Biometrika (1971), **58**, 1, *p.* 21
Printed in Great Britain

On the estimation of a harmonic component in a time series with stationary independent residuals

BY A. M. WALKER
University of Sheffield

SUMMARY

Let $\{X_t\}$ be a time series such that

$$X_t = E(X_t) + \sum_{u=0}^{\infty} g_u(\boldsymbol{\theta})\,\epsilon_{t-u},$$

where $E(X_t)$ is the sum of a finite number of simple harmonic terms of the form

$$A\cos(\omega t) + B\sin(\omega t),$$

the ϵ_t are independently and identically distributed random variables each with mean zero and finite variance, and the $g_u(\boldsymbol{\theta})$ are specified functions of a vector-valued parameter $\boldsymbol{\theta}$. Whittle (1952) proposed an approximate least squares method of simultaneously estimating $\boldsymbol{\theta}$ and the angular frequencies, sine and cosine coefficients, of each harmonic term from observations $(X_1, \ldots, X_n)$ and derived heuristically the asymptotic distribution of the estimators. This paper presents rigorous proofs of Whittle's statements concerning the asymptotic distribution, formulated precisely as limit theorems, for the special case of independent residuals, where $X_t = E(X_t) + \epsilon_t$, so that the parameter $\boldsymbol{\theta}$ disappears. The arguments used here suggest how one can deal with the general case, and proofs for this will be given in a subsequent paper.

1. INTRODUCTION

We consider a discrete parameter time series $\{X_t, t = 0, \pm 1, \ldots\}$ such that

$$X_t = m_t + Y_t, \tag{1·1}$$

where

$$m_t = E(X_t) = \sum_{r=1}^{q} \{A_r \cos(\omega_r t) + B_r \sin(\omega_r t)\}, \tag{1·2}$$

$$Y_t = \sum_{u=0}^{\infty} g_u(\boldsymbol{\theta})\,\epsilon_{t-u}. \tag{1·3}$$

Further the ϵ_t are distributed independently and identically with $E(\epsilon_t) = 0$ and $E(\epsilon_t^2) = v < \infty$, and the $g_u(\boldsymbol{\theta})$ are specified functions of a vector-valued parameter $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)$, with $g_0(\boldsymbol{\theta}) \equiv 1$, to avoid indeterminacy, and $\sum_{u=0}^{\infty} g_u^2(\boldsymbol{\theta}) < \infty$. The series $\{X_t\}$ thus has a systematic component consisting of the sum of q simple harmonic terms with angular frequencies ω_r and a residual or noise component which is a completely stationary series having spectral density

$$f(\omega, \boldsymbol{\theta}) = \frac{v}{2\pi}\left|\sum_{u=0}^{\infty} g_u(\boldsymbol{\theta})\, e^{i\omega u}\right|^2 \tag{1·4}$$

and is usually called a linear process; see, for example, Hannan (1960, p. 33).

Suppose that the values of $\boldsymbol{\theta}$, A_r, B_r and ω_r $(1 \leqslant r \leqslant q)$ and v are all unknown *a priori*. We then have a fairly general type of hidden periodicities model, the term hidden periodicity denoting a harmonic component in m_t whose frequency as well as its amplitude and phase are unknown. The restricted model obtained by taking the residual component to consist of white noise, so that $Y_t = \epsilon_t$, the parameter $\boldsymbol{\theta}$ disappearing, has been quite widely used in analyzing physical and economic data, underlying what is often referred to as periodogram analysis. In recent years it has tended to be regarded as unsatisfactory, mainly because of the misleading results that can be obtained if the white noise assumption is not a good approximation or because of doubts as to whether it is realistic to represent m_t as the sum of harmonic components.

The problem of estimating the parameters in (1·1) from data consisting of a set of observations $\mathbf{X}^{(n)} = (X_1, \ldots, X_n)$ and of determining the approximate distribution of the estimators for large n was dealt with by Whittle (1952). He used a method of estimation which was approximately equivalent to an application of the principle of least squares, becoming approximately the method of maximum likelihood when ϵ_t has a normal distribution so that $\{X_t\}$ becomes a normal or Gaussian process. Our object is to present rigorous proofs of his main results, stated precisely as limit theorems, which he obtained by purely heuristic arguments. These can be constructed by proceeding in the same way as in the classical asymptotic theory of maximum likelihood estimation, whereby consistency of the estimators is first established, and then the mean value theorem is applied to yield asymptotic normality with the aid of a central limit theorem; compare, for example, Rao (1952, pp. 157–61) or Rao (1965, pp. 299–302). The details of the proofs unfortunately turn out to be extremely complicated. This is due essentially to the presence of unknown angular frequencies ω_r in (1·2) combined with the dependence between the residuals Y_t expressed by (1·3). It, therefore, seemed best to deal with the white noise case, for which $Y_t = \epsilon_t$, and leave the general case with dependent residuals to a subsequent paper. Even in the white noise case the effect of having unknown angular frequencies is fairly complicated, but the degree of complication is substantially reduced by having independent residuals. Also it turns out that the arguments used for this, together with those used to establish consistency and asymptotic normality of the estimators of the components of $\boldsymbol{\theta}$ when $m_t = 0$, a problem first treated heuristically by Whittle (1952, 1953, 1954) and dealt with rigorously much later by Walker (1964), suggest how to proceed in the general case.

Appreciable simplification is achieved when only one harmonic component is present. For this reason, we shall take $q = 1$ in (1·2) in the proofs that follow, and then indicate the modifications required when $q > 1$ in § 5.

2. Statement of results, one harmonic component

Suppose that

$$X_t = A\cos(\omega t) + B\sin(\omega t) + \epsilon_t. \tag{2·1}$$

If the ϵ_t are normally distributed, the log likelihood function of the observations $X_1, \ldots, X_n$ is obviously

$$L_n(A, B, \omega, v) = -\tfrac{1}{2}n\log(2\pi v) - \tfrac{1}{2}S_n(A, B, \omega)/v, \tag{2·2}$$

where

$$S_n(A, B, \omega) = \sum_{t=1}^{n} \{X_t - A\cos(\omega t) - B\sin(\omega t)\}^2. \tag{2·3}$$

Maximum likelihood estimators of A, B and ω are thus obtained by minimizing the residual sum of squares S_n, and the maximum likelihood estimator of v is equal to the minimum sum of squares divided by n. We use this method of estimation whatever the distribution of ϵ_t. It is, however, convenient to modify the definition of the estimators slightly by replacing $S_n(A, B, \omega)$ by the expression

$$U_n(A,B,\omega) = \sum_{t=1}^{n} X_t^2 - 2\sum_{t=1}^{n} X_t\{A\cos(\omega t) + B\sin(\omega t)\} + \tfrac{1}{2}n(A^2+B^2). \tag{2·4}$$

This appreciably simplifies the estimation equations, and since

$$S_n(A,B,\omega) - U_n(A,B,\omega) = \frac{1}{2}\sum_{t=1}^{n}\{(A^2-B^2)\cos(2\omega t) + 2AB\sin(2\omega t)\}$$

which is $O(1)$ as $n \to \infty$ if $\omega \neq 0$ or π, we may expect the effect on the estimators to be negligible for large n provided that ω_0, the true value of ω, is not equal to 0 or π; we assume from now on that $0 < \omega_0 < \pi$.

