# Correction to <br> Stability of Nonlinear Stochastic Recursions with Application to Nonlinear AR-GARCH Models 

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A subtle error in the recent paper (Cline, Adv. Appl. Probab. 39, 462-491, 2007) has resulted in an incorrect proof of the critical Lemma 4.1. The result may be salvaged with the addition of another (quite reasonable) assumption.

First, we take note of Assumption 1.4(i) which says
There exists a set $\Theta_{\#}$, open in $\Theta=\{x \in \mathbb{X}:\|x\|=1\}$, such that $\{B(\cdot, u)\}_{|u| \leq M}$ is equicontinuous on $\Theta_{\#}$ for all finite $M$.

For the intended applications such as threshold models, $\Theta_{\#}=\bigcup_{i=1}^{N} \Theta_{i}$ where $\Theta_{1}, \ldots, \Theta_{N}$ are open, connected and disjoint. They usually have, however, common boundaries. An interpretation of the assumption was given as follows.

For each $\epsilon>0$ and $M<\infty$, there is $\delta>0$ such that $\left\|\theta-\theta^{\prime}\right\|<\delta$, $\theta, \theta^{\prime} \in \Theta_{\#}$ implies $\left\|B(\theta, u)-B\left(\theta^{\prime}, u\right)\right\|<\epsilon$ for all $|u| \leq M$.

The interpretation was meant to treat $\left\|\theta-\theta^{\prime}\right\|$ as a norm on $\Theta_{\#}$ (not on $\Theta$ ), but this may be confusing. To be correct, we define a metric $d\left(\theta, \theta^{\prime}\right)$ on $\Theta_{\#}$ that reflects the intended meaning:
$d\left(\theta, \theta^{\prime}\right)= \begin{cases}\left\|\theta-\theta^{\prime}\right\|, & \text { if there is a connected subset of } \Theta_{\#} \text { containing } \\ \left\|\theta-\theta^{\prime}\right\|+1, & \text { both } \theta \text { and } \theta^{\prime},\end{cases}$
The equicontinuity of Assumption 1.4(i) above, therefore, is reinterpreted as
(i) For each $\epsilon>0$ and $M<\infty$, there is $\delta>0$ such that $d\left(\theta, \theta^{\prime}\right)<\delta, \theta, \theta^{\prime} \in \Theta_{\#}$ implies $\left\|B(\theta, u)-B\left(\theta^{\prime}, u\right)\right\|<\epsilon$ for all $|u| \leq M$.

The point is that not only $\theta, \theta^{\prime} \in \Theta_{\#}$ is required but also (in the threshold model context) $\theta$ and $\theta^{\prime}$ must be in the same $\Theta_{i}$. In other words, if $\theta^{\prime}$ and $\theta$ are close in $\Theta$ but on opposite sides of a boundary, they are not close in $\Theta_{\#}$.

Assumption 1.4(ii) accordingly should be rephrased as well, as
(ii) For each $\epsilon>0$ there exists $L<\infty$ such that

$$
P\left(\tilde{\theta}_{1} \in \Theta_{\#}, \tilde{\theta}_{1}^{*} \in \Theta_{\#}, d\left(\tilde{\theta}_{1}, \tilde{\theta}_{1}^{*}\right)<1 \mid X_{0}=x\right)>1-\epsilon
$$

for all $x \in \mathbb{X}$ with $x /\|x\| \in \Theta_{\#}$ and $\|x\|>L$.
That is, it is highly likely that $\tilde{\theta}_{1}$ and $\tilde{\theta}_{1}^{*}$ are in $\Theta_{\#}$ and on the same side of all boundaries, when $\|x\|$ is large.

This confusion of norms/metrics also occurs in the proof of Lemma 4.1. Recalling the definition $\eta(\theta, u)=\frac{B(\theta, u)}{\|B(\theta, u)\|}$, we observe that, by (i), for each $v>$ 0 there exists $\delta$ such that $d\left(\theta, \theta^{\prime}\right)<\delta,\left|e_{1}\right| \leq M$ implies $\left\|\eta\left(\theta, e_{1}\right)-\eta\left(\theta^{\prime}, e_{1}\right)\right\|<v$. But this does not preclude the possibility that $d\left(\eta\left(\theta, e_{1}\right), \eta\left(\theta^{\prime}, e_{1}\right)\right) \geq 1$. As discussed below, this causes the existing proof of Lemma 4.1 to fail.

Before providing a resolution, we require an additional assumption, namely that $\Theta_{\#}$ may be chosen in such a way that the following holds.
(iii) For each $\epsilon>0$ and $M<\infty$ there exists $\delta>0$ such that $d\left(\theta, \theta^{\prime}\right)<\delta$, $\theta, \theta^{\prime} \in \Theta_{\#}$ implies $P\left(d\left(\eta\left(\theta, e_{1}\right), \eta\left(\theta^{\prime}, e_{1}\right)\right) \geq 1,\left|e_{1}\right| \leq M\right)<\epsilon$.
In other words, the probability that $\eta\left(\theta, e_{1}\right)$ and $\eta\left(\theta^{\prime}, e_{1}\right)$ fail to be in the same $\Theta_{i}$ is small, uniformly for $d\left(\theta, \theta^{\prime}\right)<\delta$.

