Homework 1

This homework is due in class Feb 6, Tuesday

Problem 1: Suppose f is a density on $\mathbb{R}$ which you know how to draw samples from. Let F be the cdf of f. Describe a procedure to draw samples from the distribution with cdf $G(x) = \{F(x)\}^4$.

Problem 2: The Gamma$(\alpha, 1)$ distribution for $\alpha < 1$ is unbounded near zero, causing difficulty finding a dominating density (say, to implement rejection sampling). Prove that if $X \sim$ Gamma$(\alpha+1, 1)$ and $U \sim U(0,1)$ with U and X independent, then $XU^{1/\alpha} \sim$ Gamma$(\alpha, 1)$. Comment on the utility of this result.

Problem 3: Suppose we want to calculate $p = P(X > 4)$ for $X \sim N(0,1)$. Approximate p using 1000 draws from a standard Cauchy importance density.

Problem 4: Let ϕ denote the standard normal pdf and Φ the corresponding cdf.

(i) Show that $f(x) = 2\phi(x)\Phi(x)$ is a density on $\mathbb{R}$.

(ii) Describe and implement a rejection sampler to sample from the above density using draws from a standard normal density. Overplay a histogram of iid samples from f with its density plot.

Problem 5: Let N be the number of draws from g to obtain one sample from f in a rejection sampler (same notation as in lecture notes or class notes). Find the distribution of N and its expected value, $E(N)$.

Problem 6: The Henrey–Greenstein distribution is used to model the angle θ at which photons scatter, with medical and environmental applications. Let $\eta = \cos(\theta)$. For a parameter $b \in (-1, 1)$, the density of η is given by

$$f(\eta \mid b) = \frac{1}{2} \frac{1 - b^2}{(1 + b^2 - 2b\eta)^{3/2}}, \quad \eta \in [-1, 1].$$

(i) draw a plot of the density for $b = 0.8$.

(ii) For the above value of b, use the inverse-cdf method (or any other method you prefer) to draw independent samples from f (provide a description of your method and histogram of obtained samples), and use these samples to estimate the mean of η.

Problem 7. Take a look back at Problem 1(v) from HW0. I want you to think about $f(u)$ from a different perspective now. First, convince yourself that f is the density of $u \mid t$ where $t \mid u \sim \mathcal{N}(Au, \Sigma)$ and $u \sim \mathcal{N}(0, \Omega)$. Can you identify the joint distribution of (t, u)? [Hint: I tell you that (t, u) has a joint Gaussian distribution]

You don't need to turn in the following calculation, though I recommend going through this. Once you get the joint density of (t, u) and you know it is joint Gaussian, you can directly apply the conditional distribution formula for MVNs to obtain the conditional of $u \mid t$, which we have already argued is f. See if you get the same answer as before.