We denote the modified estimators of A, B, ω and v by $\hat{A}_n$, $\hat{B}_n$, $\hat{\omega}_n$ and $\hat{v}_n$, respectively. Then clearly

$$\hat{A}_n = \frac{2}{n}\sum_{t=1}^{n} X_t\cos(\hat{\omega}_n t), \quad \hat{B}_n = \frac{2}{n}\sum_{t=1}^{n} X_t\sin(\hat{\omega}_n t), \tag{2·5}$$

where $\hat{\omega}_n$ is such that

$$I_n(\hat{\omega}_n) = \max_{0\leqslant\omega\leqslant\pi}\{I_n(\omega)\}. \tag{2·6}$$

Further

$$I_n(\omega) = \frac{2}{n}\left|\sum_{t=1}^{n} X_t e^{i\omega t}\right|^2, \tag{2·7}$$

the usual definition of the periodogram intensity function. It is easy to see that without loss of generality the range of ω may be taken to be $[0, \pi]$. Also

$$\hat{v}_n = \frac{1}{n}\left\{\sum_{t=1}^{n} X_t^2 - I_n(\hat{\omega}_n)\right\}. \tag{2·8}$$

THEOREM 1. *Let*

$$X_t = A_0\cos(\omega_0 t) + B_0\sin(\omega_0 t) + \epsilon_t \quad (0 < \omega_0 < \pi),$$

where the ϵ_t are distributed independently and identically with $E(\epsilon_t) = 0$ and $E(\epsilon_t^2) = v < \infty$. Then the estimators $\hat{A}_n$, $\hat{B}_n$, $\hat{\omega}_n$ and *$\hat{v}_n$ are all consistent as $n \to \infty$.*

The restriction $\omega_0 \neq 0$ or π is not important. In practice one might have to allow for a constant term in m_t, replacing (2·1) by

$$X_t = C + A\cos(\omega t) + B\sin(\omega t) + \epsilon_t,$$

but then it can be shown that the results still hold if in the definition of the estimators X_t is replaced by $X_t - n^{-1}\sum_{t=1}^{n} X_t$; this extension, which also applies to Theorem 2, is discussed briefly at the end of the paper in § 6.

THEOREM 2. *Under the conditions of Theorem* 1, $\{n^{\frac{1}{2}}(\hat{A}_n - A_0), n^{\frac{1}{2}}(\hat{B}_n - B_0), n^{\frac{3}{2}}(\hat{\omega}_n - \omega_0)\}$ *converges in law to $N(\mathbf{0}, 2v\mathbf{W}_0^{-1})$, when $n \to \infty$, a multinormal distribution with mean* $(0, 0, 0)$ *and covariance matrix $2v\mathbf{W}_0^{-1}$, where*

$$\mathbf{W}_0 = \begin{bmatrix} 1 & 0 & \tfrac{1}{2}B_0 \\ 0 & 1 & -\tfrac{1}{2}A_0 \\ \tfrac{1}{2}B_0 & -\tfrac{1}{2}A_0 & \tfrac{1}{3}(A_0^2+B_0^2) \end{bmatrix}. \tag{2·9}$$

This gives us the asymptotic distribution of the estimators $\hat{A}_n$, $\hat{B}_n$ and $\hat{\omega}_n$. Evaluating the inverse of $\mathbf{W}_0$ explicitly, we easily see that

$$\mathbf{W}_0^{-1} = \frac{1}{A_0^2+B_0^2}\begin{bmatrix} A_0^2+4B_0^2 & -3A_0B_0 & -6B_0 \\ -3A_0B_0 & 4A_0^2+B_0^2 & 6A_0 \\ -6B_0 & 6A_0 & 12 \end{bmatrix}. \tag{2·10}$$

An equivalent way of stating our result is therefore that the distribution of $(\hat{A}_n, \hat{B}_n, \hat{\omega}_n)$ is asymptotically normal with mean (A_0, B_0, ω_0) and covariance matrix

$$\frac{2v}{A_0^2+B_0^2}\begin{bmatrix} n^{-1}(A_0^2+4B_0^2) & -3n^{-1}A_0B_0 & -6n^{-2}B_0 \\ -3n^{-1}A_0B_0 & n^{-1}(4A_0^2+B_0^2) & 6n^{-2}A_0 \\ -6n^{-2}B_0 & 6n^{-2}A_0 & 12n^{-3} \end{bmatrix}. \tag{2·11}$$

Compare Whittle [1952, p. 53, (4·14); 1954, p. 224, (11)] where, however, the factor in the bottom diagonal element of the matrix is given incorrectly as $\frac{1}{6}$ instead of $\frac{1}{3}$. Approximate confidence intervals for A_0, B_0 and ω_0 and functions of these such as the phase of the harmonic component $\tan^{-1}(-B_0/A_0)$ can be obtained in the usual way by using the consistent estimator of the covariance matrix given by substituting $\hat{A}_n$, $\hat{B}_n$, $\hat{\omega}_n$ and $\hat{v}_n$ for A_0, B_0, ω_0 and v respectively in (2·11); for example, an approximate 95 % confidence interval for ω_0 is $(\hat{\omega}_n - 1{\cdot}96[24\hat{v}_n/\{n^3(\hat{A}_n^2+\hat{B}_n^2)\}]^{\frac{1}{2}}, \hat{\omega}_n + 1{\cdot}96[24\hat{v}_n/\{n^3(\hat{A}_n^2+\hat{B}_n^2)\}]^{\frac{1}{2}})$. An unusual feature here is the norming factor $n^{\frac{3}{2}}$ for $\hat{\omega}_n - \omega_0$, the asymptotic variance of $\hat{\omega}_n$ being proportional to n^{-3} instead of the expected n^{-1}. As we shall see, this is due to the sharpness of the largest peak of the periodogram intensity function I_n. It should be noted that G. R. Hext in an unpublished Stanford report gives a formula for the asymptotic variance of $\hat{\omega}_n$, according to which it is proportional to n^{-4} instead of n^{-3}. Hext's formula, however, is incorrect, the reason being that the effects of certain remainder terms in his expressions for means and variances of sample autocovariances cannot be neglected, as he assumes.

