Homework 0

This is a collection of practice problems meant as review for some concepts needed for this course. The contents of all the problems will be revisited in some form during the course, which makes this a very important homework for your preparation. You should work on the problems within the first week. If you have questions, I shall be happy to discuss during office hours. This homework is due in class in one week. Answer concisely and clearly.

Problem 1: (i) Consider the univariate density

$$f(x) \propto e^{-\frac{1}{2}(qx^2 - 2bx)}, \quad x \in \mathbb{R},$$

where $q > 0$. Write down a full expression for the density, i.e., with the constants. Can you identify what density this is?

(ii) Consider the univariate density

$$f(u) \propto e^{-\frac{\tau}{2}\left[N(t-u)^2 + \lambda(u-\xi)^2\right]}, \quad u \in \mathbb{R},$$

where $N, \lambda, \tau > 0$. Write down a full expression for the density, i.e., with the constants. Can you identify what density this is?

(iii) Consider the d-variate density,

$$f(x) \propto e^{-\frac{1}{2}(x'Qx - 2b'x)}, \quad x \in \mathbb{R}^d,$$

where Q is a p.d. matrix and $b \in \mathbb{R}^d$. Write down a full expression for the density, i.e., with the constants. Can you identify what density this is?

(iv) Show that X from the following algorithm produces a sample from the density in (iii).

- Perform a Cholesky decomposition $Q = LL^{\mathrm{T}}$, where L is lower triangular.
- Draw $z \sim N(0, \mathrm{I}_d)$, solve $L^{\mathrm{T}}y = z$.
- Solve $L^{\mathrm{T}}\theta = v$, where $Lv = b$.
- Set $X = y + \theta$.

(v) Consider the d-variate density,

$$f(u) \propto e^{-\frac{1}{2}\left[(t-Au)'\Sigma^{-1}(t-Au) + u'\Omega^{-1}u\right]},$$

where $t \in \mathbb{R}^N$, A is an $N \times d$ matrix, Σ is an $N \times N$ positive definite matrix, and Ω is a $d \times d$ positive definite matrix. Can you identify what distribution this is (along with the parameters)?

Problem 2:

(i) Suppose $X \mid \tau \sim N(\mu, \tau^{-1}\sigma^2)$, and $\tau \sim \text{Gamma}(a, b)$. Write down the marginal density of X and identify it. What happens when $a = b = \nu/2$ for some $\nu > 0$?

(ii) Suppose $X \mid \lambda \sim \text{Poisson}(\lambda)$, and $\lambda \sim \text{Gamma}(r, (1-p)/p)$ for some $p \in (0, 1)$ and integer r. Find the marginal distribution of X, and calculate its mean and variance (avoid extra work!). Can you identify this distribution?

Problem 3. Suppose $X = (X_1, \ldots, X_5) \sim \text{Dirichlet}(\alpha_1, \ldots, \alpha_5, \alpha_6)$. Find the marginal distribution of X_1, the joint distribution of (X_2, X_4) and $(X_1, X_2 + X_3, X_4 + X_5)$.

Problem 4.

(i) If $U \sim U(0,1)$, show that $-\log(U) \sim \text{Expo}(1)$.

(ii) Let $E_1, E_2, \ldots$ be i.i.d. Expo(1) random variables. Fix $\lambda > 0$. Let X be the smallest integer such that $\sum_{i=1}^{X+1} E_i > \lambda$. Show that $X \sim \text{Poisson}(\lambda)$. [**Hint:** Try calculating $P(X = 0), P(X = 1)$ and so on. Do you see a pattern?]

Problem 5. (i) Suppose we want to sample from a truncated distribution with density proportional to $f\mathbb{1}_{(u,v)}$, where f is a (absolutely continuous) density on the real line, with cdf F, and $-\infty \leq u < v \leq \infty$. Draw $U \sim U(F(u), F(v))$ and set $X = F^{-1}(U)$. Then, show that $X \sim f\mathbb{1}_{(u,v)}$.

(ii) Use this to draw 1000 samples from a standard normal distribution truncated to $(0, \infty)$, and draw a histogram of the obtained samples.