Returning specifically to the model (1.2) investigated in section 5 of the paper, we define $\Theta_{\#}$ as was done in the proof of Theorem 5.3. Now suppose that $\delta$ is chosen to ensure that $d\left(\theta, \theta^{\prime}\right)<\delta$ implies $\left\|\eta\left(\theta, e_{1}\right)-\eta\left(\theta^{\prime}, e_{1}\right)\right\|<v$ when $\left|e_{1}\right| \leq M$. Note that $\theta$ and $\theta^{\prime}$ are in the same $\Theta_{i}$. One may see that, under this scenario, the event

$$
d\left(\eta\left(\theta, e_{1}\right), \eta\left(\theta^{\prime}, e_{1}\right)\right) \geq 1 \quad \text { and } \quad\left|e_{1}\right| \leq M
$$

is contained in a finite union of events of the form $\left|r_{i} e_{1}-s_{i}\right|<v$, with $r_{i}, s_{i}$ chosen independently of $\theta, \theta^{\prime}$. (The correct definition of $\Theta_{\#}$ is important here.) Therefore, since the density of $e_{1}$ is assumed bounded, it follows that

$$
P\left(d\left(\eta\left(\theta, e_{1}\right), \eta\left(\theta^{\prime}, e_{1}\right)\right) \geq 1,\left|e_{1}\right| \leq M\right)<L_{1} v
$$

for some constant $L_{1}$. Thus, (iii) holds with the choice $v=\epsilon / L_{1}$. Similarly, the probability in (ii) differs from 1 by no more than $L_{2} /\|x\|$ for some other constant $L_{2}$, and so (ii) holds as well with $L=L_{2} / \epsilon$.

Now we will revise the proof of Lemma 4.1. Suppose $q$ is a bounded function, uniformly continuous (with respect to $d$ ) on $\Theta_{\#}$. The error in the proof of Lemma 4.1 occurs after (4.4) where it is claimed incorrectly that assumption (i) above "implies $\left\{q(\eta(\theta, u))(w(\theta, u))^{\zeta}\right\}_{|u| \leq M}$ likewise is equicontinuous on $\Theta_{\#}$." (Note: $w(\theta, u)=\|B(\theta, u)\|$.) This is because $\eta(\theta, u)$ may cross a boundary with even a small change in $\theta$.

What we may say instead, based on (i) and the uniform continuity of $q$, is that there exists sufficiently small $\delta$ and $v$ such that $d\left(\theta, \theta^{\prime}\right)<\delta$ implies $\left\|\eta(\theta, u)-\eta\left(\theta^{\prime}, u\right)\right\|<v$ and

$$
\left|q(\eta(\theta, u))(w(\theta, u))^{\zeta}-q\left(\eta\left(\theta^{\prime}, u\right)\right)\left(w\left(\theta^{\prime}, u\right)\right)^{\zeta}\right|<\epsilon / 4
$$

$$
\begin{equation*}
\text { if }|u| \leq M, d\left(\eta(\theta, u), \eta\left(\theta^{\prime}, u\right)\right)<1 \tag{1}
\end{equation*}
$$

Additionally, let $K=\bar{b}(1+M)^{\zeta} \sup _{\theta \in \Theta} q(\theta)$ (cf. Assumption 1.2(ii)). By (iii), we actually may choose $\delta$ so that also

$$
\begin{equation*}
P\left(d\left(\eta\left(\theta, e_{1}\right), \eta\left(\theta^{\prime}, e_{1}\right)\right) \geq 1,\left|e_{1}\right| \leq M\right)<\epsilon /(8 K) \tag{2}
\end{equation*}
$$

From (1) and (2), it follows with little calculation that

$$
\begin{aligned}
& \left|E\left(q\left(\theta_{1}^{*}\right)\left(W_{1}^{*}\right)^{\zeta} 1_{\left|e_{1}\right| \leq M} \mid \theta_{0}^{*}=\theta\right)-E\left(q\left(\theta_{1}^{*}\right)\left(W_{1}^{*}\right)^{\zeta} 1_{\left|e_{1}\right| \leq M} \mid \theta_{0}^{*}=\theta^{\prime}\right)\right| \\
& \quad \leq E\left(\left|q\left(\eta\left(\theta, e_{1}\right)\right)\left(w\left(\theta, e_{1}\right)\right)^{\zeta}-q\left(\eta\left(\theta^{\prime}, e_{1}\right)\right)\left(w\left(\theta^{\prime}, e_{1}\right)\right)^{\zeta}\right| 1_{\left|e_{1}\right| \leq M}\right) \\
& \quad<\epsilon / 2
\end{aligned}
$$

which is (4.5) in the paper. The rest of the proof then proceeds as originally stated.

The remainder of the results and proofs may be allowed to stand essentially as they are, with the understanding that $d$ is to be used as the metric on $\Theta_{\#}$. Note that (ii) justifies its use in the proofs of Lemma 4.3 and Theorem 3.4, in particular.

