LARGE DEVIATION PROBABILITIES FOR SUMS OF RANDOM VARIABLES WITH HEAVY OR SUBEXPONENTIAL TAILS

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Abstract.

Let S_n be the sum of independent random variables with distribution F. Under the assumption that $-\log(1 - F(x))$ is slowly varying, conditions for

$$\lim_{n \to \infty} \sup_{s \geq t_n} \left| \frac{P[S_n > s]}{n(1 - F(s))} - 1 \right| = 0$$

are given. These conditions extend and strengthen a series of previous results. Additionally, a connection with subexponential distributions is demonstrated. That is, F is subexponential if and only if the condition above holds for some t_n and

$$\lim_{t \to \infty} \frac{1 - F(t + x)}{1 - F(t)} \to 1 \qquad \text{for each real } x.$$

1. Subexponentiality and large deviations.

Let X_i, $i \geq 1$, be independent and identically distributed random variables with distribution function F, and $S_n = \sum_{i=1}^n X_i$. Suppose $\{t_n\}$ is a sequence of positive constants such that $S_n/t_n \xrightarrow{P} 0$. It is often of interest to consider the rate at which the large deviation probabilities $P[|S_n| > t_n]$ and $P[S_n > t_n]$ tend to zero. This topic is of traditional importance in probability theory, and has numerous statistical applications.

Following Cramér's pioneering work (1938), the study of large deviation probabilities at first was confined to distributions F satisfying Cramér's condition, namely, for some constant $\epsilon > 0$,

$$\int_{-\infty}^{\infty} e^{cx} F(dx) < \infty \quad \text{for all } c \in [-\epsilon, \epsilon]. \tag{1.1}$$

For example, (1.1) implies

$$\lim_{n \to \infty} n^{-1} \log P[S_n > nx] = \sup_{\lambda \geq 0} [\lambda x - L(\lambda)] \quad \text{for all } x > E(X_1),$$

and

$$\lim_{n \to \infty} n^{-1} \log P[S_n < nx] = \sup_{\lambda \leq 0} [\lambda x - L(\lambda)] \quad \text{for all } x < E(X_1),$$

where $L(\lambda) = \log E e^{\lambda X_1}$.

It was not until the 1960's that attention was given to heavy–tailed distributions. In that connection, the most noticeable work was by Linnik (1961), Heyde (1967a,b,1968), A.V. Nagaev (1969a,b) and S.V. Nagaev (1973). (See also the review paper by S.V. Nagaev (1979).) While Linnik, A. V. Nagaev, and S. V. Nagaev mostly considered distributions with finite variance, Heyde focused on distributions with infinite variance, including the ones that are attracted to nonnormal stable laws. However, their results contain a common implication for the heavy tailed distributions they considered: namely, if t_n tends to ∞ fast enough then one has

$$0 < \liminf_{n \to \infty} \frac{P[S_n > t_n]}{nP[X > t_n]} \leq \limsup_{n \to \infty} \frac{P[S_n > t_n]}{nP[X > t_n]} < \infty \tag{1.2}$$

and, in some cases, even

$$\lim_{n \to \infty} \frac{P[S_n > t_n]}{nP[X > t_n]} = 1. \tag{1.3}$$

Typically the results require at least $n(1 - F(t_n)) \to 0$. Note that if this is so then (1.2) and (1.3) state that the probabilities of large deviations for the sum and the maximum are asymptotically comparable.

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What class of distributions can be expected to have these properties? The most natural notion is that of subexponentiality. Specifically, the subexponential class $\mathcal{S}$ consists of those probability distributions F satisfying

$$\lim_{t\to\infty} \frac{P[X_1 + X_2 > t]}{\overline{F}(t)} \quad \text{exists finite} \tag{1.4}$$

and

$$\lim_{t\to\infty} \frac{\overline{F}(t+x)}{\overline{F}(t)} = 1 \quad \text{for each real } x, \tag{1.5}$$

where $\overline{F}(x) = 1 - F(x)$ and X_1 and X_2 are independent random variables distributed according to F. This class was first studied by Chistyakov (1964) and by Chover, Ney and Wainger (1973a) in the case $P[X \geq 0] = 1$, with application to branching processes. Additional work is found in Embrechts and Goldie (1980) and Cline (1987). Extensions for the case $P[X \geq 0] < 1$ are given by Willekens (1986) and numerous references and applications are provided in Embrechts and Goldie (1982).

Two properties of $\mathcal{S}$ are of interest to us here. First, if $F \in \mathcal{S}$ then (1.1) fails to hold (cf. Embrechts and Goldie (1982)). Second, the limit in (1.4) equals 2 and furthermore

$$\lim_{t\to\infty} \frac{P[S_n > t]}{n\overline{F}(t)} = 1 \quad \text{for every } n \geq 1 \tag{1.6}$$

(cf. Chistyakov (1964), with extension by Willekens (1986)). Immediately one has that, for some sequence $\{t_n\}$, (1.3) holds for distributions in the subexponential family. In fact, we may characterize $\mathcal{S}$ as follows.

