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LIMIT THEOREMS FOR THE SHIFTING LEVEL PROCESS

DAREN B. H. CLINE,* *Colorado State University*

Abstract

This paper studies the asymptotic properties of moment estimators for the general shifting level process (SLP). A law of large numbers and a weak convergence theorem are obtained under conditions involving the unobservable processes which make up SLP. Specific conditions about those underlying processes are added to give explicit results, applicable to a large class of moment estimators. Actual formulae for asymptotic variances, etc. are obtained for a simple example, the GNN model.

SHIFTING LEVEL PROCESS; LAW OF LARGE NUMBERS; WEAK CONVERGENCE; MOMENT ESTIMATORS; ERGODICITY; ϑ-MIXING; l-DEPENDENCE

1. Introduction

The shifting level process (SLP) is an alternative to the familiar ARMA time series model. The process was first formulated by Boes and Salas (1978) as a possible explanation of the Hurst phenomenon in hydrological time series. Special cases of the model were suggested earlier by Hurst (1957) and Klemes (1974). Boes and Salas showed also that the covariance structure of SLP is in some cases identical to that of ARMA, providing additional rationale for the use of SLP in time series modeling.

Random changes ('shifts') in the attributes ('levels') of the model characterize the shifting level process, thus giving it its name. In addition to the moments and covariance structure, therefore, the frequency and the extent of the level shifts require estimation. Maximum likelihood estimation is difficult, however, because the density is a complex mixture. Moment estimators may thus provide cheaper, if not more precise, estimates. This paper establishes basic results concerning the asymptotic behavior of moment estimators, namely a law of large numbers and a central limit theorem. (On the other hand, it does not attempt to

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* Postal address: Department of Statistics, Colorado State University, Fort Collins, CO 80523, U.S.A.
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discuss estimation procedures or even to justify the use of moment estimators in a general model.)

The second section discusses the law of large numbers for sample moments, while the third section concentrates on weak convergence. Calculations are covered in Section 4 and a simple model illustrates applications in Section 5.

The shifting level process. Let $(\boldsymbol{X}, \mathscr{X})$, $(\boldsymbol{\Lambda}, \mathscr{L})$ and $(\boldsymbol{N}, \mathscr{N})$ be measurable spaces, with $\boldsymbol{N} = \{1, 2, 3, \cdots\}$. Also let $(\Omega, \mathscr{F}, P)$ be the underlying probability space.

Definition 1.0. If $\{\{X_j^{(\lambda)}\}_{j=1}^{\infty}, \lambda \in \boldsymbol{\Lambda}\}$ is a family of stochastic processes on $(\Omega, \mathscr{F}, P)$ with elements in $\boldsymbol{X}$, and $\{N_n, \Lambda_n\}_{n=1}^{\infty}$ is a stochastic process on $(\Omega, \mathscr{F}, P)$ with elements in $\boldsymbol{N} \times \boldsymbol{\Lambda}$, then the process

$$\begin{aligned}\{Y_\alpha\}_{\alpha=1}^{\infty} &= \{X_1^{(\Lambda_1)}, X_2^{(\Lambda_1)}, \cdots, X_{N_1}^{(\Lambda_1)}, X_1^{(\Lambda_2)}, \cdots, X_{N_2}^{(\Lambda_2)}, X_1^{(\Lambda_3)}, \cdots\} \\ &= \{\{X_j^{(\Lambda_n)}\}_{j=1}^{N_n}\}_{n=1}^{\infty}\end{aligned}$$

is called a *shifting level process* with shift epochs $\{T_n\}_{n=1}^{\infty} = \{N_1 + \cdots + N_n\}_{n=1}^{\infty}$, levels $\{\Lambda_n\}_{n=1}^{\infty}$ and underlying processes $\{X_j^{(\lambda)}\}_{j=1}^{\infty}$, $\lambda \in \boldsymbol{\Lambda}$.

Essentially, the process $\{Y_\alpha\}$ is constructed as a concatenation of segments of random length, randomly selected from the family of processes $\{X_j^{(\lambda)}\}$. The example which will be used at the end of this paper is the geometric–normal–normal (GNN) model, where $\{N_n\} \sim$ i.i.d. $G(\pi)$, $\{\Lambda_n\} \sim$ i.i.d. $N(\mu, \rho\sigma^2)$, $\{X_j^{(\lambda)}\} \sim$ i.i.d. $N(\lambda, (1-\rho)\sigma^2)$ with $\rho, \pi \in (0, 1)$ and all processes are independent. In this case, the process shifts randomly from mean to mean with Λ_n as the mean of the nth segment. The parameters of chief interest in estimation are the overall mean μ, the composite variance σ^2, the relative frequency of shifts π, and the relative variance of shift ρ.

Sample moments. In practice $\{N_n, \Lambda_n\}$ is unobservable. Thus estimators will consist entirely of functions of $\{Y_\alpha\}$. In particular, they can be functions of the sample moments:

$$\frac{1}{\alpha} S_\alpha^f = \frac{1}{\alpha} \sum_{i=1}^{\alpha} f(Y_i, Y_{i+1}, \cdots, Y_{i+k}), \quad f : \boldsymbol{X}^{k+1} \to \boldsymbol{R}^P.$$

It will suffice to consider only real-valued sample moments as estimators, because the convergence theorems are easily extended to vector-valued moments and to continuous functions of the sample moments. It is assumed that the parameters of interest are actually estimable by using sample moments. In certain models this may not be the case. For purposes of the following limit theorems, here are defined the (unobservable, auxiliary) random variables,

$$R_n^f = \sum_{i=T_{n-1}+1}^{T_n} f(Y_i, \cdots, Y_{i+k}) = \sum_{j=1}^{N_n} f(X_j^{(\Lambda_n)}, X_{I_{j+1}}^{(L_{j+1})}, \cdots, X_{I_{j+k}}^{(L_{j+k})})$$

where

$$L_j = \Lambda_{m_j}, \quad I_j = j - (T_{m_j} - T_n) \quad \text{and} \quad m_j \text{ satisfies } (T_{m_j} - T_n) < j \leqq (T_{m_{j+1}} - T_n).$$