As regards the numerical computation of the estimators, the only part presenting any difficulty is the determination of $\hat{\omega}_n$ from (2·6). We hope to deal with this problem in detail elsewhere. Meanwhile we note that the following procedure should usually be satisfactory. Obtain a first approximation, $\tilde{\omega}_n^{(1)}$ say, by carrying out a standard periodogram analysis, which yields values $I_{n,k} = I_n(2\pi k/n)$, where k runs through all nonnegative integers not exceeding $\frac{1}{2}n$, and taking $\tilde{\omega}_n^{(1)} = 2\pi k(n)/n$, the angular frequency giving the largest of these periodogram intensities $I_{n,k(n)}$. Strictly we should write $k\{n, \mathbf{X}^{(n)}\}$ here rather than $k(n)$. Next, obtain a second approximation, $\tilde{\omega}_n^{(2)}$, by applying parabolic interpolation based on the reciprocals of $I_{n,k(n)-1}$, $I_{n,k(n)}$ and $I_{n,k(n)+1}$; if we write y_1, y_2 and y_3 respectively for these reciprocals, this gives

$$\tfrac{1}{2}n\{\tilde{\omega}_n^{(2)} - \tilde{\omega}_n^{(1)}\}/\pi = -\tfrac{1}{2}(y_3-y_1)/(y_3+y_1-2y_2).$$

Finally, starting from $\tilde{\omega}_n^{(2)}$, obtain further approximations by iteration based on Newton's rule in the usual way. For the iteration to converge the difference between the starting value and ω_0 must be of smaller order than n^{-1} in probability; this is the reason $\tilde{\omega}_n^{(2)}$ is used rather than $\tilde{\omega}_n^{(1)}$.

3. Consistency of estimators

We begin the proof of Theorem 1 by showing that, as $n \to \infty$,

$$\hat{\omega}_n - \omega_0 = o_p(n^{-1}). \tag{3·1}$$

This result is much stronger than we need to establish consistency of $\hat{\omega}_n$, but without it, it is not clear how one can obtain consistency of $\hat{A}_n$ and $\hat{B}_n$.

From the definition (2·7) of $I_n(\omega)$, we have

$$\tfrac{1}{2}nI_n(\omega) = \left|\sum_{t=1}^{n} e^{i\omega t}\{A_0 \cos(\omega_0 t) + B_0 \sin(\omega_0 t) + \epsilon_t\}\right|^2. \tag{3·2}$$

Write

$$A_0 \cos(\omega_0 t) + B_0 \sin(\omega_0 t) = D_0 e^{i\omega_0 t} + D_0^* e^{-i\omega_0 t}, \tag{3·3}$$

where

$$D_0 = \tfrac{1}{2}(A_0 - iB_0), \quad D_0^* = \tfrac{1}{2}(A_0 + iB_0).$$

Then from (3·2), if we write

$$M_n(u) = \sum_{t=1}^{n} e^{iut} = \begin{cases} \exp\{\tfrac{1}{2}i(n+1)u\}\sin(\tfrac{1}{2}nu)/\sin(\tfrac{1}{2}u) & (0 < u < 2\pi), \\ n & (u = 0 \text{ or } 2\pi), \end{cases} \tag{3·4}$$

$$\tfrac{1}{2}nI_n(\omega) = \left|\sum_{t=1}^{n} \epsilon_t e^{i\omega t}\right|^2 + 2\mathscr{R}\left[\sum_{t=1}^{n} \epsilon_t e^{-i\omega t}\{D_0 M_n(\omega+\omega_0) + D_0^* M_n(\omega-\omega_0)\}\right] + |D_0 M_n(\omega+\omega_0) + D_0^* M_n(\omega-\omega_0)|^2. \tag{3·5}$$

When $\omega = \omega_0$, (3·5) is dominated by the term

$$|D_0^* M_n(0)|^2 = \tfrac{1}{4}n^2(A_0^2 + B_0^2).$$

In fact, since the real and imaginary parts of $\sum_{t=1}^{n} \epsilon_t e^{i\omega t}$ each have variance $\tfrac{1}{2}nv + O(1)$, so that

$$\sum_{t=1}^{n} \epsilon_t e^{i\omega_0 t} = O_p(n^{\frac{1}{2}}),$$

and $M_n(2\omega_0) = O(1)$, we see that

$$\tfrac{1}{2}nI_n(\omega_0) = \tfrac{1}{4}n^2(A_0^2 + B_0^2) + O_p(n^{\frac{3}{2}}),$$

or

$$I_n(\omega_0) = \tfrac{1}{2}n(A_0^2 + B_0^2) + O_p(n^{\frac{1}{2}}). \tag{3·6}$$

We now obtain an estimate of

$$\max_{|\omega-\omega_0| \geq n^{-1}\delta} \{I_n(\omega)\} = K(n, \delta), \tag{3·7}$$

say, where δ can be arbitrarily small. For this we require the result

$$\max_{0 \leq \omega \leq \pi} \left(\left|\sum_{t=1}^{n} \epsilon_t e^{i\omega t}\right|^2\right) = O_p(n^{\frac{3}{2}}). \tag{3·8}$$

To establish (3·8), we note that

$$\left|\sum_{t=1}^{n} \epsilon_t e^{i\omega t}\right|^2 = \sum_{|s| \leq n-1} e^{i\omega s} \sum_{t=1}^{n-|s|} \epsilon_t \epsilon_{t+|s|}$$

$$\leq \sum_{|s| \leq n-1} \left|\sum_{t=1}^{n-|s|} \epsilon_t \epsilon_{t+|s|}\right|,$$

whose expectation does not exceed

$$E\left(\sum_{t=1}^{n} \epsilon_t^2\right) + 2\sum_{s=1}^{n-1}\left\{E\left(\sum_{t=1}^{n-|s|} \epsilon_t \epsilon_{t+s}\right)^2\right\}^{\frac{1}{2}} = v\left\{n + 2\sum_{s=1}^{n-1}(n-s)^{\frac{1}{2}}\right\} < 2vn^{\frac{3}{2}};$$

compare Walker [1965, p. 112, (29)].