THEOREM 1.1.   *$F \in \mathcal{S}$ if and only if (1.5) holds and there exists t_n such that*

$$\limsup_{n\to\infty} \sup_{s \geq t_n} \frac{P[S_n > s]}{n\overline{F}(s)} \leq 1. \tag{1.7}$$

Moreover, when $F \in \mathcal{S}$ there exists t_n such that

$$\lim_{n\to\infty} \sup_{s \geq t_n} \left| \frac{P[S_n > s]}{n\overline{F}(s)} - 1 \right| = 0. \tag{1.8}$$

Our proof of Theorem 1.1 (in Section 4), however, does not construct a specific choice of $\{t_n\}$ for (1.8) to hold. Therefore our second goal is to consider how to choose the constants t_n for a particular subexponential distribution.

The subexponential distributions that we will focus on are those F for which $-\log \overline{F}(x)$ is slowly varying as $x \to \infty$. These are the distributions whose tails are on

the heavy side in $\mathcal{S}$, and they include distributions that are attracted to the nonnormal stable distributions as well as distributions with finite variance, such as the lognormal. Large deviation results for this class are few and scattered. For example, there are no results for distributions whose tails behave like the Cauchy and lognormal distributions. We show that for these distributions, it is possible to have a unified treatment based on a truncation argument and Markov's inequality. We not only obtain non-trivial extensions of results previously proved by using different methods, but we also fill some holes in the literature. The main result is Theorem 2.1 in the next section which is followed by a discussion on the conditions of the result. In Section 3 we give some examples that can be derived directly from Theorem 2.1, or from slight modifications of Theorem 2.1. We also describe how existing results relate to the examples.

The paper does not cover "semiexponential" distributions, e.g. $F(x) = 1 - e^{-x^\rho}$ with $0 < \rho < 1$, which are also in the subexponential class (cf. Cline (1986) and Goldie and Resnick (1988)). A.V. Nagaev (1969b) and S.V. Nagaev (1973) have considered large deviations for special cases of semiexponential distributions. We do not know of a unified approach that will work for all subexponential distributions.

For clarity, all the proofs and technical details are collected in Section 4.

2. Main Result.

For what follows, let $\psi(t) = -\log \overline{F}(t)$. When (1.5) holds, $\overline{F}(\log t)$ is slowly varying and thus ψ has the representation

$$\psi(t) = b(t) + \int_0^t \eta(u)du, \qquad \text{for all } t \geq 0, \tag{2.1}$$

where $\eta(t)$ and $b(t)$ are measurable and $\eta(t) \to 0$, $b(t) \to b \in \mathbb{R}$, as $t \to \infty$. Because of this, a represenation like (2.1) will be the starting point for our theorems.

Also, define

$$\mu_1(t) = \int_{|x| \leq t} x dF(x) \quad \text{and} \quad \mu_2(t) = \int_{|x| \leq t} x^2 dF(x), \quad t > 0.$$

We now state the main result of this paper.

THEOREM 2.1.   *Let $a = a_n(s) = \psi(s) - \log n$. Assume (2.1) where $\eta(t) \downarrow 0$ and $b(t)$ is measurable, bounded and satisfies*

$$\lim_{y \to 1, t \to \infty} (b(yt) - b(t)) = 0. \tag{2.2}$$

Suppose t_n are constants increasing to ∞ and $\lambda \in (0, 1)$ such that, as $n \to \infty$,

$$\sup_{s \geq t_n} \inf_{w \geq s/a} n \left(\frac{|\mu_1(w)| + a\mu_2(w)/s}{s \wedge \eta^{-1}(\lambda s)} + F(-w) \right) \to 0, \tag{2.3}$$

$$\sup_{s \geq t_n} \frac{1 \vee s\eta(\lambda s) \log s\eta(\lambda s)}{a} \to 0, \tag{2.4}$$

and

$$\sup_{s \geq t_n} n(1 \vee s\eta(\lambda s))\overline{F}(s/a) \to 0, \tag{2.5}$$

where $\eta^{-1}(\lambda s) = 1/\eta(\lambda s)$. Then

$$\lim_{n \to \infty} \sup_{s \geq t_n} \left| \frac{P[S_n > s]}{n\overline{F}(s)} - 1 \right| = 0. \tag{2.6}$$

The conditions (2.3)-(2.5) attempt to cover a variety of situations. While they do not offer much insight into the large deviation problem at this stage, it is worth mentioning that they are sharp since they form the weakest condition for (2.6) to hold in the case where the tail probabilities of F are regularly varying with index in $(-1, 0]$ (cf. Theorem 3.3). The following remarks are useful.