Basically, R_n^f is the sum of the functional values between the $(n-1)$th and the nth shift epochs. For example, for

$$f: \boldsymbol{X} \to \boldsymbol{R}, \quad R_n^f = \sum_{i=T_{n-1}+1}^{T_n} f(Y_i) = \sum_{j=1}^{N_n} f(X_j^{(\Lambda_n)}),$$

and for

$$f: \boldsymbol{X}^2 \to \boldsymbol{R}, \quad R_n^f = \sum_{j=1}^{N_n - 1} f(X_j^{(\Lambda_n)}, X_{j+1}^{(\Lambda_n)}) + f(X_{N_n}^{(\Lambda_n)}, X_1^{(\Lambda_{n+1})}).$$

For convenience in the succeeding sections, let

$$f_j^{(n)} = f(X_j^{(\Lambda_n)}, X_{I_{j+1}}^{(L_{j+1})}, \cdots, X_{I_{j+k}}^{(L_{j+k})})$$

and

$$U_n^f = \sum_{j=1}^{n} R_j^f.$$

Note that R_n^f is measurable with respect to the field $\sigma(\{X_j^{(\lambda)}\}_{j=1}^{\infty}, \lambda \in \{\Lambda_i\}_{i=n}^{n+k}, \{N_i\}_{i=n}^{n+k})$. Hence, when $\{X_j^{(\lambda)}\}$ are independent processes and independent of $\{N_n, \Lambda_n\}$ which itself is l-dependent (l-order Markov), then $\{N_n, R_n^f\}$ is $(l+k)$-dependent ($(l+k)$-order Markov with respect to a σ-field expanded to include $\{X_j^{(\lambda)}\}$, $\lambda \in \Lambda$).

The sample moments $(1/\alpha)S_\alpha^f$ will generally behave similarly to the sample means of the process $\{N_n, R_n^f\}$. This is a reasonable expectation, because the quantity $(1/\alpha)S_\alpha^f$ will be something like

$$\frac{1}{\alpha} S_\alpha^f \approx \frac{R_1^f + R_2^f + \cdots + R_M^f}{N_1 + N_2 + \cdots + N_M}$$

for some random index M. Furthermore, because of the relationship $R_n^f = \sum_{j=1}^{N_n} f_j^{(n)}$, it is reasonable to expect R_n^f and N_n to behave similarly. Therefore, the major theorems following will assume convergence for the sample mean of the process $\{N_n, R_n^f\}$ and the corollaries will demonstrate conditions on the underlying processes $\{N_n, \Lambda_n\}$, $\{X_j^{(\lambda)}\}$ to satisfy this.

2. Law of large numbers

A law of large numbers will show that the sample moments of SLP are consistent under general conditions. Theorem 2.0 states the convergence of the sample moment $(1/\alpha)S_\alpha^f$ generally, but with the assumption of convergence for

$(1/n)T_n = (1/n)\Sigma_{j=1}^n N_j$ and for $(1/n)U_n^f = (1/n)\Sigma_{j=1}^n R_j^f$. The remaining part of the section discusses convergence for particular cases.

Theorem 2.0. Let $\{Y_\alpha\}_{\alpha=1}^\infty = \{\{X_j^{(\Lambda_n)}\}_{j=1}^{N_n}\}_{n=1}^\infty$ be a shifting level process. If $(1/n)T_n \to N$ almost surely (in probability), $P[N < \infty] = 1$ and $(1/n)U_n^f \to R$ almost surely (in probability) and $(1/n)R_n^{|f|} \to 0$ almost surely (in probability), then $(1/\alpha)S_\alpha^f \to RN^{-1}$ almost surely (in probability).

Proof. The proof is identical in the cases of almost sure convergence or convergence in probability. Let $M_n = \max\{j : T_j \leqq n\}$. Both $M_n \to \infty$ and $T_n \to \infty$. Also note $(1/n)T_n \geqq 1$ almost surely so that $N > 0$ almost surely.

Then, too,

$$\frac{T_{M_n}}{M_n} \leqq \frac{n}{M_n} < \frac{T_{M_n+1}}{M_n}.$$

Therefore, $(1/n)T_n \to N$ implies $(1/n)M_n \to N^{-1}$.

Thus,

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{T_{M_n}} f(Y_i, \cdots, Y_{i+k}) &= \frac{1}{n}\sum_{j=1}^{M_n}\sum_{i=1}^{N_j} f_i^{(j)} \\ &= \frac{1}{n}\sum_{j=1}^{M_n} R_j^f \\ &= \left(\frac{1}{M_n} U_{M_n}\right)\left(\frac{M_n}{n}\right) \\ &\to RN^{-1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left|\frac{1}{n}\sum_{i=T_{M_n}+1}^{n} f(Y_i, \cdots, Y_{i+k})\right| &= \frac{1}{n}\left|\sum_{i=1}^{n-T_{M_n}} f_i^{(M_n+1)}\right| \\ &\leqq \frac{1}{n}\sum_{i=1}^{N_{M_n+1}} |f_i^{(M_n+1)}| \\ &= \left(\frac{M_n+1}{n}\right)\left(\frac{1}{M_n+1} R_{M_n+1}^{|f|}\right) \\ &\to 0, \quad \text{since } \frac{1}{n} R_n^{|f|} \to 0.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{n} S_n^f &= \frac{1}{n}\sum_{i=1}^{T_{M_n}} f(Y_i, \cdots, Y_{i+k}) + \frac{1}{n}\sum_{i=T_{M_n}+1}^{n} f(Y_i, \cdots, Y_{i+k}) \\ &\to RN^{-1}.\end{aligned}$$

In Theorem 2.0, the condition $(1/n)R_n^{|f|} \to 0$ is required, but in most examples it can easily be shown. Before giving an example, two lemmas are provided. The first is given without proof. For the second, refer to Billingsley (1968) and Breiman (1968), respectively, for definitions of ϑ-mixing and ergodicity.

Lemma 2.1. Assume $\{Z_n\}$ is a real-valued sequence of random variables with l-dependence. Let $\bar{Z}_n = (1/n)\Sigma_{j=1}^n Z_j$.

1. If $\{Z_n\}$ are identically distributed, with $E[Z_n] = \theta < \infty$, then $\bar{Z}_n \to \theta$ almost surely.
2. If $E[Z_n] = \theta_n \to \theta$ and there exists $\beta < 1$, $K > 0$ such that $V[Z_n] = \gamma_n^2 \leqq Kn^\beta$, then $\bar{Z}_n \to \theta$ almost surely.

