870)$
	$j = 1$	$(0, 0.1295)$	$(0, 0.1158)$	$(0, 0.1295)$
	$j = 0$	$(0, 0.0870)$	$(0, 0.1295)$	$(0, 0.4033)$
student t	$(\alpha^*_{i,j}, \gamma^*_{i,j})$	$i = 0$	$i = 1$	$i = 2$
	$j = 2$	$(0.4108, 0.5892)$	$(0.5087, 0.4913)$	$(0.6402, 0.3958)$
	$j = 1$	$(0.5087, 0.4913)$	$(0.5175, 0.4825)$	$(0.5087, 0.4913)$
	$j = 0$	$(0.6402, 0.3958)$	$(0.5087, 0.4913)$	$(0.4108, 0.5892)$
FGM	$(\alpha^*_{i,j}, \gamma^*_{i,j})$	$i = 0$	$i = 1$	$i = 2$
	$j = 2$	$(0.0279, 0)$	$(0.0169, 0)$	$(0.0113, 0)$
	$j = 1$	$(0.0169, 0)$	$(0.0169, 0)$	$(0.0169, 0)$
	$j = 0$	$(0.0113, 0)$	$(0.0169, 0)$	$(0.0279, 0)$
Gumbel	$(\alpha^*_{i,j}, \gamma^*_{i,j})$	$i = 0$	$i = 1$	$i = 2$
	$j = 2$	$(0.0147, 0)$	$(0.0656, 0)$	$(0.5006, 0)$
	$j = 1$	$(0.0643, 0)$	$(0.1014, 0)$	$(0.0656, 0)$
	$j = 0$	$(0.2271, 0)$	$(0.0643, 0)$	$(0.0147, 0)$
Clayton	$(\alpha^*_{i,j}, \gamma^*_{i,j})$	$i = 0$	$i = 1$	$i = 2$
	$j = 2$	$(0, 0.1210)$	$(0, 0)$	$(0, 0)$
	$j = 1$	$(0, 0.2871)$	$(0, 0)$	$(0, 0)$
	$j = 0$	$(0, 0.4450)$	$(0, 0.2871)$	$(0, 0.1210)$
mixture	$(\alpha^*_{i,j}, \gamma^*_{i,j})$	$i = 0$	$i = 1$	$i = 2$
	$j = 2$	$(0.5035, 0.4965)$	$(0.4973, 0.5027)$	$(0.6222, 0.3778)$
	$j = 1$	$(0.5198, 0.4802)$	$(0.5256, 0.4744)$	$(0.4973, 0.5027)$
	$j = 0$	$(0.6028, 0.3972)$	$(0.5198, 0.4802)$	$(0.5035, 0.4965)$

Figure 5.1: The coutours of a Clayton copula with a non-strict generator and various copula approximations with $m = 3$. The solid lines represent the Clayton copula and the dashed lines represent approximations.



6.1. Pricing Rainbow Options

Rainbow options refer to options whose payoff depends on more than one underlying risky assets and each asset is referred to as a color of the rainbow. Options on the minimum or the maximum of two risky assets are useful for pricing a wide variety of contingent claims in finance. Analytical formulas for European put and call options on the minimum or the maximum of two risky assets were provided by Stulz (1982) and extended to the case of several risky assets by Johson (1987).

Let $S_1(t)$ and $S_2(t)$ represent the price processes of two assets, and the instantaneous rate of interest r is constant through time. The payoff for a bivariate rainbow option on the minimum with the strike price K and maturity T is