REMARK 1. Note that (2.3) is equivalent to

$$\sup_{s \geq t_n} n \left(\frac{|\mu_1(w)| + a\mu_2(w)/s}{s \wedge \eta^{-1}(\lambda s)} + F(-w) \right) \to 0 \text{ for some } w = w_n(s) \geq s/a. \tag{2.3$'$}$$

REMARK 2. The conditions require

$$nP[|X| > w_n] \to 0, \quad \frac{n\mu_1(w_n)}{t_n \wedge \eta^{-1}(\lambda t_n)} \to 0, \text{ and } \frac{n\mu_2(w_n)}{(t_n \wedge \eta^{-1}(\lambda t_n))^2} \to 0,$$

for some w_n and hence

$$\frac{S_n}{t_n \wedge \eta^{-1}(\lambda t_n)} \xrightarrow{P} 0,$$

by a standard argument using truncation and Chebyshev's inequality.

REMARK 3. Condition (2.4) is possible for some t_n if and only if $(t\eta(t) \log t\eta(t))/\psi(t) \to 0$, as $t \to \infty$, which implies that ψ is slowly varying. Condition (2.5) is possible for some t_n if and only if $t\eta(t)\overline{F}(t/\psi(t)) \to 0$. The following lemma gives sufficient conditions.

LEMMA 2.2.   *Assume (2.1) with $\eta(t) \downarrow 0$ and $b(t) \to b$, as $t \to \infty$.*

 (i) *If $\exp(\log^2 \psi(t))$ is slowly varying then*

$$\lim_{t\to\infty} \frac{t\eta(t)\log\psi(t)}{\psi(t)} = 0.$$

 (ii) *If*

$$\lim_{t\to\infty} \frac{t\eta(t)}{\psi(t)} = 0 \tag{2.7}$$

 and

$$\liminf_{t\to\infty} \frac{\eta(t/\psi(t))}{\eta(t)} > 1 \tag{2.8}$$

  then

$$\lim_{t\to\infty} t\eta(t)\overline{F}(t/\psi(t)) = 0.$$

REMARK 4. In particular, (2.7) and (2.8) hold if η is monotone and regularly varying with index -1. Some examples are given in Section 3.

3. Examples and ramifications.

For our first example we consider a class of finite variance distributions for which the condition on t_n is simply expressed. This class includes the lognormal distribution and some log-Weibull distributions.

THEOREM 3.1.   *Suppose F has mean 0 and a finite $2 + \delta$ moment for some positive δ. Suppose also that (2.1) and (2.2) hold with b bounded and measurable, $\eta(t) \downarrow 0$, $t\eta(t)$ bounded away from 0, and*

$$\lim_{t\to\infty} \frac{t\eta(t)\log t\eta(t)}{\psi(t)} = 0. \tag{3.1}$$

If $\lambda \in (0,1)$ and $t_n \uparrow \infty$ satisfy

$$\sup_{s\geq t_n} \frac{n\psi(s)\eta(\lambda s)}{s} \to 0, \tag{3.2}$$

as $n \to \infty$, then (2.6) holds.

For example, suppose F is the centered lognormal distribution

$$F(x) = \Phi\left(\frac{\log(x+\beta)}{\sigma}\right), \qquad x > -\beta,$$

where Φ is the standard normal distribution and $\beta = e^{\sigma^2/2}$. Then

$$\overline{F}(x) \sim \frac{\sigma}{\sqrt{2\pi}} e^{-(\log(x+\beta))^2/2\sigma^2 - \log\log(x+\beta)} \sim \frac{\sigma}{\sqrt{2\pi}} e^{-(\log x)^2/2\sigma^2 - \log\log x}.$$

and we can take

$$\eta(x) = \frac{\log x}{\sigma^2 x} + \frac{1}{x \log x} \sim \frac{\log x}{\sigma^2 x}.$$

Hence if t_n is such that

$$\lim_{n\to\infty} \frac{n \log^3 t_n}{t_n^2} = 0$$

then (2.6) holds.

If b is bounded in Theorem 2.1 but does not satisfy (2.2) then F may not be subexponential. Nevertheless, useful limiting bounds are available. From the proof of Theorem 2.1 one may deduce that if all the other assumptions are met then

$$e^{-\gamma} \leq \liminf_{n\to\infty} \inf_{s\geq t_n} \frac{P[S_n > s]}{n\overline{F}(s)} \leq \limsup_{n\to\infty} \sup_{s\geq t_n} \frac{P[S_n > s]}{n\overline{F}(s)} \leq e^{\gamma}, \tag{3.3}$$

where

$$\gamma = \lim_{y\downarrow 1} \limsup_{t\to\infty} \sup_{1\leq x\leq y} (b(xt) - b(t)).$$

In a slightly different direction, the proof of Theorem 2.1 can be specialized to handle distributions with dominated varying tails. To that end, we recall some basic definitions (cf. Bingham, Goldie, and Teugels (1987), p. 61–76, and Cline (1994)).