Using (3·8) and

$$\max_{0\leqslant\omega\leqslant\pi}\{|M_n(\omega+\omega_0)|\} = O(1), \quad \max_{0\leqslant\omega\leqslant\pi}\{|M_n(\omega-\omega_0)|\} = n,$$

we see from (3·5) that

$$\max_{0\leqslant\omega\leqslant\pi}\{|\tfrac{1}{2}nI_n(\omega) - |D_0^* M_n(\omega-\omega_0)|^2|\} = O_p(n^{\frac{3}{2}}) + O_p(n^{\frac{7}{4}}) + O(n),$$

which gives

$$\max_{0\leqslant\omega\leqslant\pi}\{|I_n(\omega) - \tfrac{1}{2}n^{-1}(A_0^2+B_0^2)\,|M_n(\omega-\omega_0)|^2|\} = O_p(n^{\frac{3}{4}}). \tag{3·9}$$

If we were to add the assumption that $E(|\epsilon_t|^r) < \infty$ for some $r > 4$, we could use a much more powerful, but by no means elementary result of Whittle [1959, p. 180, (44)], according to which the factor $n^{\frac{3}{2}}$ in (3·8) can be reduced to $n\log n$, so that (3·9) becomes $O_p\{(n\log n)^{\frac{1}{2}}\}$, but it turns out that this is not necessary.

Now the function $|M_n(u)|^2 = \sin^2(\frac{1}{2}nu)/\sin^2(\frac{1}{2}u)$ $(0 < u < 2\pi)$ decreases monotonically from its absolute maximum of n^2 at $u = 0$ to a minimum of zero at $u = 2\pi/n$. For the derivative of $\log\{|M_n(u)|^2\}$ is

$$n\cot(\tfrac{1}{2}nu) - \cot(\tfrac{1}{2}u) = (2/u)\{\psi(\tfrac{1}{2}nu) - \psi(\tfrac{1}{2}u)\},$$

where $\psi(x) = x\cot x$, and

$$\psi'(x) = \tfrac{1}{2}\operatorname{cosec}^2 x\{\sin(2x) - 2x\} < 0 \quad (x > 0).$$

Hence for any δ which is sufficiently small to make $\{\sin(\frac{1}{2}\delta)/(\frac{1}{2}\delta)\}^2 > 1/\pi^2$,

$$\max_{|\omega-\omega_0|\geqslant n^{-1}\delta}\{|M_n(\omega-\omega_0)|^2\} = \sin^2(\tfrac{1}{2}\delta)/\sin^2(\tfrac{1}{2}n^{-1}\delta) \tag{3·10}$$

when n is sufficiently large, since further local maxima of this function must be less than $\operatorname{cosec}^2(\pi/n)$, and

$$\lim_{n\to\infty}[\sin^2(\tfrac{1}{2}\delta)/\{\sin^2(\tfrac{1}{2}n^{-1}\delta)\operatorname{cosec}^2(\pi/n)\}] = \{\pi\sin(\tfrac{1}{2}\delta)/(\tfrac{1}{2}\delta)\}^2.$$

It follows from (3·9) that

$$K(n,\delta) \leqslant \tfrac{1}{2}n^{-1}(A_0^2+B_0^2)\sin^2(\tfrac{1}{2}\delta)/\sin^2(\tfrac{1}{2}n^{-1}\delta) + O_p(n^{\frac{3}{4}})$$

if $n > n_0(\delta)$, say. Therefore, with probability tending to unity as $n\to\infty$,

$$\begin{aligned} n^{-1}K(n,\delta) &\leqslant \tfrac{1}{2}(A_0^2+B_0^2)\sin^2(\tfrac{1}{2}\delta)\lim_{n\to\infty}[\{n^2\sin^2(\tfrac{1}{2}n^{-1}\delta)\}^{-1}] \\ &< \tfrac{1}{2}(A_0^2+B_0^2) \\ &= p\lim_{n\to\infty}\{n^{-1}I_n(\omega_0)\}, \end{aligned}$$

from (3·6). This implies that

$$\lim_{n\to\infty}[\operatorname{pr}\{K(n,\delta) < I_n(\omega_0)\}] = 1,$$

and, therefore,

$$\lim_{n\to\infty}\{\operatorname{pr}(n|\hat{\omega}_n - \omega_0| < \delta)\} = 1. \tag{3·11}$$

Since δ can be arbitrarily small, (3·11) is equivalent to (3·1).

We now establish that

$$p\lim_{n\to\infty} \hat{A}_n = A_0, \quad p\lim_{n\to\infty} \hat{B}_n = B_0. \tag{3·12}$$

From the definition (2·5) of $\hat{A}_n$ and $\hat{B}_n$,

$$\hat{A}_n + i\hat{B}_n = \frac{2}{n}\sum_{t=1}^{n} (D_0 e^{i\omega_0 t} + D_0^* e^{-i\omega_0 t} + \epsilon_t) e^{i\hat{\omega}_n t},$$

and so

$$\hat{A}_n - A_0 + i(\hat{B}_n - B_0) = \frac{2}{n}\left[D_0 M_n(\hat{\omega}_n + \omega_0) + D_0^*\{M_n(\hat{\omega}_n - \omega_0) - n\} + \sum_{t=1}^{n} \epsilon_t e^{i\hat{\omega}_n t}\right].$$

Thus

$$|\hat{A}_n - A_0 + i(\hat{B}_n - B_0)| \leqslant \frac{2|D_0|}{n}\{|M_n(\hat{\omega}_n + \omega_0)| + |M_n(\hat{\omega}_n - \omega_0) - n|\} + \frac{2}{n}\left|\sum_{t=1}^{n} \epsilon_t e^{i\hat{\omega}_n t}\right|. \tag{3·13}$$

Now consistency of $\hat{\omega}_n$ clearly gives

$$p\lim_{n\to\infty}\{n^{-1}|M_n(\hat{\omega}_n + \omega_0)|\} = 0. \tag{3·14}$$

Also, applying the mean value theorem to the real and imaginary parts of

$$M_n(\hat{\omega}_n - \omega_0) - n = M_n(\hat{\omega}_n - \omega_0) - M_n(0),$$

and using $|M_n'(\omega)| = \left|\sum_{t=1}^{n} t\, e^{i\omega t}\right| < n^2$, for all ω, we see that

$$|n^{-1}\{M_n(\hat{\omega}_n - \omega_0) - n\}| < 2^{\frac{1}{2}} n |\hat{\omega}_n - \omega_0|.$$

Hence, from (3·1),

$$p\lim_{n\to\infty}\{n^{-1}|M_n(\hat{\omega}_n - \omega_0) - n|\} = 0. \tag{3·15}$$

Finally, from (3·8), $\left|\sum_{t=1}^{n} \epsilon_t e^{i\hat{\omega}_n t}\right| = O_p(n^{\frac{3}{4}})$, which gives

$$p\lim_{n\to\infty}\left(\frac{2}{n}\left|\sum_{t=1}^{n} \epsilon_t e^{i\hat{\omega}_n t}\right|\right) = 0. \tag{3·16}$$

From (3·13)–(3·16) we thus obtain $p\lim_{n\to\infty} |\hat{A}_n - A_0 + i(\hat{B}_n - B_0)| = 0$, which is equivalent to (3·12).