Lemma 2.2. Let $(\boldsymbol{T}, \mathscr{T})$ and $(\boldsymbol{\Delta}, \mathscr{D})$ be measurable spaces, with $\mathscr{D}$ the Borel class of events in $\boldsymbol{\Delta}$. Let $\Delta = \{\Delta_\tau, \tau \in \boldsymbol{T}\}$ be a family of independent random elements in $\boldsymbol{\Delta}$, and independent of $\{\tau_n\}$, a stationary process of elements in $\boldsymbol{T}$, such that $P[\tau_n = \tau_m] = 0$ for $n \neq m$. Then the process $\{\Delta_{\tau_n}\}$ is stationary and

1. if $\{\tau_n\}$ is ϑ-mixing, then $\{\Delta_{\tau_n}\}$ is ϑ-mixing;
2. if $\{\tau_n\}$ is ergodic, then $\{\Delta_{\tau_n}\}$ is ergodic.

Proof. The measurable functional $\psi(t, \delta(\cdot)) = \delta(t)$ gives $\Delta_{\tau_n} = \psi(\tau_n, \Delta)$ for every n. Letting $A_1, A_2, \cdots, A_l \in \mathscr{D}$,

$$\begin{aligned}
P[\Delta_{\tau_i} \in A_i\,; i = 1, \cdots, l] &= P[(\tau_i, \Delta) \in \psi^{-1}(A_i); i = 1, \cdots, l] \\
&= \int P[(\tau_i, \delta) \in \psi^{-1}(A_i); i = 1, \cdots, l] P[\Delta \in d\delta] \\
&\qquad \text{by independence of } \Delta \text{ and } \{\tau_n\}. \\
&= \int P[(\tau_{n+i}, \delta) \in \psi^{-1}(A_i); i = 1, \cdots, l] P[\Delta \in d\delta] \\
&\qquad \text{by stationarity of } \{\tau_n\}. \\
&= P[\Delta_{\tau_{n+i}} \in A_i\,; i = 1, \cdots, l], \quad \text{for every } n.
\end{aligned}$$

Therefore, $\{\Delta_{\tau_n}\}$ is stationary.

If $\{\tau_n\}$ is ϑ-mixing, a similar calculation shows that $\{\Delta_{\tau_n}\}$ is also. Let $A_1, A_2, \cdots \in \mathscr{D}$. Then

$$\begin{aligned}
p(n,j) &= P[\Delta_{\tau_i} \in A_i\,; i \geqq n+j \mid \Delta_{\tau_i} \in A_i\,; i \leqq n] - P[\Delta_{\tau_i} \in A_i\,; i \geqq n+j] \\
&= \int \{P[(\tau_i, \delta) \in \psi^{-1}(A_i); i \geqq n+j \mid (\tau_i, \delta) \in \psi^{-1}(A_i); i \leqq n] \\
&\qquad - P[(\tau_i, \delta) \in \psi^{-1}(A_i); i \geqq n+j]\} P[\Delta \in d\delta] \quad \text{by independence.}
\end{aligned}$$

Therefore, using the ϑ-mixing of $\{\tau_n\}$,

$$|p(n,j)| \leqq \int \vartheta(j) P[\Delta \in d\delta] = \vartheta(j).$$

Hence $\{\Delta_{\tau_n}\}$ is ϑ-mixing.

Now suppose that $\{\tau_n\}$ is stationary ergodic, and let the random variable Z be bounded and invariant for $\{\Delta_{\tau_n}\}$. Thus for some function β_1, $Z = \beta_1(\Delta_{\tau_n}, \Delta_{\tau_{n+1}}, \cdots)$ for all n. Let $B \in \mathscr{B}$, the Borel class on $\boldsymbol{R}$, $\{t_n\}$ a sequence in $\boldsymbol{\Delta}$ with $t_n \neq t_m$.

$$\begin{aligned} P[Z \in B \mid \tau_m = t_m\,; m \geqq n] &= P[(\Delta_{\tau_n}, \Delta_{\tau_{n+1}}, \cdots) \in \beta_1^{-1}(B) \mid \tau_m = t_m\,; m \geqq n] \\ &= P[(\Delta_{t_n}, \Delta_{t_{n+1}}, \cdots) \in \beta_1^{-1}(B)] \quad \text{by independence} \\ &= 0 \text{ or } 1 \text{ by Kolmogorov's 0–1 law since } \{\Delta_{t_n}\} \text{ is an} \\ &\quad \text{independent sequence for } t_n \neq t_m. \end{aligned}$$

The conditional probability distribution of Z, given $\tau_n, \tau_{n+1}, \cdots$, being trivial, it follows that a function β_2 exists so that $Z = \beta_2(\tau_n, \tau_{n+1}, \cdots)$ almost surely for every n. That is, Z is invariant for $\{\tau_n\}$. But $\{\tau_n\}$ is ergodic and hence Z is almost surely constant. This shows that $\{\Delta_{\tau_n}\}$ is ergodic.

Theorem 2.0 can now be applied to SLP in the following two particular cases.

Corollary 2.3. Let $\{Y_\alpha\} = \{\{X_j^{(\Lambda_n)}\}_{j=1}^{N_n}\}$ be a shifting level process such that

(i) $\{N_n, \Lambda_n\}$ is a sequence of random elements in $\boldsymbol{N} \times \boldsymbol{\Lambda}$ with $P[\Lambda_n = \Lambda_m] = 0$, $n \neq m$.

(ii) $\{X_j^{(\lambda)}\}$, $\lambda \in \boldsymbol{\Lambda}$ is a family of independent stochastic processes and independent of $\{N_n, \Lambda_n\}$.

Let $f : \boldsymbol{X}^{k+1} \to \boldsymbol{R}$ and define R_n^f and S_α^f as before.

1. If $\{N_n, \Lambda_n\}$ is stationary, ergodic and $E[N_n] = \eta < \infty$, $E[R_n^f] = \eta\theta < \infty$, then $(1/\alpha) S_\alpha^f \to \theta$ almost surely.

2. If $\{N_n, \Lambda_n\}$ is l-dependent, and $E[N_n] \to \eta$, $E[R_n^f] \to \eta\theta$, $E[R_n^{|f|}] \to \eta\zeta$ and $V[R_n^f] \leqq Kn^\beta$, $V[R_n^{|f|}] \leqq Kn^\beta$, $V[N_n] \leqq Kn^\beta$, $\beta < 1$, then $(1/\alpha) S_\alpha^f \to \theta$ almost surely.