For a distribution F, $\overline{F}$ is *regularly varying* ($\overline{F} \in RV_{-\alpha}$) if, for some nonnegative α,

$$\lim_{t\to\infty} \frac{\overline{F}(\lambda t)}{\overline{F}(t)} = \lambda^{-\alpha}, \quad \text{for all } \lambda \geq 1.$$

$\overline{F}$ is *intermediate regularly varying* if

$$\lim_{\lambda\to 1, t\to\infty} \frac{\overline{F}(\lambda t)}{\overline{F}(t)} = 1.$$

$\overline{F}$ is *dominated varying* if

$$\liminf_{t\to\infty} \frac{\overline{F}(\lambda t)}{\overline{F}(t)} > 0, \quad \text{for some } \lambda > 1.$$

THEOREM 3.2. *Suppose $P[|X_1| > x]$ is dominated varying and $S_n/t_n \xrightarrow{P} 0$. If*

$$\limsup_{n\to\infty} \sup_{s\geq t_n} \frac{na\mu_2(s)}{s^2} = 0 \tag{3.4}$$

then (3.3) holds with

$$\gamma = -\log\lim_{\lambda\downarrow 1}\liminf_{t\to\infty} \frac{\overline{F}(\lambda t)}{\overline{F}(t)}.$$

Moreover, if F is intermediate regular varying then (2.6) holds.

Theorem 3.2 shows that distributions with intermediate regularly varying tails are subexponential. In fact it is known that the subexponential class contains those F with dominated varying right tails and satisfying (1.5) (Goldie (1978)).

THEOREM 3.3. *Suppose $P[|X| > x] \in RV_{-\alpha}$, where $\alpha \geq 0$, $p = \sup_{x\geq 0} \frac{F(-x)}{\overline{F}(x)} < \infty$ and t_n satisfies $n\overline{F}(t_n) \to 0$. Then any one of the following implies (2.6):*

    (i) $0 \leq \alpha < 1$.
    (ii) $1 \leq \alpha < 2$ *and* $\lim_{n\to\infty} \frac{n}{t_n}\mu_1(t_n) = 0$.
    (iii) $\alpha \geq 2$ *and* $\lim_{n\to\infty} \frac{nE(X)}{t_n} = \lim_{n\to\infty} \frac{n\mu_2(t_n)\log t_n}{t_n^2} = 0$.

In connection with Theorem 3.3, Heyde (1968) considered the infinite variance case and Nagaev (1969b) considered the finite variance case. Our method, which is a mixture of theirs, has several advantages. First we are able to study these two cases in a unifying way, whereas the methods in Heyde (1968) and Nagaev (1969b) are essentially different. For the infinite variance case, our result is new since Heyde (1968) only considered large deviations of $|S_n|$. Also neither author considered the cases where $\alpha = 0, 1$ or 2.

4. Proofs.

PROOF OF THEOREM 1.1. If $F \in \mathcal{S}$, then (1.5) holds by definition. To show both (1.7) and (1.8) we simply choose t_n by (1.6) to satisfy

$$\sup_{s\geq t_n}\left|\frac{P[S_n > s]}{n\overline{F}(s)} - 1\right| \leq \frac{1}{n}.$$

Suppose instead that (1.5) holds and such t_n exist that (1.7) holds. Choose n_0 so that for each $n \geq n_0$,

$$\sup_{s \geq t_n} \frac{P[S_n > s]}{n\overline{F}(s)} \leq 1 + \epsilon.$$

Let $F_n(x) = P[S_n \leq x]$. Equation (1.5) is also satisfied with F_n replacing F (cf. Embrechts and Goldie (1980); with extension by Willekens (1986)). Using this fact and Fatou's Lemma,

$$\liminf_{x \to \infty} \frac{\overline{F}_{2n}(x)}{\overline{F}_2(x)} \geq \liminf_{x \to \infty} \int_{-\infty}^{x/2} \frac{\overline{F}_{2n-2}(x-u)}{\overline{F}_2(x)} F_2(du) + \liminf_{x \to \infty} \int_{-\infty}^{x/2} \frac{\overline{F}_2(x-u)}{\overline{F}_2(x)} F_{2n-2}(du)$$

$$\geq \liminf_{x \to \infty} \frac{\overline{F}_{2n-2}(x)}{\overline{F}_2(x)} + 1. \tag{4.1}$$

By induction, $\liminf_{x \to \infty} \frac{\overline{F}_{2n}(x)}{n\overline{F}_2(x)} \geq 1$. Choose successively, therefore, $y_n \geq \max(t_{2n}, y_{n-1})$ so that

$$\inf_{y \geq y_n} \frac{\overline{F}_{2n}(y)}{n\overline{F}_2(y)} \geq 1 - \epsilon.$$