Lastly, from (2·8) we have

$$\hat{v}_n = n^{-1}\sum_{t=1}^{n} X_t^2 - \tfrac{1}{2}(\hat{A}_n^2 + \hat{B}_n^2), \tag{3·17}$$

since

$$I_n(\hat{\omega}_n) = \tfrac{1}{2} n(\hat{A}_n^2 + \hat{B}_n^2).$$

Substituting $X_t = A_0 \cos(\omega_0 t) + B_0 \sin(\omega_0 t) + \epsilon_t$ in (3·17), we see that

$$\hat{v}_n = n^{-1}\sum_{t=1}^{n} \epsilon_t^2 + 2n^{-1}\left\{A_0 \sum_{t=1}^{n} \epsilon_t \cos(\omega_0 t) + B_0 \sum_{t=1}^{n} \epsilon_t \sin(\omega_0 t)\right\} + \tfrac{1}{2}(A_0^2 + B_0^2 - \hat{A}_n^2 - \hat{B}_n^2) + o_p(1). \tag{3·18}$$

In (3·18) the first term converges in probability to v by the weak law of large numbers, and the second and third terms clearly converge in probability to zero. Hence v_n is consistent.

4. Asymptotic normality of the estimators

Since $U_n(A, B, \omega)$ defined by (2·4) is a minimum when $A = \hat{A}_n$, $B = \hat{B}_n$ and $\omega = \hat{\omega}_n$, an application of the mean value theorem gives

$$(U_n)_{A_0} = (U_n)_{A_n^* A_n^*}(A_0 - \hat{A}_n) + (U_n)_{A_n^* B_n^*}(B_0 - \hat{B}_n) + (U_n)_{A_n^* \omega_n^*}(\omega_0 - \hat{\omega}_n), \tag{4·1}$$

$$(U_n)_{B_0} = (U_n)_{A_n^* B_n^*}(A_0 - \hat{A}_n) + (U_n)_{B_n^* B_n^*}(B_0 - \hat{B}_n) + (U_n)_{B_n^* \omega_n^*}(\omega_0 - \hat{\omega}_n), \tag{4·2}$$

$$(U_n)_{\omega_0} = (U_n)_{A_n^* \omega_n^*}(A_0 - \hat{A}_n) + (U_n)_{B_n^* \omega_n^*}(B_0 - \hat{B}_n) + (U_n)_{\omega_n^* \omega_n^*}(\omega_0 - \hat{\omega}_n), \tag{4·3}$$

where $(U_n)_{A_0}$ denotes $\partial U_n(A_0, B_0, \omega_0)/\partial A_0$, $(U_n)_{A_n^* B_n^*}$ denotes $\partial U_n(A_n^*, B_n^*, \omega_n^*)/(\partial A_n^* \partial B_n^*)$, etc., and we use the generic notation $(A_n^*, B_n^*, \omega_n^*)$ for a point on the line joining (A_0, B_0, ω_0) and $(\hat{A}_n, \hat{B}_n, \hat{\omega}_n)$, so that

$$(A_n^*, B_n^*, \omega_n^*) = \lambda(A_0, B_0, \omega_0) + (1-\lambda)(\hat{A}_n, \hat{B}_n, \hat{\omega}_n) \quad (0 < \lambda < 1). \tag{4·4}$$

The points $(A_n^*, B_n^*, \omega_n^*)$ in (4·1), (4·2) and (4·3) will in general not be the same, but to distinguish them would complicate the notation, and no ambiguity will arise by not doing so.

Now

$$\begin{aligned}(U_n)_{A_0} &= nA_0 - 2\sum_{t=1}^{n} X_t \cos(\omega_0 t) \\ &= nA_0 - 2\sum_{t=1}^{n} \{A_0 \cos^2(\omega_0 t) + B_0 \sin(\omega_0 t)\cos(\omega_0 t) + \epsilon_t \cos(\omega_0 t)\} \\ &= -2\sum_{t=1}^{n} \epsilon_t \cos(\omega_0 t) + O_p(1),\end{aligned} \tag{4·5}$$

and similarly

$$(U_n)_{B_0} = -2\sum_{t=1}^{n} \epsilon_t \sin(\omega_0 t) + O_p(1), \tag{4·6}$$

$$(U_n)_{\omega_0} = 2\sum_{t=1}^{n} \epsilon_t t\{A_0 \sin(\omega_0 t) - B_0 \cos(\omega_0 t)\} + O_p(n). \tag{4·7}$$

The sums in (4·5)–(4·7) are all of the form $\sum_{t=1}^{n} k_t \epsilon_t$, where

$$\lim_{n\to\infty}\left[\max_{1\leqslant t\leqslant n}\left\{|k_t| \Big/ \left(\sum_{t=1}^{n} k_t^2\right)^{\frac{1}{2}}\right\}\right] = 0. \tag{4·8}$$

For example, with (4·7),

$$\begin{aligned}\sum_{t=1}^{n} k_t^2 &= 4\sum_{t=1}^{n} t^2 [\tfrac{1}{2}A_0^2\{1 - \cos(2\omega_0 t)\} + \tfrac{1}{2}B_0^2\{1 + \cos(2\omega_0 t)\} - A_0 B_0 \sin(2\omega_0 t)] \\ &= 2(A_0^2 + B_0^2)\sum_{t=1}^{n} t^2 + O(n^2) \\ &= \tfrac{2}{3}n^3(A_0^2 + B_0^2) + O(n^2).\end{aligned}$$

It follows that the central limit theorem can be applied to these. For (4·8) implies the Lindeberg–Feller condition

$$\lim_{n\to\infty}\left\{\sum_{t=1}^{n}\int_{|\eta|>\delta\sigma_n} \eta^2\, dG_t(\eta)/\sigma_n^2\right\} = 0, \tag{4·9}$$

where G_t is the distribution function of $\eta_t = k_t \epsilon_t$ and $\sigma_n = \left(v \sum_{t=1}^{n} k_t^2\right)^{\frac{1}{2}}$ (see, for example, Rao, 1965, p. 108), since the integral on the left hand side of (4·9) is equal to

$$\frac{1}{\sigma_n^2} \sum_{t=1}^{n} k_t^2 \int_{|\epsilon| > \delta\sigma_n/|k_t|} \epsilon^2 \, dF(\epsilon),$$

where F is the distribution function of ϵ_t, and therefore does not exceed

$$v^{-1} \int_{|\epsilon| > \delta\sigma_n/\max|k_t|} \epsilon^2 \, dF(\epsilon).$$

We, therefore, see that $n^{-\frac{1}{2}}(U_n)_{A_0}$, $n^{-\frac{1}{2}}(U_n)_{B_0}$ and $n^{-\frac{3}{2}}(U_n)_{\omega_0}$ converge in law respectively to $N(0, 2v)$, $N(0, 2v)$ and $N\{0, \frac{2}{3}(A_0^2 + B_0^2) v\}$ when $n \to \infty$.