Case 1. By Lemma 2.2, setting $\tau_n = (N_n, \Lambda_n)$ and $\Delta_{(\nu,\lambda)} = (\nu, \{X_j^{(\lambda)}\}_{j=1}^{\infty})$ the random sequence $\{\Delta_{\tau_n}\}_{n=1}^{\infty} = \{N_n, \{X_j^{(\Lambda_n)}\}_{j=1}^{\infty}\}_{n=1}^{\infty}$ is stationary ergodic. It is clear that $(N_n, R_n^f, R_n^{|f|})$ relies on $\{N_m, \{X_j^{(\Lambda_m)}\}\}_{m=n}^{n+k}$ in the same way for every n, i.e., that ψ exists such that

$$(N_n, R_n^f, R_n^{|f|}) = \psi(\Delta_{\tau_n}, \Delta_{\tau_{n+1}}, \cdots, \Delta_{\tau_{n+k}}) \quad \text{for every } n.$$

Therefore, $\{N_n, R_n^f, R_n^{|f|}\}$ is also stationary ergodic. The ergodic theorem thus gives

$$\frac{1}{n}\sum_{j=1}^{n} N_j \to \eta \quad \text{a.s.}$$

$$\frac{1}{n}\sum_{j=1}^{n} R_j^f \to \eta\theta \quad \text{a.s.}$$

$$\frac{1}{n}\sum_{j=1}^{n} R_j^{|f|} \to \eta\zeta \quad \text{a.s.}$$

where $\zeta = (1/\eta)E[R_n^{|f|}]$. In particular, $(1/n)R_n^{|f|} \to 0$ a.s. Applying Theorem 2.0,

$$\frac{1}{\alpha} S_\alpha^f \to (\eta\theta)\eta^{-1} = \theta \quad \text{a.s.}$$

Case 2. Since $\{\Lambda_n\}$ are almost surely distinct and $\{X_j^{(\lambda)}\}$ are independent processes, then $\{X_j^{(\Lambda_n)}\}$ and $\{X_j^{(\Lambda_m)}\}$ are independent processes whenever $|n-m|>l$. Also $R_n^f \in \sigma(\{X_i^{(\Lambda_j)}\}_{i=1}^{N_j};\ j=n,n+1,\cdots,n+k)$. It follows, then, that $\{R_n^f, N_n\}$ is $k+l$-dependent.

Lemma 2.1.2 implies that

$$\frac{1}{n}\sum_{j=1}^{n} R_j^f \to \eta\theta \quad \text{a.s.} \quad \text{and} \quad \frac{1}{n}\sum_{j=1}^{n} N_j \to \eta \quad \text{a.s.}$$

and

$$\frac{1}{n}\sum_{j=1}^{n} R_j^{|f|} \to \eta\zeta \quad \text{so that } \frac{1}{n} R_n^{|f|} \to 0 \quad \text{a.s.}$$

By Theorem 2.0, then, $(1/\alpha)S_\alpha^f \to \theta$ a.s.

Note that the condition $P[\Lambda_n = \Lambda_m] = 0$ presents no problem since $\Lambda_n' = (\Lambda_n, n)$ may be used instead.

Thus sample moments of SLP are consistent estimators in these two special cases. When SLP is built from different types of processes, similar theorems may be applied to get the consistency, such as whenever $\{N_n, \Lambda_n\}$ behaves well and appropriate moments of R_n^f exist.

3. Weak convergence

This section covers the convergence in distribution of the sample moments of a shifting level process, thus helping to determine the precision of the estimators. The section is organized like Section 2. It begins with a general theorem requiring convergence conditions on $\{R_n^f, N_n\}$ and continues with particular cases when the conditions are met.

Let $\boldsymbol{D}$ be the space of right-continuous functions on $[0,\infty)$ with finite left limits, and let $T_n' = \sum_{j=1}^{n}(N_j - \eta)$ and $U_n' = \sum_{j=1}^{n}(R_j^f - \eta\theta)$, where $f : \boldsymbol{X}^{k+1} \to \boldsymbol{R}$. Also, define $f-\theta$ to be the function with values $f(x_1,\cdots,x_{k+1}) - \theta$. $[n\,\cdot\,]$ represents the function with values equal to the integer part of (nt).

Theorem 3.0. If, as jointly distributed random functions on $\boldsymbol{D}^3$ (using the Skorohod topology),

$$\left\{\frac{U'_{[n\cdot]}}{a_n}, \frac{T'_{[n\cdot]}}{a_n}, \frac{R^{|f-\theta|}_{[n\cdot]}}{a_n}\right\} \to \{U(\cdot), T(\cdot), 0\} \quad \text{in distribution,}$$

with U and T stochastically continuous and $a_n = o(n)$, then $(S^f_{[n\cdot]} - [n\cdot]\theta)/a_n \to U(\cdot/\eta) - \theta T(\cdot/\eta)$ in distribution.

Proof. Since $a_n = o(n)$, then $T'_{[n\cdot]}/n$ must converge to 0 in probability. That is,

$$\frac{1}{n} T_{[nt]} = \frac{1}{n} \sum_{j=1}^{[nt]} N_j \to \eta t \quad \text{in probability for all } t > 0.$$

But as shown in the proof of Theorem 2.0,

$$\frac{[nt]}{M_{[nt]}} \sim \frac{T_{M_{[nt]}}}{M_{[nt]}} \sim \eta, \quad \text{so that } \frac{1}{n} M_{[nt]} \to t/\eta \text{ in probability for all } t > 0.$$

Since this limit is degenerate and continuous,

$$\left\{\frac{U'_{[n\cdot]}}{a_n}, \frac{T'_{[n\cdot]}}{a_n}, \frac{M_{[n\cdot]}}{n}, \frac{R^{|f-\theta|}_{[n\cdot]}}{a_n}\right\} \to \{U(\cdot), T(\cdot), (\cdot)/\eta, 0\} \quad \text{in distribution.}$$

Now apply the continuous functional $\psi_1 : \boldsymbol{D}^4 \to \boldsymbol{D}$, $\psi_1(w, x, y, z) = w \circ y - \theta x \circ y$, so that

$$\frac{U'_{[M_{[n\cdot]}]} - \theta T'_{[M_{[n\cdot]}]}}{a_n} \to U(\cdot/\eta) - \theta T(\cdot/\eta) \quad \text{in distribution,}$$

by the continuous mapping theorem, since U and T are stochastically continuous.