Now let $x_k \to \infty$ and define $m = \sup\{n : x_k \geq y_n\}$. Then for k large enough so that $2m > n_0$,

$$\frac{\overline{F}_2(x_k)}{2\overline{F}(x_k)} = \frac{m\overline{F}_2(x_k)}{\overline{F}_{2m}(x_k)} \frac{\overline{F}_{2m}(x_k)}{2m\overline{F}(x_k)} \leq \frac{1+\epsilon}{1-\epsilon}.$$

That is,

$$\limsup_{k \to \infty} \frac{\overline{F}_2(x_k)}{2\overline{F}(x_k)} \leq 1.$$

By the same use of Fatou's Lemma that gave (4.1),

$$\liminf_{k \to \infty} \frac{\overline{F}_2(x_k)}{2\overline{F}(x_k)} \geq 1.$$

Since the sequence x_k is arbitrary, then (1.4) holds and $F \in \mathcal{S}$. $\qquad\square$

We need the following lemma in the proof of Theorem 2.1.

LEMMA 4.1. Let $1 \leq y_n \leq x_n$. There exists z_n such that $x_n/z_n \to \infty$, $z_n/y_n \to \infty$ and $x_n/(y_n \log z_n) \to \infty$ if and only if $x_n/y_n \to \infty$ and $x_n/(y_n \log y_n) \to \infty$.

PROOF: The sufficiency is obvious. For the necessary part, let

$$v_n = \min\left(\left(\frac{x_n \log y_n}{y_n}\right)^{1/2}, \frac{1}{2}(\log x_n + \log y_n)\right)$$

and $z_n = e^{v_n}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

PROOF OF THEOREM 2.1. We shall proceed by first proving

$$\limsup_{n\to\infty} \sup_{s\geq t_n} \frac{P[S_n > s]}{n\overline{F}(s)} \leq 1 \qquad\qquad (4.2)$$

and then

$$\liminf_{n\to\infty} \inf_{s\geq t_n} \frac{P[S_n > s]}{n\overline{F}(s)} \geq 1. \qquad\qquad (4.3)$$

Let $\epsilon = \epsilon_n(s)$ and $\epsilon' = \epsilon'_n(s)$ be functions which vanish uniformly in $s \geq t_n$, as $n \to \infty$. Specific choices for ϵ and ϵ' will be made later. Define

$$m = m_n(s) = (1 - \epsilon')s.$$

Let $S'_n = \sum_{i=1}^{n} X_i 1_{X_i \leq m}$. Then

$$\frac{P[S_n > s]}{n\overline{F}(s)} \leq \frac{nP[X_1 > m] + P[S'_n > s]}{n\overline{F}(s)}$$

$$= \frac{\overline{F}(m)}{\overline{F}(s)} + \frac{P[S'_n > s]}{n\overline{F}(s)}. \qquad\qquad (4.4)$$

By Markov's inequality,

$$\frac{P[S'_n > s]}{n\overline{F}(s)} \leq \exp\left(n \int_{-\infty}^{m} (e^{cx} - 1)F(dx) - cs + a \right)$$

for any $c > 0$. In particular, the choice $c = (1 + \epsilon)a/s$ gives

$$\frac{P[S'_n > s]}{n\overline{F}(s)} \leq \exp\left(n \int_{-\infty}^{m} (e^{\frac{(1+\epsilon)ax}{s}} - 1)F(dx) - \epsilon a \right) \qquad\qquad (4.5)$$

$$\leq \exp\left(n \int_{-w}^{s/a} (e^{\frac{(1+\epsilon)ax}{s}} - 1)F(dx) + n \int_{s/a}^{m} (e^{\frac{(1+\epsilon)ax}{s}} - 1)F(dx) - \epsilon a \right)$$

for any $w > 0$. In the following, let $w = w_n(s)$ be chosen by $(2.3')$. In view of (4.4) and (4.5), condition (4.2) follows from choosing ϵ and ϵ' so that

$$\lim_{n\to\infty} \sup_{s\geq t_n} \frac{\overline{F}(m)}{\overline{F}(s)} = 1, \qquad\qquad (4.6)$$

$$\limsup_{n\to\infty} \sup_{s\geq t_n} \frac{n}{\epsilon a} \int_{-w}^{s/a} (e^{\frac{(1+\epsilon)ax}{s}} - 1)F(dx) \leq \delta \qquad\qquad (4.7)$$

$$\limsup_{n\to\infty} \sup_{s\geq t_n} \frac{n}{\epsilon a} \int_{s/a}^{m} (e^{\frac{(1+\epsilon)ax}{s}} - 1)F(dx) < 1 - \delta \qquad\qquad (4.8)$$