For the limiting joint distribution we consider the random variable

$$V_n(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 n^{-\frac{1}{2}}(U_n)_{A_0} + \lambda_2 n^{-\frac{1}{2}}(U_n)_{B_0} + \lambda_3 n^{-\frac{3}{2}}(U_n)_{\omega_0},$$

where the λ_i ($i = 1, 2, 3$) are arbitrary real numbers. Now

$$V_n(\lambda_1, \lambda_2, \lambda_3) = 2 \sum_{t=1}^{n} \epsilon_t [\lambda_3 n^{-\frac{3}{2}} t \{A_0 \sin(\omega_0 t) - B_0 \cos(\omega_0 t)\} - n^{-\frac{1}{2}} \{\lambda_1 \cos(\omega_0 t) + \lambda_2 \sin(\omega_0 t)\}] + O_p(n^{-\frac{1}{2}}) \quad (4{\cdot}10)$$

and the sum in (4·10) is of the form $\sum_{t=1}^{n} k_{n,t} \epsilon_t$, where

$$\lim_{n\to\infty} \left[\max_{1 \leqslant t \leqslant n} \left\{ |k_{n,t}| \Big/ \left(\sum_{t=1}^{n} k_{n,t}^2 \right)^{\frac{1}{2}} \right\} \right] = 0, \quad (4{\cdot}11)$$

the numerator in (4·11) being $O(n^{-\frac{1}{2}})$ and the denominator $O(1)$. Hence the central limit theorem will apply to this sum also by the generalized Lindeberg–Feller condition (see, for example, Loève, 1963, p. 295), which is implied by (4·8). A straightforward calculation shows that

$$\lim_{n\to\infty} \left(\sum_{t=1}^{n} k_{n,t}^2 \right) = 2(\lambda_1^2 + \lambda_2^2) + \tfrac{2}{3}(A_0^2 + B_0^2)\lambda_3^2 + 2B_0\lambda_1\lambda_3 - 2A_0\lambda_2\lambda_3. \quad (4{\cdot}12)$$

Thus $V_n(\lambda_1, \lambda_2, \lambda_3)$ converges in law to a normal distribution with mean zero and variance (4·12). Consequently, by using the equivalence of convergence in law and pointwise convergence of characteristic functions (see, for example, Rao, 1965, p. 103), we see that the joint distribution of $n^{-\frac{1}{2}}(U_n)_{A_0}$, $n^{-\frac{1}{2}}(U_n)_{B_0}$ and $n^{-\frac{3}{2}}(U_n)_{\omega_0}$ converges to that of $N\{(0,0,0), 2v\mathbf{W}_0\}$, where $\mathbf{W}_0$ is given by (2·9).

Next, we look at the behaviour of the second order partial derivatives occurring on the right hand sides of (4·1)–(4·3). For three of these no analysis is required, as

$$(U_n)_{AA} = (U_n)_{BB} = n, \quad (U_n)_{AB} = 0. \quad (4{\cdot}13)$$

Now $(U_n)_{A\omega} = 2 \sum_{t=1}^{n} X_t t \sin(\omega t)$. Therefore,

$$\begin{aligned}(U_n)_{A_n^* \omega_n^*} &= 2 \sum_{t=1}^{n} \{A_0 \cos(\omega_0 t) + B_0 \sin(\omega_0 t) + \epsilon_t\} t \sin(\omega_n^* t) \\ &= B_0 \sum_{t=1}^{n} t \cos\{(\omega_n^* - \omega_0) t\} + A_0 t \sin\{(\omega_n^* - \omega_0) t\} + 2 \sum_{t=1}^{n} \epsilon_t t \sin(\omega_n^* t) + O_p(n). \quad (4{\cdot}14)\end{aligned}$$

From (3·1) and (4·4), we have $\omega_n^* - \omega_0 = o_p(n^{-1})$.

Now by applying the mean value theorem to the real and imaginary parts of

$$M'_n(\omega_n^* - \omega_0) - M'_n(0),$$

and using $|M''_n(\omega)| \leqslant \sum_{t=1}^{n} t^2 \sim \frac{1}{3}n^3$, for all ω, we see that

$$i\Sigma t \exp\{i(\omega_n^* - \omega_0)t\} = \tfrac{1}{2}n(n+1)i + o_p(n^2),$$

so that

$$p\lim_{n\to\infty}\left[n^{-2}\sum_{t=1}^{n} t \exp\{i(\omega_n^* - \omega_0)t\}\right] = \tfrac{1}{2}. \tag{4·15}$$

Also, employing an argument similar to that following equation (3·8) in § 3, we have

$$\left|\sum_{t=1}^{n}\epsilon_t t\, e^{i\omega t}\right|^2 = \sum_{|s|\leqslant n-1} e^{i\omega s}\sum_{t=1}^{n-|s|}\epsilon_t\epsilon_{t+|s|}t(t+|s|)$$

$$\leqslant \sum_{|s|\leqslant n-1}\left|\sum_{t=1}^{n-|s|}\epsilon_t\epsilon_{t+|s|}t(t+|s|)\right|,$$

so that

$$E\left\{\max_{0\leqslant\omega\leqslant\pi}\left(\left|\sum_{t=1}^{n}\epsilon_t t\, e^{i\omega t}\right|^2\right)\right\}$$

$$\leqslant E\left(\sum_{t=1}^{n} t^2\epsilon_t^2\right) + 2\sum_{s=1}^{n-1}\left\{E\left[\left\{\sum_{t=1}^{n-s}\epsilon_t\epsilon_{t+s}t(t+s)\right\}^2\right]\right\}^{\frac{1}{2}}$$

$$= v\left[\tfrac{1}{6}n(n+1)(2n+1) + 2\sum_{s=1}^{n-1}\left\{\sum_{t=1}^{n-s}t^2(t+s)^2\right\}^{\frac{1}{2}}\right]$$

$$< v\{n^3 + 2(n-1)n^{\frac{5}{2}}\}$$

$$< 3vn^{\frac{7}{2}}. \tag{4·16}$$

Expression (4·16) is rather a crude inequality, but it suffices for our purpose here. It follows from (4·16) that

$$\left|\sum_{t=1}^{n}\epsilon_t t \sin(\hat{\omega}_n t)\right| \leqslant \max_{0\leqslant\omega\leqslant\pi}\left(\left|\sum_{t=1}^{n}\epsilon_t t\, e^{i\omega t}\right|^2\right)^{\frac{1}{2}} = O_p(n^{\frac{7}{4}}). \tag{4·17}$$