$$\max[\min\{S_1(T), S_2(T)\} - K, 0].$$

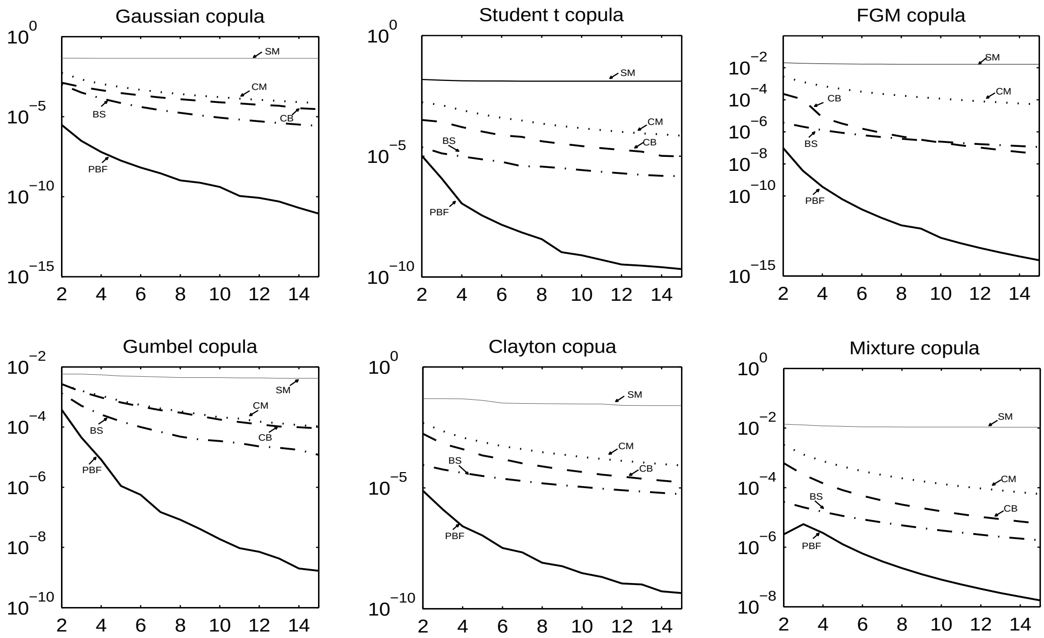
Suppose the price processes $S_1(t)$ and $S_2(t)$ satisfy

$$dS_1(t)/S_1(t) = rdt + \sigma_1 dW_1(t), \qquad dS_2(t)/S_2(t) = rdt + \sigma_2 dW_2(t), \tag{6.1}$$

where $W_1(t)$ and $W_2(t)$ are standard Wiener processes with the correlation coefficient ρ which is assumed to be constant through time. The real price of the option at time t is

$$G(S_1, S_2, t; K, T) = E\{e^{-r(T-t)} \max[\min\{S_1(T), S_2(T)\} - K, 0] | S_1(s), S_2(s), s \leq t\}. \tag{6.2}$$

Figure 5.2: The L_2 errors of various patched copula approximations vs m for six copula functions. PBF, SM, CM, CB and BS represent respectively the patched bivariate Fréchet, straight shuffle of min, checkmin, checkerboard, and Bernstein approximations.



Denote $\tau = T - t$. It follows that the connecting copula C_t of $(S_1(T), S_2(T))$ conditional on $\{S_1(s), S_2(s), s \leq t\}$ is a Gaussian copula with the correlation coefficient ρ. Detailed calculation yields that

$$G(S_1, S_2, t; K, T) = S_1(t)\Phi_{\rho_1}(d_1 + \sigma_1\sqrt{\tau}, \{\ln(S_2(t)/S_1(t)) - \frac{1}{2}\sigma^2\tau\}/\sigma\sqrt{\tau})$$
$$+S_2(t)\Phi_{\rho_2}(d_2 + \sigma_2\sqrt{\tau}, \{\ln(S_1(t)/S_2(t)) - \frac{1}{2}\sigma^2\tau\}/\sigma\sqrt{\tau}) \qquad - K\exp(-r\tau)\Phi_\rho(d_1, d_2), \tag{6.3}$$

where

$$\rho_1 = (\rho\sigma_2 - \sigma_1)/\sigma, \quad \rho_2 = (\rho\sigma_1 - \sigma_2)/\sigma, \quad \sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2,$$

and

$$d_1 = \{ln(S_1(t)/K) + (r - \frac{1}{2}\sigma_1^2)\tau\}/\sigma_1\sqrt{\tau}, \quad d_2 = \{\ln(S_2(t)/K) + (r - \frac{1}{2}\sigma_2^2)\tau\}/\sigma_2\sqrt{\tau}.$$

Next, we give an expression of the option price in terms of a copula and apply various copula approximations for computation. We can verify that

$$G(S_1, S_2, t; K, T) = G(S_1, t; K, T) + G(S_2, t; K, T) - e^{-r(T-t)}\int_K^\infty \{1 - C_t(F_1(u|t), F_2(u|t))\}\, du, \tag{6.4}$$

where $G(S_i, t; K, T)$ denotes the price of the call option on S_i at time t, i.e.,

$$G(S_i, t; K, T) = E\{e^{-r(T-t)} \max[S_i(T) - K, 0] | S_1(s), S_2(s), s \leq t\} = S_i(t)\Phi(d_i + \sigma_i \sqrt{\tau}) - e^{-r\tau} K\Phi(d_i) \quad (6.5)$$

for $i = 1, 2$, and $F_1(u|t)$, $F_2(u|t)$ are the marginal distributions of $(S_1(T), S_2(T))$ conditional on $\{S_1(s), S_2(s), s \leq t\}$. Replacing the Gaussian copula in (6.4) by various approximations, we obtain approximated values of the option price, which can be compared with the real price given in (6.3) to evaluate the quality of the approximations.