and

$$\epsilon a \to \infty \text{ as } n \to \infty \tag{4.9}$$

for some $\delta \in (0, 1)$. To this end, define

$$B = 1 + \sup_{u \geq t \geq 0} |b(t) - b(u)|$$

and choose $z_n(s)$ according to Lemma 4.1 with $x_n(s) = a_n(s)$ and $y_n(s) = 1 \vee s\eta(\lambda s)$. Fix $\delta \in (0, 1 - e^{-2})$ and define

$$\epsilon = \max\left(z_n(s)^{-1}, ne^{B+3}\overline{F}(s/a), \frac{2n}{\delta}\left(\frac{|\mu_1(w)|}{s} + \frac{e^2\mu_2(w)}{s^2/a}\right)\right). \tag{4.10}$$

Using $(2.3')$, (2.4) and (2.5) and Lemma 4.1 we have, uniformly for $s \geq t_n$,

$$(1 \vee s\eta(\lambda s))\epsilon \leq \max\left(\frac{y_n(s)}{z_n(s)}, ne^{B+3}(1 \vee s\eta(\lambda s))\overline{F}(s/a), \frac{2n(|\mu_1(w)| + e^2 a\mu_2(w)/s)}{\delta(s \wedge \eta^{-1}(\lambda s))}\right)$$
$$\to 0, \tag{4.11}$$

$$\epsilon a \geq \frac{x_n(s)}{z_n(s)} \to \infty \tag{4.12}$$

and

$$\frac{(-\log\epsilon)(1 \vee s\eta(\lambda s))}{a} \leq \frac{y_n(s)\log z_n(s)}{x_n(s)} \to 0. \tag{4.13}$$

Now choose

$$\epsilon' = \frac{\epsilon + (2B - \log\epsilon)/a}{1 - s\eta(\lambda s)/a}.$$

Thus

$$(\epsilon' - \epsilon)a - \epsilon' s\eta(\lambda s) - 2B = -\log\epsilon. \tag{4.14}$$

In addition, from (2.4), (4.11) and (4.13) we have $\epsilon' \to 0$ and with the monotonicity of η,

$$s\eta(m)\epsilon' \leq (1 \vee s\eta(\lambda s))\epsilon'$$
$$= \frac{(1 \vee s\eta(\lambda s))(\epsilon + (2B - \log\epsilon)/a)}{1 - s\eta(\lambda s)/a} \to 0. \tag{4.15}$$

We now show (4.6)–(4.8), since (4.9) already follows from (4.12). From (4.15) and the assumption on b,

$$\psi(s) - \psi(m) \leq b(s) - b(m) + \epsilon' s\eta(m) \to 0, \tag{4.16}$$

uniformly for $s \geq t_n$, which is (4.6). Using $w \geq s/a$, (4.10) and a Taylor expansion, for large enough n,

$$n\int_{-w}^{s/a}(e^{\frac{(1+\epsilon)ax}{s}} - 1)F(dx) \leq 2n\frac{\mu_1(w)}{s/a} + 2ne^2\frac{\mu_2(w)}{(s/a)^2} \leq \delta a\epsilon,$$

uniformly for $s \geq t_n$, which is (4.7).

Integrating by parts, we get

$$n \int_{s/a}^{m} (e^{\frac{(1+\epsilon)ax}{s}} - 1)F(dx) \leq n\frac{(1+\epsilon)a}{s} \int_{s/a}^{m} e^{\frac{(1+\epsilon)ax}{s} - \psi(x)} \, dx + ne^{(1+\epsilon)}\overline{F}(s/a). \qquad (4.17)$$

Since η is decreasing, $\frac{(1+\epsilon)ax}{s} - (\psi(x) - b(x))$ is convex and hence

$$\sup_{s/a \leq x \leq m} e^{\frac{(1+\epsilon)ax}{s} - \psi(x)} \leq \max\left(e^{\frac{(1+\epsilon)ma}{s} - \psi(m) + B - 1}, e^{B+\epsilon}\overline{F}(s/a) \right). \qquad (4.18)$$

By (4.17), (4.18) and the fact that $(1+\epsilon)m \leq s$, we obtain the bound

$$n \int_{s/a}^{m} (e^{\frac{(1+\epsilon)ax}{s}} - 1)F(dx) \leq na \max\left(e^{\frac{(1+\epsilon)ma}{s} - \psi(m) + B - 1}, e^{B+\epsilon}\overline{F}(s/a) \right) + ne^{(1+\epsilon)}\overline{F}(s/a)$$

$$\leq a \max\left(e^{-(\epsilon'-\epsilon)a + \epsilon's\eta(\lambda s) + 2B - 2}, e^{B+1}n\overline{F}(s/a) \right) + ne^2\overline{F}(s/a)$$

$$= e^{-2}\epsilon a + ne^2\overline{F}(s/a), \qquad (4.19)$$

where the final equality comes from (4.10) and (4.14). The bound in (4.8) follows from (2.5) and that completes the proof of (4.2).