Next consider

$$\begin{aligned} S_{[nt]} - [nt]\theta &= \sum_{i=1}^{T_M} (f(Y_i, \cdots, Y_{i+k}) - \theta) + \sum_{i=T_M+1}^{[nt]} (f(Y_i, \cdots, Y_{i+k}) - \theta) \\ &= \sum_{j=1}^{M} \sum_{i=1}^{N_j} (f_i^{(j)} - \theta) + Q_{[nt]} \end{aligned}$$

where M denotes $M_{[nt]}$ and $Q_{[nt]}$ is the remainder. Then,

$$\begin{aligned} S_{[nt]} - [nt]\theta &= \sum_{j=1}^{M} R_j^f - \theta \sum_{j=1}^{M} N_j + Q_{[nt]} \\ &= U'_{[M_{[nt]}]} - \theta T'_{[M_{[nt]}]} + Q_{[nt]}. \end{aligned}$$

Also

$$\begin{aligned}
\frac{1}{a_n}\,|Q_{[nt]}| &= \frac{1}{a_n}\left|\sum_{i=T_M+1}^{[nt]}(f(Y_i,\cdots,Y_{i+k})-\theta)\right| \\
&= \frac{1}{a_n}\left|\sum_{i=1}^{[nt]-T_M}(f_i^{(M+1)}-\theta)\right| \\
&\leqq \frac{1}{a_n}\sum_{i=1}^{N_{M+1}}|f_i^{(M+1)}-\theta| \\
&= \frac{1}{a_n}R^{|f-\theta|}_{M_{[nt]}+1} \\
&\to 0 \quad \text{for all } t>0,
\end{aligned}$$

by the continuous mapping theorem with $\psi_2(w,x,y,z)=z\circ y$. $(1/a_n)Q_{[n\cdot]}$ is easily accommodated into the convergence statement since the limit is both continuous and degenerate. Therefore,

$$\begin{aligned}
\frac{S_{[n\cdot]}-[n\cdot]\theta}{a_n} &= \frac{U'_{[M_{[n\cdot]}]}-\theta T'_{[M_{[n\cdot]}]}}{a_n}+\frac{Q_{[n\cdot]}}{a_n} \\
&\to U(\cdot/\eta)-\theta T(\cdot/\eta) \quad \text{in distribution.}
\end{aligned}$$

Note that the joint convergence of U'_n and T'_n is quite reasonable considering the relationship $R^f_n=\sum_{j=1}^{N_n}f_j^{(n)}$, so that one might expect R^f_n and N_n to behave similarly.

Corollary 3.1. Let $\{Y_\alpha\}=\{\{X_j^{(\Lambda_n)}\}_{j=1}^{N_n}\}$ be a shifting level process and $f:\boldsymbol{X}^{k+1}\to\boldsymbol{R}$ such that

(i) $\{N_n,\Lambda_n\}$ is a strictly stationary, ϑ-mixing sequence of random elements in $\boldsymbol{N}\times\boldsymbol{\Lambda}$ with $P[\Lambda_m=\Lambda_n]=0$ for $n\neq m$ and $\sum_{j=1}^{\infty}\vartheta(j)^{1/2}<\infty$.

(ii) $\{X_j^{(\lambda)}\}$, $\lambda\in\boldsymbol{\Lambda}$ is a family of independent stochastic processes and independent of $\{N_n,\Lambda_n\}$.

(iii) $V[R^f_n]<\infty$, $V[N_n]<\infty$, $V[R^{|f-\theta|}_n]<\infty$, letting $\eta=E[N_n]$, $\eta\theta=E[R^f_n]$, $\eta\chi_j=\mathrm{Cov}[R^f_n-\theta N_n, R^f_{n+j}-\theta N_{n+j}]$, $j\geqq 0$.

Then

$$\sqrt{\alpha}\left(\frac{1}{\alpha}S^f_\alpha-\theta\right)\to N(0,\gamma^2)\quad\text{in distribution, } \gamma^2=\chi_0+2\sum_{j=1}^{\infty}\chi_j.$$

Proof. Using Lemma 2.2, as in the proof of Corollary 2.3, the sequence

$$\{\Delta_{(N_n,\Lambda_n)}\}_{n=1}^{\infty}=\{N_n,\{X_j^{(\Lambda_n)}\}_{j=1}^{\infty}\}_{n=1}^{\infty}$$

is stationary and ϑ-mixing. Apply the functional $\psi(\Delta_{(N_n,\Lambda_n)},\cdots,\Delta_{(N_{n+k},\Lambda_{n+k})})=(N_n,R^f_n,R^{|f-\theta|}_n)$. Clearly, ψ does not upset the stationarity, but it perturbs the ϑ-mixing by overlapping k variables. The outcome of this is that $\{N_n,R^f_n,R^{|f|}_n\}$ is stationary, ϑ_1-mixing with $\vartheta_1(j)=\vartheta(j-k)$ for $j>k$ and $\vartheta_1(j)=1$ for $j\leqq k$.

Furthermore, $\Sigma_{j=1}^{\infty} \vartheta_1(j)^{1/2} < \infty$, so that by Billingsley's theorem (cf., Billingsley (1968), p. 174) setting

$$U'_n = \sum_{j=1}^{n} (R_j^f - \eta\theta), \quad V'_n = \sum_{j=1}^{n} (R_j^{|f-\theta|} - \eta E[R_j^{|f-\theta|}]), \quad T'_n = \sum_{j=1}^{n} (N_j - \eta),$$

then

$$\left\{\frac{U'_{[n\cdot]}}{\sqrt{n}}, \frac{T'_{[n\cdot]}}{\sqrt{n}}, \frac{V'_{[n\cdot]}}{\sqrt{n}}\right\} \to \{W_1(\cdot), W_2(\cdot), W_3(\cdot)\} \quad \text{in distribution,}$$

a trivariate Wiener process. In particular,

$$\left\{\frac{U'_{[n\cdot]}}{\sqrt{n}}, \frac{T'_{[n\cdot]}}{\sqrt{n}}, \frac{R_{[n\cdot]}^{|f-\theta|}}{\sqrt{n}}\right\} \to \{W_1(\cdot), W_2(\cdot), 0\} \quad \text{in distribution.}$$

By Theorem 3.0, then,

$$\frac{S_{[n\cdot]}^f - [n\cdot]\theta}{\sqrt{n}} \to W_1(\cdot/\eta) - \theta W_2(\cdot/\eta) \quad \text{in distribution.}$$

Billingsley's theorem further gives the variances of these processes, so that

$$\begin{aligned} V[W_1(t) - \theta W_2(t)] &= t\left(V[R_1^f - \theta N_1] + 2\sum_{j=1}^{\infty} \operatorname{Cov}[R_1^f - \theta N_1, R_{1+j}^f - \theta N_{1+j}]\right) \\ &= t\eta\left(\chi_0 + 2\sum_{j=1}^{\infty} \chi_j\right) \\ &= t\eta\gamma^2, \quad \text{for all } t > 0. \end{aligned}$$

And hence,

$$\frac{S_\alpha^f - \alpha\theta}{\sqrt{\alpha}} \to W_1(1/\eta) - \theta W_2(1/\eta) \sim N(0, \gamma^2) \quad \text{in distribution.}$$

The following is given without proof.