Table 6.2 presents a comparison in a setting when $r = 0.02$, $t = 0$, $T = 2$, $S_1(0) = S_2(0) = 1$, $\rho = 0.6322$, $\sigma_1 = 0.3286$, and $\sigma_2 = 0.43$. The numerical integration is done in Matlab using the composite-midpoint method. It is clear that the PBF approximation performs the best, and the quality of approximation is high even when m is small.

Table 6.2: Comparison of approximated prices of a rainbow option using various copula approximations when $m = 4$. PBF, SM, CM, CB and BS represent respectively the patched bivariate Fréchet, straight shuffle of min, checkmin, checkerboard, and Bernstein approximations. The numbers shown are the actual numbers multiplied by 10^4.

Strike Price	Real Price	PBF	SM	CM	CB	BS
$K = 1$	1104	1112	2268	1637	901.8	388.1
$K = 2$	95.27	98.06	202.6	145.5	37.15	34.34
$K = 3$	10.39	9.491	24.18	12.85	1.865	1.850

6.2. Loss and the allocated expenses

Consider the stop-loss premium for the loss amount and allocated adjustment expenses (ALAE) on a single claim. Let X and Y denote the loss amount and ALAE, and their distributions are denoted as F_1 and F_2 respectively. The stop-loss premium is defined as

$$E(X + Y - d)_+ = E\{F_1^{-1}(U) + F_2^{-1}(V) - d\}_+, \quad (6.6)$$

where d is a constant to be determined by the insurer, the connecting copula of (U, V) is $C(u, v; \theta)$, and U, V are uniform $[0, 1]$ random variables. Following Frees and Valdez (1998), the marginal distributions are assumed to be Pareto distributions

$$F_i(x) = 1 - (1 + \xi_i \frac{x}{\gamma_i})^{-\frac{1}{\xi_i}}, \quad i = 1, 2,$$

and the connecting copula is the Gumbel copula

$$C(u, v; \theta) = \exp\{-[(-ln(u))^\theta + (-ln(v))^\theta]^{\frac{1}{\theta}}\},$$

where $\theta \geq 1$. Cebrián et al. (2002) fitted the above model using a data set with 1,466 uncensored observed values of the random vector (X, Y) and obtained the parameter estimates $\hat{\xi}_1 = 0.760$, $\hat{\gamma}_1 = 12816.9$, $\hat{\xi}_2 = 0.425$,$\hat{\gamma}_2 = 6756.5$, and $\hat{\theta} = 1.425$.

For these estimated parameter values, we calculated the stop-loss preminum using various patched copula approximations with $m = 3$. The probabilistic representation of the copula approximations were applied in our calculation. In particular, we used (3.1) and (3.2) for the PBF approximation, (5.1) for the straight shuffle of min approximation, (5.3) for the checkmin approximation, (5.5) for the checkerboard approximation, and (5.7) for the Bernstein approximation. It is clear from Table 6.3 that the values based on the optimal PBF approximation is closer to the actual values than all other approximations.

Table 6.3: Comparison of the values of the stop-loss premium based on the original copula and various approximations with $m = 3$. PBF, SM, CM, CB and BS represent respectively the patched bivariate Fréchet, straight shuffle of min, checkmin, checkerboard, and Bernstein approximations.

	Original	PBF	SM	CM	CB	BS
$d = 10^3$	56156	56328	57981	57218	54484	50592
$d = 10^4$	48468	48657	50935	49605	47025	43666

7. Conclusions

We develop a new approximation scheme for bivariate copulas where a copula is written as a convex combination of locally supported distributions and BF copulas are used to characterize the local dependence. We have developed a bound for the approximation error and showed that it goes to zero when the number of local patches goes to infinity. We have also given a probabilistic representation of a PBF copula and discussed how to use the representation for computing integrals with respect to the PBF copula. Our numerical examples show that the PBF copulas outperform some existing copula approximations. In particular, the PBF copulas provide good characterization of the local dependence while the other approximations mentioned in this paper fail to do so.

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