Finally, we verify (4.3). Let $m' = s + \zeta(s \wedge \eta^{-1}(\lambda s))$ where ζ is any fixed positive constant. By Bonferroni's inequality,

$$P[S_n > s] \geq P[S_n > s, \max_{1 \leq i \leq n} X_i > m']$$

$$\geq \sum_{i=1}^{n} P[S_n > s, \, X_i > m'] - \sum_{1 \leq i < j \leq n} P[S_n > s, \, X_i > m', \, X_j > m']$$

$$\geq n\overline{F}(m') \left(P[S_{n-1} > -\zeta(s \wedge \eta^{-1}(\lambda s))] - \frac{n}{2}\overline{F}(s) \right). \qquad (4.20)$$

Note that $n\overline{F}(s) \to 0$ by (2.4), and if we let

$$\gamma(\zeta) = \limsup_{t \to \infty} \sup_{0 \leq u \leq \zeta} (b((1+u)t) - b(t))$$

then

$$\liminf_{n \to \infty} \inf_{s \geq t_n} \frac{\overline{F}(m')}{\overline{F}(s)} \geq e^{-\zeta - \gamma(\zeta)} \qquad (4.21)$$

(cf. the derivation of (4.16)).

By Remark 2 of Section 2,

$$\sup_{s \geq t_n} \frac{S_{n-1}}{s \wedge \eta^{-1}(\lambda s)} \xrightarrow{P} 0.$$

Hence

$$\inf_{s \geq t_n} P[S_{n-1} \geq -\zeta(s \wedge \eta^{-1}(\lambda s))] \to 1$$

and by (4.20) and (4.21),

$$\liminf_{n \to \infty} \inf_{s \geq t_n} \frac{P[S_n > s]}{n\overline{F}(s)} \geq \liminf_{n \to \infty} \inf_{s \geq t_n} \frac{\overline{F}(m')}{\overline{F}(s)} \left(P[S_{n-1} > -\zeta(s \wedge \eta^{-1}(\lambda s))] - \frac{n}{2}\overline{F}(s) \right)$$
$$\geq e^{-\zeta - \gamma(\zeta)}.$$

Condition (4.3) follows from this since $\zeta > 0$ is arbitrary and $\gamma(\zeta) \to 0$ as $\zeta \to 0$ by (2.2).

$\square$

PROOF OF LEMMA 2.2. Define $B_1 = \inf_{u \geq 0} b(u)$ and $\psi_1(t) = \psi(t) - b(t) + B_1$. Also define $B_2 = \sup_{u \geq 0} b(u)$ and $\psi_2(t) = \psi(t) - b(t) + B_2$.

(i) Let $\xi(t) = \exp(\log^2 \psi_1(t))$. Then, for large enough t,

$$\frac{2\eta(2t) \log \psi(2t)}{\psi(2t)} \leq \frac{2}{\xi(t)} \int_t^{2t} \frac{\eta(u)(\log \psi_1(u))\xi(u)}{\psi_1(u)} \, du$$
$$= \frac{\xi(2t)}{\xi(t)} - 1$$
$$\leq \frac{e^{\log^2 \psi(2t)}}{e^{\log^2 \psi(t)}} e^{-2 \log \psi(t) \log(1-(B_2-B_1)/\psi(t))} - 1 \to 0.$$

(ii) From (2.7) and (2.8) choose t_0 so that

$$\left(1 - \frac{t\eta(t)}{\psi_2(t)} \right) \frac{\eta(t/\psi_2(t))}{\eta(t)} \geq 1, \qquad \text{for all } t \geq t_0.$$

Then

$$\psi(t/\psi(t)) \geq \psi_2(t/\psi_2(t)) + B_1 - B_2$$
$$= \psi_2(t_0/\psi_2(t_0)) + B_1 - B_2 + \int_{t_0}^t \left(1 - \frac{u\eta(u)}{\psi_2(u)} \right) \frac{\eta(u/\psi_2(u))}{\psi_2(u)} \, du$$
$$\geq \psi_2(t_0/\psi_2(t_0)) + B_1 - B_2 + \int_{t_0}^t \frac{\eta(u)}{\psi_2(u)} \, du$$
$$\geq \psi_2(t_0/\psi_2(t_0)) + B_1 - B_2 - \log \psi_2(t_0) + \log \psi(t).$$

Hence

$$t\eta(t)\overline{F}(t/\psi(t)) = \frac{t\eta(t)}{\psi(t)} e^{\log \psi(t) - \psi(t/\psi(t))} \to 0.$$

