Answer booklet

April 5, 2018 **Name:** ....

STAT 632 (Anirban Bhattacharya) **UIN:** ....

1	2	3	$\sum$

The exam is out of 60 points. For each problem a maximum of 20 points can be earned. You have 65 minutes to complete the exam.

- **Work your problems in the space provided.**
- **Show all work clearly and concisely.**
- **Justify your answers. State results you use.**
- **Box your $\boxed{\text{final answer}}$ whenever applicable.**
- $\mathbb{1}_A(x)$ denotes the indicator function of the set A, that is, $\mathbb{1}_A(x) = 1$ if $x \in A$ and is zero otherwise.

Distribution name	pdf/pmf	mean	variance
Binomial(n,p)	$\binom{n}{x} p^x (1-p)^{n-x}, x \in \{0,1,\ldots,n\}$	np	$np(1-p)$
Poisson(λ)	$\frac{e^{-\lambda}\lambda^x}{x!}, x \in \{0,1,\ldots\}$	λ	λ
Gamma(α,β)	$\frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta x} x^{\alpha-1}, x \in (0,\infty)$	α/β	α/β^2
Beta(α,β)	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, x \in (0,1)$	$\alpha/(\alpha+\beta)$	
$\mathrm{N}(\mu,\sigma^2)$	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x \in \mathbb{R}$	μ	σ^2
$\mathrm{N}_d(\mu,\Sigma)$	$(2\pi)^{-d/2}\|\Sigma\|^{-1/2} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}, x \in \mathbb{R}^d$	μ	Σ

1. Suppose $x_1, \ldots, x_n$ are independent observations drawn from a $\mathrm{N}(\lambda_1\eta, 1)$ distribution, and $y_1, \ldots, y_m$ are independent observations drawn from a $\mathrm{N}(\lambda_2\eta, 1)$ distribution. Further, assume that $\mathrm{x} = (x_1, \ldots, x_n)$ and $\mathrm{y} = (y_1, \ldots, y_m)$ are conditionally independent given $\lambda_1, \lambda_2, \eta$. Assume independent $\mathrm{N}(0, 1)$ priors on $\lambda_1, \lambda_2, \eta$.

(a) Clearly outline the steps of a **Gibbs sampler** to sample from the posterior distribution of $(\lambda_1, \lambda_2, \eta) \mid (\mathrm{x}, \mathrm{y})$, specifying the necessary **full conditional distributions**.

(b) Suppose you run the Gibbs sampler for 11,000 iterations and discard the first 1,000 iterations as burn-in. Describe how you shall use the output of the Gibbs sampler to obtain a point estimate and 95% confidence interval for $\lambda_1\lambda_2$.

2. Suppose we observe $\mathrm{y} = (y_1, \ldots, y_n)$, with $y_i \mid \mu_i \sim \mathrm{N}(\mu_i, 1)$ independently for $i = 1, \ldots, n$. Given $\omega \in (0, 1)$, assume independent $\mathrm{N}(0, (1-\omega)/\omega)$ priors on the μ_is. [**Note:** $(1-\omega)/\omega$ is the prior variance, which is guaranteed to be positive since $\omega \in (0, 1)$]

Assume a Beta$(1/2, 1/2)$ prior on ω.

(a) Show that the marginal posterior distribution of $\omega \mid \mathrm{y}$ is of the form

$$\pi(\omega \mid \mathrm{y}) \propto \omega^{A-1}(1-\omega)^{B-1}e^{-C\omega}, \quad \omega \in (0, 1).$$

Identify A, B, C.

(b) Describe the steps of a Metropolis algorithm to sample from $\pi(\omega \mid \mathrm{y})$ that performs a normal random walk (with standard deviation σ_{MH}) on $\log\left(\frac{\omega}{1-\omega}\right)$. In particular, your answer should contain an explicit expression for the acceptance probability.

(c) Describe the steps of a Bootstrap filter to obtain a discrete approximation to $\pi(\omega \mid \mathrm{y})$. How shall you use this to estimate $P(\omega > 0.5 \mid \mathrm{y})$?

3. For p and q two probability densities, recall the total variation distance

$$d_{TV}(p,q) = \frac{1}{2}\int |p-q| = \int_{p>q}(p-q) = \int_{q>p}(q-p),$$

Let $p(x) = e^{-x}\mathbb{1}_{(0,\infty)}(x)$ and $q(x) = \lambda e^{-\lambda x}\mathbb{1}_{(0,\infty)}(x)$ be the densities of exponential distributions with rates 1 and $\lambda > 1$ respectively.

(a) Find $d_{TV}(p,q)$.

(b) Let q_n be the density of an exponential distribution with rate $\lambda_n = (1+1/n)$. Show that $\lim_{n\to\infty} d_{TV}(p,q_n) = 0$. [**Hint:** You may need to use the fact $\lim_{n\to\infty}(1+1/n)^n = e$]