Hence, from (4·14), (4·15), and (4·17),

$$p\lim_{n\to\infty}\{n^{-2}(U_n)_{A_n^*\omega_n^*}\} = \tfrac{1}{2}B_0. \tag{4·18}$$

Similarly, since

$$(U_n)_{B_n^*\omega_n^*} = -2\sum_{t=1}^{n}\{A_0\cos(\omega_0 t) + B_0\sin(\omega_0 t) + \epsilon_t\}\, t\cos(\omega_n^* t)$$

$$= -A_0\sum_{t=1}^{n} t\cos\{(\omega_n^* - \omega_0)t\} - 2\sum_{t=1}^{n}\epsilon_t t\cos(\omega_n^* t) + O_p(n),$$

we obtain

$$p\lim_{n\to\infty}\{n^{-2}(U_n)_{B_n^*\omega_n^*}\} = -\tfrac{1}{2}A_0. \tag{4·19}$$

Finally,

$$(U_n)_{\omega_n^*\omega_n^*} = 2\sum_{t=1}^{n} X_t t^2\{A_n^*\cos\omega_n^* t + B_n^*\sin(\omega_n^* t)\}$$

$$= 2\sum_{t=1}^{n} t^2\{A_0\cos(\omega_0 t) + B_0\sin(\omega_0 t)\}\{A_n^*\cos(\omega_n^* t) + B_n^*\sin(\omega_n^* t)\}$$

$$+ 2A_n^*\sum_{t=1}^{n}\epsilon_t t^2\cos(\omega_n^* t) + 2B_n^*\sum_{t=1}^{n}\epsilon_t t^2\sin(\omega_n^* t). \tag{4·20}$$

We can show that

$$E\left\{\max_{0\leqslant\omega\leqslant\pi}\left(\left|\sum_{t=1}^{n}\epsilon_t t^2 e^{i\omega t}\right|^2\right)\right\} = O(n^{\frac{11}{2}})$$

in the same way as we obtained (4·16), and hence, (4·4) giving consistency of A_n^* and B_n^*, that the last two terms of (4·20) are each $O_p(n^{\frac{11}{4}})$. Also the first term of (4·20) is equal to

$$\sum_{t=1}^{n} t^2(A_0A_n^*+B_0B_n^*)\cos\{(\omega_n^*-\omega_0)t\}+O_p(n^2) = \tfrac{1}{3}n^3(A_0^2+B_0^2)+o_p(n^3) \qquad (4·21)$$

if we apply the mean value theorem to the real part of

$$-\sum_{t=1}^{n} t^2\exp\{i(\omega_n^*-\omega_0)t\} = M_n''(\omega_n^*-\omega_0)-M_n''(0).$$

Hence

$$p\lim_{n\to\infty}\{n^{-3}(U_n)_{\omega_n^*\omega_n^*}\} = \tfrac{1}{3}(A_0^2+B_0^2). \qquad (4·22)$$

Thus if

$$\mathbf{W}_n^* = \begin{bmatrix} n^{-1}(U_n)_{A_n^*A_n^*} & n^{-1}(U_n)_{A_n^*B_n^*} & n^{-2}(U_n)_{A_n^*\omega_n^*} \\ n^{-1}(U_n)_{A_n^*B_n^*} & n^{-1}(U_n)_{B_n^*B_n^*} & n^{-2}(U_n)_{B_n^*\omega_n^*} \\ n^{-2}(U_n)_{A_n^*\omega_n^*} & n^{-2}(U_n)_{B_n^*\omega_n^*} & n^{-3}(U_n)_{\omega_n^*\omega_n^*} \end{bmatrix} \qquad (4·23)$$

we have, from (4·13), (4·18), (4·19) and (4·22),

$$p\lim_{n\to\infty}\mathbf{W}_n^* = \mathbf{W}_0. \qquad (4·24)$$

Now from (4·1) to (4·3), we see that

$$\{n^{-\frac{1}{2}}(U_n)_{A_0}, n^{-\frac{1}{2}}(U_n)_{B_0}, n^{-\frac{3}{2}}(U_n)_{\omega_0}\} = -\{n^{\frac{1}{2}}(\hat{A}_n-A_0), n^{\frac{1}{2}}(\hat{B}_n-B_0), n^{\frac{3}{2}}(\hat{\omega}_n-\omega_0)\}\mathbf{W}_n^*,$$

so that, assuming $\mathbf{W}_n^*$ to be nonsingular, which will be true with probability tending to unity as $n\to\infty$ by (4·24), since $\mathbf{W}_0$ is easily verified to be nonsingular, we have

$$\{n^{\frac{1}{2}}(\hat{A}_n-A_0), n^{\frac{1}{2}}(\hat{B}_n-B_0), n^{\frac{3}{2}}(\hat{\omega}_n-\omega_0)\} = -\{n^{\frac{1}{2}}(U_n)_{A_0}, n^{\frac{1}{2}}(U_n)_{B_0}, n^{\frac{3}{2}}(U_n)_{\omega_0}\}\mathbf{W}_n^{*-1}. \quad (4·25)$$

We showed above that the row vector on the right hand side of (4·24) converges in law to $N(\mathbf{0}, 2v\mathbf{W}_0)$ when $n\to\infty$, where $\mathbf{0} = (0,0,0)$. It will therefore follow that the row vector on the left hand side converges in law to $N(\mathbf{0}, 2v\mathbf{W}_0^{-1})$, by using an obvious generalization of a well known elementary limit theorem [see, for example, Rao, 1965, p. 102, (x*b*)], namely that if $\mathbf{Y}_n$ and $\mathbf{Y}$ are row vector-valued and $\mathbf{Z}_n$ matrix valued random variables such that when $n\to\infty$, $\mathbf{Y}_n\to\mathbf{Y}$ in law and $p\lim\mathbf{Z}_n = \mathbf{C}$, then $\mathbf{Y}_n\mathbf{Z}_n\to\mathbf{YC}$ in law. We have thus established Theorem 2.