$\square$

PROOF OF THEOREM 3.1. Condition (3.2), $t\eta(t)$ bounded away from 0 and $t^{2+\delta}\overline{F}(t) \to 0$ imply

$$\limsup_{n\to\infty} \sup_{s\geq t_n} \frac{\log n}{\psi(s)} \leq \frac{2}{2+\delta}$$

so that (2.4) follows from (3.1). In addition, $\psi(t)$ must be slowly varying so

$$\limsup_{s\geq t_n} ns\eta(\lambda s)\left(\overline{F}(s/a) + F(-s/a)\right) \leq \frac{na\eta(\lambda s)}{s}\frac{\psi^{1+\delta}(s)}{s^\delta}(s/a)^{2+\delta}\left(\overline{F}(s/a) + F(-s/a)\right)$$
$$\to 0.$$

Furthermore,

$$\sup_{s\geq t_n} \frac{na\mu_2(s)/s}{s \wedge \eta^{-1}(\lambda s)} \to 0.$$

Therefore, (2.3) and (2.5) hold. □

PROOF OF THEOREM 3.2. By the dominated variation assumption, (2.1) holds with b measurable and bounded and η measurable and satisfying

$$0 \leq t\eta(t) \leq A \text{ for all } t > 0, \tag{4.23}$$

for some finite A (cf. Bingham, Goldie and Teugels (1987), Theorem 2.2.7).

Conditions (2.3) and (2.4) are easily checked from (3.4) and the assumption $\frac{S_n}{t_n} \xrightarrow{P} 0$ (cf. Loève (1955), p. 317). It is actually not necessary to check (2.5) for this proof though in fact something stronger holds (see below). The basic structure in the argument for Theorem 2.1 may be followed, with a slight difference. We focus on the relevant distinction here.

Since η is not necessarily monotone, the bound in (4.19) must be modified. In fact, using (4.23),

$$n\int_{s/a}^m (e^{\frac{(1+\epsilon)ax}{s}} - 1)F(dx) \leq ne^{\frac{(1+\epsilon)am}{s}}\overline{F}(s/a)$$
$$\leq e^{-(\epsilon'-\epsilon)a+\psi(s)-\psi(s/a)}$$
$$\leq e^{-(\epsilon'-\epsilon)a+B-1+A\log a}.$$

So if we let

$$\epsilon = \max\left(z_n(s)^{-1}, \frac{2n(|\mu_1(w)| + e^2 a\mu_2(w)/s)}{\delta(s \wedge \eta^{-1}(\lambda s))}\right)$$

and

$$\epsilon' = \epsilon + \frac{B-1+A\log a - \log\epsilon}{a}$$

and we use the idea in (3.3), then the rest of the proof follows readily. □

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PROOF OF THEOREM 3.3. We need only to check the assumptions of Theorem 3.2 since $P[|X| > x]$ is regularly varying. Note that (2.2) holds. Consider the following three cases separately.

i) If $\alpha < 1$, we make use of the well known relationships (cf. Bingham, Goldie and Teugels (1987), Theorem 8.1.2) between $\overline{F}$, μ_1 and μ_2 to obtain

$$\limsup_{t\to\infty} \frac{|\mu_1(t)|}{t\overline{F}(t)} \leq \frac{(1+p)\alpha}{1-\alpha} < \infty \tag{4.24}$$

and

$$\limsup_{t\to\infty} \frac{\mu_2(t)}{t^2\overline{F}(t)} \leq \frac{(1+p)\alpha}{2-\alpha} < \infty.$$

Hence $n\overline{F}(t_n) \to \infty$ implies $S_n/t_n \xrightarrow{P} 0$ and

$$\lim_{n\to\infty} \sup_{s\geq t_n} \frac{na\mu_2(s)}{s^2} \leq \frac{(1+p)\alpha}{2-\alpha} \lim_{n\to\infty} \sup_{s\geq t_n} n\overline{F}(s)(-\log n\overline{F}(s)) = 0.$$

ii) In case $1 \leq \alpha < 2$, the only difference is that (4.24) fails so the appropriate centering of S_n must be assumed. Otherwise, the proof is as above.

iii) For $\alpha \geq 2$, the extra conditions imply $S_n/t_n \xrightarrow{P} 0$. Also,

$$\lim_{t\to\infty} \frac{-\log \overline{F}(t)}{\log t} = \alpha,$$

so that for some finite C,

$$-\log n\overline{F}(t) \leq C \log t.$$

Furthermore, $\mu_2(x)\log x/x^2 \in RV_{-2}$ and therefore is almost decreasing (cf. Bingham, Goldie and Teugels (1987), p. 41). That is,

$$\lim_{n\to\infty} \sup_{s\geq t_n} \frac{\mu_2(s)\log s/s^2}{\mu_2(t_n)\log t_n/t_n^2} < \infty.$$

Hence (3.4) follows readily. □

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