Corollary 3.2. If instead of requiring that $\{N_n, \Lambda_n\}$ is stationary, one requires that $\{N_n\}$ is a stationary renewal process and that $\{N_n, \Lambda_n\}_{n=2}^{\infty}$ is stationary, then the conclusions of both Corollary 2.3.1 and Corollary 3.1 are valid.

This completes the sections on the limit theorems for sample moments of SLP. When the underlying processes $\{N_n, \Lambda_n\}$ and $\{X_j^{(\Lambda)}\}$ have different characteristics, similar theorems may be devised.

4. Calculating moments

Corollaries 2.3 and 3.1 require the calculation of the moments of $\{Y_n\}$, $\{R_n^f\}$, etc. This section provides formulae (without derivation) under the following assumptions:

1. $\{X_j^{(\lambda)}\}$, $\lambda \in \boldsymbol{\Lambda}$ is a family of independent stochastic processes, each of which is a sequence of exchangeable random elements of $\boldsymbol{X}$.

2. $\{N_n\}$ and $\{\Lambda_n\}$ are sequences of independent and identically distributed random elements of $\boldsymbol{N}$ and $\boldsymbol{\Lambda}$, respectively, and are independent of each other and of $\{X_j^{(\lambda)}\}$.

For $f : \boldsymbol{X}^k \to \boldsymbol{R}$,

$$E[f(Y_1, \cdots, Y_k)]$$
$$= \sum_{m=1}^{k} \sum_{j_1+\cdots+j_m=k} E[f(X_1^{(\Lambda_1)}, \cdots, X_{j_1}^{(\Lambda_1)}, X_1^{(\Lambda_2)}, \cdots, X_{j_m}^{(\Lambda_m)})] \prod_{i=1}^{m-1} P[N_i = j_i] P[N_m \geqq j_m].$$

For $k = 1, 2$ the latter formula is

$$E[f(Y_1)] = E[X_1^{(\Lambda_1)}],$$
$$E[f(Y_1, Y_2)] = P[N_1 = 1] E[f(X_1^{(\Lambda_1)}, X_1^{(\Lambda_2)})] + P[N_1 > 1] E[f(X_1^{(\Lambda_1)}, X_2^{(\Lambda_1)})].$$

Also

$$E[f(Y_k)] = \sum_{j=1}^{k} P[T_{j-1} < k \leqq T_j] E[X_1^{(\Lambda_j)}]$$

and for $i \geqq 0$

$$\mathrm{Cov}[f(Y_k), f(Y_{k+i})] = \sum_{j=1}^{k+i} P[T_{j-1} < k \leqq k+i \leqq T_j] \mathrm{Cov}[f(X_1^{(\Lambda_j)}), f(X_2^{(\Lambda_j)})].$$

If the limit theorems are to be applied, then the moments of R_n^f are required. Let $\theta = E[R_n^f]/E[N_n]$ and $(x)_+ = \max(x, 0)$.

For $f : \boldsymbol{X} \to \boldsymbol{R}$,

$$E[R_n^f] = E[N_n] E[f(X_1^{(\Lambda_n)})]$$

and

$$\begin{aligned} V[R_n^{f-\theta}] &= V[R_n^f - \theta N_n] \\ &= E[N_n] V[f(X_1^{(\Lambda_n)})] + E[N_n(N_n - 1)] \mathrm{Cov}[f(X_1^{(\Lambda_n)}), f(X_2^{(\Lambda_n)})]. \end{aligned}$$

For $f : \boldsymbol{X}^2 \to \boldsymbol{R}$

$$\begin{aligned} E[R_n^f] &= E[N_n - 1] E[f(X_1^{(\Lambda_n)}, X_2^{(\Lambda_n)})] + E[f(X_1^{(\Lambda_n)}, X_1^{(\Lambda_{n+1})})] \\ V[R_n^{f-\theta}] &= E[N_n - 1] V[f(X_1^{(\Lambda_n)}, X_2^{(\Lambda_n)})] \\ &\quad + 2E[(N_n - 2)_+] \mathrm{Cov}[f(X_1^{(\Lambda_n)}, X_2^{(\Lambda_n)}), f(X_2^{(\Lambda_n)}, X_3^{(\Lambda_n)})] \\ &\quad + E[(N_n - 2)_+(N_n - 3)_+] \mathrm{Cov}[f(X_1^{(\Lambda_n)}, X_2^{(\Lambda_n)}), f(X_3^{(\Lambda_n)}, X_4^{(\Lambda_n)})] \\ &\quad + V[f(X_1^{(\Lambda_n)}, X_1^{(\Lambda_{n+1})})] \end{aligned}$$

$$
\begin{aligned}
&+2P[N_n>1]\mathrm{Cov}[f(X_1^{(\Lambda_n)},X_2^{(\Lambda_n)}),f(X_2^{(\Lambda_n)},X_1^{(\Lambda_{n+1})})]\\
&+2E[(N_n-2)_+]\mathrm{Cov}[f(X_1^{(\Lambda_n)},X_2^{(\Lambda_n)}),f(X_3^{(\Lambda_n)},X_1^{(\Lambda_{n+1})})]\\
&+V[N_n](E[f(X_1^{(\Lambda_n)},X_2^{(\Lambda_n)})-\theta])^2
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Cov}[R_n^{f-\theta},R_{n+1}^{f-\theta}]&=P[N_{n+1}=1]\mathrm{Cov}[f(X_1^{(\Lambda_n)},X_1^{(\Lambda_{n+1})}),f(X_1^{(\Lambda_{n+1})},X_1^{(\Lambda_{n+2})})]\\
&+P[N_{n+1}>1]\mathrm{Cov}[f(X_1^{(\Lambda_n)},X_1^{(\Lambda_{n+1})}),f(X_1^{(\Lambda_{n+1})},X_2^{(\Lambda_{n+1})})]\\
&+E[(N_{n+1}-2)_+]\mathrm{Cov}[f(X_1^{(\Lambda_n)},X_1^{(\Lambda_{n+1})}),f(X_2^{(\Lambda_{n+1})},X_3^{(\Lambda_{n+1})})]\\
&+P[N_{n+1}>1]\mathrm{Cov}[f(X_1^{(\Lambda_n)},X_1^{(\Lambda_{n+1})}),f(X_2^{(\Lambda_{n+1})},X_1^{(\Lambda_{n+2})})].
\end{aligned}
$$

5. The GNN model

The first section introduced a very elementary shifting level model which is briefly defined again here. Let $\{N_n\}\sim$ i.i.d. geometric(π), $\{\Lambda_n\}\sim$ i.i.d. normal($\mu,\rho\sigma^2$) and $\{X_j^{(\lambda)}\}\sim$ i.i.d. normal($\lambda,(1-\rho)\sigma^2$), all processes independent, $0<\rho<1, 0<\pi<1$. As usual, the observable process is $\{Y_\alpha\}=\{\{X_j^{(\Lambda_n)}\}_{j=1}^{N_n}\}$. This example is elementary enough so that $\{f(Y_\alpha,\cdots,Y_{\alpha+k})\}$ is stationary and ϑ-mixing for virtually any square integrable function f, so Billingsley's theorem applies directly. For purposes of illustration, however, Corollary 3.1 will be applied.

Corollary 3.1 may indeed be applied because

(i) $\{N_n,\Lambda_n\}$ is i.i.d., hence stationary and ϑ-mixing with $\vartheta^{1/2}(k)$ summable, and $P[\Lambda_n=\Lambda_m]=0,\ n\neq m$.

(ii) $\{X_j^{(\lambda)}\}$ is a family of independent processes and independent of $\{N_n,\Lambda_n\}$.

(iii) $E[N_n]=1/\pi<\infty$ and $V[N_n]=(1-\pi)/\pi^2<\infty$.

Therefore, if f is such that $V[R_n^f]<\infty$ then $(1/\alpha)S_\alpha^f$ is approximately $N(\theta,\gamma^2/\alpha)$, where $\theta=\pi E[R_n^f]=E[f(Y_\alpha,\cdots,Y_{\alpha+k})]$ and

$$
\begin{aligned}
\gamma^2&=\pi\left(V[R_n^{f-\theta}]+2\sum_{j=1}^{k}\mathrm{Cov}[R_n^{f-\theta},R_{n+j}^{f-\theta}]\right)\\
&=V[f(Y_\alpha,\cdots,Y_{\alpha+k})]+2\sum_{j=1}^{\infty}\mathrm{Cov}[f(Y_\alpha,\cdots,Y_{\alpha+k}),f(Y_{\alpha+j},\cdots,Y_{\alpha+k+j})].
\end{aligned}
$$

In addition (i), (ii) and (iii) satisfy the conditions for Corollary 2.3 whenever $\theta=\pi E[R_n^f]<\infty$, so that $(1/\alpha)S_\alpha^f\to\theta$ a.s.

First- and second-moment estimators will be considered for the parameters μ, σ^2 and $\gamma_k=\mathrm{Cov}[Y_n,Y_{n+k}]$. These estimators will involve the sample moments,

$$
\frac{1}{n}S_n=\frac{1}{n}\sum_{j=1}^{n}Y_j,\quad \frac{1}{n}S_n^{f_k}=\frac{1}{n}\sum_{j=1}^{n}Y_jY_{j+k},\qquad k\geqq 0.
$$

The functions f, then, are the identity and the kth lag product (denoted f_k).

Sample mean and variance. The sample mean is $\hat{\mu}_n = (1/n)S_n = (1/n)\Sigma_{j=1}^n Y_j$.

$$E[R_n] = E[N_n]E[X_1^{(\Lambda_n)}] = \frac{1}{\pi}\mu.$$

$$V[R_n - \mu N_n] = E[N_n]V[X_1^{(\Lambda_n)}] + E[N_n(N_n-1)]\mathrm{Cov}[X_1^{(\Lambda_n)}, X_2^{(\Lambda_n)}]$$
$$= \frac{1}{\pi}\sigma^2 + 2\frac{1-\pi}{\pi^2}\rho\sigma^2.$$

Hence, $\hat{\mu}_n \to \mu$ almost surely and $\sqrt{n}(\hat{\mu}_n - \mu) \to N(0, (1+2\rho((1-\pi)/\pi))\sigma^2)$ in distribution.

$$E[R_n^{f_0}] = \frac{1}{\pi}(\sigma^2 + \mu^2).$$

$$V[R_n^{f_0} - (\sigma^2+\mu^2)N_n] = \frac{1}{\pi}(2\sigma^4 + 4\mu^2\sigma^2) + 2\frac{1-\pi}{\pi^2}(2\rho^2\sigma^4 + 4\mu^2\rho\sigma^2).$$

$$\mathrm{Cov}[R_n, R_n^{f_0} - (\sigma^2+\mu^2)N_n] = \frac{1}{\pi}(2\mu\sigma^2) + 2\frac{1-\pi}{\pi^2}(2\mu\rho\sigma^2).$$

Thus, $(1/n)S_n^{f_0} \to \sigma^2 + \mu^2$ almost surely and letting

$$\hat{\sigma}_n^2 = \frac{1}{n}\sum_{j=1}^n (Y_j - \bar{Y})^2 = \frac{1}{n}S_n^{f_0} - \left(\frac{1}{n}S_n\right)^2,$$

then $\hat{\sigma}_n^2 \to \sigma^2$ almost surely and

$$\begin{bmatrix}\sqrt{n} & (\hat{\mu}_n - \mu)\\ \sqrt{n} & (\hat{\sigma}_n^2 - \sigma^2)\end{bmatrix} \to N\left[\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\left(1+2\rho\frac{1-\pi}{\pi}\right)\sigma^2 & 0\\ 0 & 2\left(1+2\rho^2\frac{1-\pi}{\pi}\right)\sigma^4\end{bmatrix}\right],$$

in distribution.

Sample autocovariances. The autocovariances for GNN are $\gamma_k = \rho(1-\pi)^k$ and the sample moments are given by $\hat{\gamma}_k = (1/n)S_n^{f_k} - ((1/n)S_n)^2$. In view of the definition of R_n^f and S_n^f, $f_k(X_1, \cdots, X_{k+1}) = X_1X_{k+1}$ must be considered as a function of $k+1$ variables, even though it involves only two variables. The point is that $\{R_n^{f_k}\}$ is k-dependent and moments are correspondingly more complicated. The only asymptotic variances given are for $\hat{\gamma}_1$ and $\hat{\gamma}_2$.

$$E[R_n^{f_1}] = E[N_n - 1]E[X_1^{(\Lambda_n)}, X_2^{(\Lambda_n)}] + E[X_1^{(\Lambda_n)}, X_1^{(\Lambda_{n+1})}] = \frac{1}{\pi}(\gamma_1^2 + \mu^2).$$

$$V[R_n^{f_1} - (\gamma_1 + \mu^2)N_n] = \frac{1}{\pi}(4 - 2\pi)\left(1 + 2\rho\frac{1-\pi}{\pi}\right)\mu^2\sigma^2$$
$$+ \frac{1}{\pi}(1 + 2\rho(1-\pi)^2 + \rho^2(1-\pi^2))\sigma^4$$
$$+ 2\rho^2\frac{(1-\pi)^2 + (1-\pi)^3}{\pi^2}\sigma^4.$$

$$\mathrm{Cov}[R_n^{f_1} - (\gamma_1 + \mu^2)N_n, R_{n+1}^{f_1} - (\gamma_1 + \mu^2)N_{n+1}] = \left(1 + 2\rho\frac{1-\pi}{\pi}\right)\mu^2\sigma^2.$$

$$\mathrm{Cov}[R_n - \mu N_n, R_n^{f_1} - (\gamma_1 + \mu)^2 N_n] = \frac{1}{\pi}\left((2-\pi) + \rho\left(\frac{4-\pi}{\pi}\right)\right)\mu\sigma^2.$$

$$\mathrm{Cov}[R_{n+1} - \mu N_{n+1}, R_n^{f_1} - (\gamma_1 + \mu)^2 N_n] = \mu\sigma^2 + \frac{1-\pi}{\pi}(\rho\mu\sigma^2).$$

Of course, since the variances are finite, $\hat{\gamma}_1 = (1/n)S_n^{f_1} - ((1/n)S_n)^2$ converges to a normal distribution. The results above give the asymptotic variance

$$V[\hat{\gamma}_1] \approx \frac{\pi}{n}[V[R_n^{f_1} - (\gamma_1 + \mu^2)N_n]$$
$$+ 2\,\mathrm{Cov}[R_n^{f_1} - (\gamma_1 + \mu^2)N_n, R_{n+1}^{f_1} - (\gamma_1 + \mu^2)N_{n+1}]$$
$$- 4\mu(\mathrm{Cov}[R_n - \mu N_n, R_n^{f_1} - (\gamma_1 + \mu^2)N_n]$$
$$+ \mathrm{Cov}[R_{n+1} - \mu N_{n+1}, R_n^{f_1} - (\gamma_1 + \mu^2)N_n])$$
$$+ 4\mu^2 V[R_n - \mu N_n]]$$
$$= \frac{1}{n}\left(1 + (2\rho - 3\rho^2)(1-\pi)^2 + 2\rho^2\frac{(1-\pi)(2-\pi)}{\pi}\right)\sigma^4.$$

Also

$$\mathrm{Cov}[\hat{\gamma}_1, \hat{\mu}] \approx \frac{\pi}{n}[\mathrm{Cov}[R_n - \mu N_n, R_n^{f_1} - (\gamma_1 + \mu^2)N_n]$$
$$+ \mathrm{Cov}[R_{n+1} - \mu N_{n+1}, R_n^{f_1} - (\gamma_1 + \mu^2)N_n] - 2\mu V[R_n - \mu N_n]]$$
$$= 0.$$

Further calculations show that the asymptotic variance-covariance matrix for $(\hat{\sigma}^2, \hat{\gamma}_1, \hat{\gamma}_2)$ is $\sigma^4 C$ where C is given by

$$C_{00} = 2\left(1 + 2\rho^2\frac{1-\pi}{\pi}\right)$$

$$C_{01} = 4\rho(1-\pi)\left(1 + \rho\frac{1-\pi}{\pi}\right)$$

$$C_{02} = 4\rho(1-\pi)^2\left(1+\rho\frac{2-\pi}{2\pi}\right)$$

$$C_{11} = 1+(2\rho-3\rho^2)(1-\pi)^2+2\rho^2\frac{(1-\pi)(2-\pi)}{\pi}$$

$$C_{12} = 2\rho(1-\pi)\left(1+(1-2\rho)(1-\pi)^2+2\rho\frac{1-\pi}{\pi}\right)$$

$$C_{22} = 1+(2\rho-5\rho^2)(1-\pi)^4+2\rho^2\left(\frac{(1-\pi)^2+(2-\pi)^2-(1-\pi)^4}{\pi(2-\pi)}\right).$$

Also, $\hat{\mu}$ is asymptotically independent of the autocovariance estimators.

Moment estimates for the shift parameters. The simplest estimates are given by the relations

$$\hat{\rho} = \frac{\hat{\gamma}_1^2}{\hat{\gamma}_2\hat{\sigma}^2}, \quad \hat{\pi} = 1-\frac{\hat{\gamma}_2}{\hat{\gamma}_1}.$$

So the covariance matrix for $(\hat{\rho}, \hat{\pi})$ is JCJ' where

$$J = \begin{bmatrix} -\rho & \dfrac{2}{(1-\pi)} & \dfrac{-1}{(1-\pi)^2} \\ 0 & \dfrac{1}{\rho} & \dfrac{-1}{\rho(1-\pi)} \end{bmatrix}.$$

Summary

Although there is a great deal of structure in the shifting level process, its mixture nature has thus far hindered dramatic exploitation of that structure. Nevertheless, this paper utilizes the limiting behavior of the sample moments of $\{R_n^f, N_n\}$, which is based directly on the processes making up SLP, in order to get at the limiting behavior of SLP itself. The results confirm one's intuition that the convergence is not disturbed by combining the processes together, even if in a relatively complicated manner. Even when the underlying processes are stationary or ϑ-mixing, SLP is not necessarily so, yet it maintains the limiting properties required for estimation.

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