5. The case of several harmonic components

Suppose now that the model in Theorem 1 is generalized to

$$X_t = \sum_{r=1}^{q}\{A_{r,0}\cos(\omega_{r,0}t)+B_{r,0}\sin(\omega_{r,0}t)\}+\epsilon_t, \qquad (5·1)$$

where $q>1$. The function corresponding to (2·4) whose minimization yields estimators $\hat{A}_{r,n}$, $\hat{B}_{r,n}$ and $\hat{\omega}_{r,n}$ $(1\leqslant r\leqslant q)$ becomes

$$\sum_{t=1}^{n}X_t^2-2\sum_{r=1}^{q}\sum_{t=1}^{n}X_t\{A_r\cos(\omega_r t)+B_r\sin(\omega_r t)\}+\tfrac{1}{2}n\sum_{r=1}^{q}(A_r^2+B_r^2). \qquad (5·2)$$

We, therefore, obtain

$$\hat{A}_{r,n} = \frac{2}{n}\sum_{t=1}^{n} X_t \cos(\hat{\omega}_{r,n}t), \quad \hat{B}_{r,n} = \frac{2}{n}\sum_{t=1}^{n} X_t \sin(\hat{\omega}_{r,n}t). \tag{5·3}$$

If we write $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_q)$ and $\hat{\boldsymbol{\omega}}_n = (\hat{\omega}_{1,n}, \ldots, \hat{\omega}_{q,n})$,

$$\phi_n(\boldsymbol{\omega}) = \sum_{r=1}^{q} I_n(\omega_r) \tag{5·4}$$

is a maximum when $\boldsymbol{\omega} = \hat{\boldsymbol{\omega}}_n$.

Here, however, since to obtain (5·2) from the residual sum of squares

$$\sum_{t=1}^{n}\left[X_t - \sum_{r=1}^{q}\{A_r\cos(\omega_r t) + B_r \sin(\omega_r t)\}\right]^2,$$

terms of the form $A_r A_s \Sigma_t \cos(\omega_r t)\cos(\omega_s t)$ and $B_r B_s \Sigma_t \sin(\omega_r t)\sin(\omega_s t)$ have to be omitted, the maximization of (5·4) cannot be unrestricted; some condition must be imposed to keep the ω_r from being too close together and thus prevent estimators of two angular frequencies from converging in probability to the same value. In fact unrestricted maximization obviously makes the $\hat{\omega}_{r,n}$ all equal to the angular frequency for which I_n attains its absolute maximum, and it is quite easy to see that this will converge in probability to that ω_r for which the corresponding amplitude $(A_r^2 + B_r^2)^{\frac{1}{2}}$ is greatest. The required condition is

$$\lim_{n\to\infty} \min_{1 \leqslant r \neq s \leqslant q} (n|\omega_r - \omega_s|) = \infty. \tag{5·5}$$

We might, therefore, for example, maximize (5·4) subject to

$$\min_{r \neq s}(|\omega_r - \omega_s|) = n^{-\frac{1}{2}}. \tag{5·6}$$

When (5·5) holds, then in the relevant domain, S_n say, of the function ϕ_n in ω space, only q of the q^2 differences $\omega_s - \omega_{r,0}$ can be $O(n^{-1})$. If we label the components of $\boldsymbol{\omega}$ so that these differences are $\omega_r - \omega_{r,0}$, we see that the behaviour of

$$\tfrac{1}{2}n\phi_n(\omega) = \sum_{r=1}^{q}\left|\sum_{s=1}^{q}\{D_{s,0}M_n(\omega_r + \omega_{s,0}) + D_{s,0}^{*}\, M_n(\omega_r - \omega_{s,0})\} + \sum_{s=1}^{n} \epsilon_t e^{i\omega_r t}\right|^2,$$

where $D_{s,0} = \frac{1}{2}(A_{s,0} - iB_{s,0})$, $D_{s,0}^{*} = \frac{1}{2}(A_{s,0} + iB_{s,0})$, $A_{s,0}$ and $B_{s,0}$ denote the true values of A_s and B_s, are controlled by the sum of terms $|D_{r,0}^{*}M_n(\omega_r - \omega_{r,0})|^2$ when $\omega_r - \omega_{r,0}$ $(1 \leqslant r \leqslant q)$ are small. In fact, we can show, just as in § 3, that if we take a sequence of sets $\{S_n\}$ in $\boldsymbol{\omega}$ space for which (5·5) holds, then

$$\max_{\boldsymbol{\omega}\in S_n}\left\{\left|\phi_n(\boldsymbol{\omega}) - \tfrac{1}{2}n^{-1}\sum_{r=1}^{q}(A_{r,0}^2 + B_{r,0}^2)\,|M_n(\omega_r - \omega_{r,0})|^2\right|\right\} = O_p(n^{\frac{3}{4}}).$$

Now let $R_{n,\delta} = \{\boldsymbol{\omega}\colon |\omega_r - \omega_{r,0}| < n^{-1}\delta\ (1 \leqslant r \leqslant q)\}$, which will certainly be contained in S_n for sufficiently large n, and $R_{n,\delta}^{(c)} = S_n - R_{n,\delta}$, the complement of $R_{n,\delta}$ with respect to S_n. Then for sufficiently small δ,

$$p\lim_{n\to\infty}[n^{-1}\sup_{\boldsymbol{\omega}\in R_{n,\delta}^{(c)}}\{\phi_n(\boldsymbol{\omega})\}] < \frac{1}{2}\sum_{r=1}^{q}(A_{r,0}^2 + B_{r,0}^2) = p\lim_{n\to\infty} n^{-1}\phi_n(\boldsymbol{\omega}_0), \tag{5·7}$$

where $\boldsymbol{\omega}_0 = (\omega_{1,0}, \ldots, \omega_{q,0})$. We, therefore, have

$$p\lim_{n\to\infty}\{n(\hat{\omega}_{r,n} - \omega_{r\,0})\} = 0. \tag{5·8}$$

Since ϕ_n is symmetrical in its q arguments, a means of determining which component of $\boldsymbol{\omega}$ is associated with a particular angular frequency has to be found. We can, however, obtain this from the fact that

$$p \lim_{n\to\infty} \{n^{-1} I_n(\hat{\omega}_{r,n})\} = \tfrac{1}{2}(A_{r,0}^2 + B_{r,0}^2), \tag{5·9}$$

which is fairly readily demonstrated by using Taylor's theorem and (5·8). If, therefore, the $\omega_{r,0}$ are labelled so that

$$A_{1,0}^2 + B_{1,0}^2 \geqslant \ldots \geqslant A_{q,0}^2 + B_{q,0}^2,$$

then with probability tending to unity as $n \to \infty$,

$$I_n(\hat{\omega}_{1,n}) \geqslant \ldots \geqslant I_n(\hat{\omega}_{q,n}).$$

Thus if we determine the $\hat{\omega}_{r,n}$ as the q largest local maxima of the periodogram intensity function subject to a separation condition satisfying (5·5), these will, for sufficiently large n, almost certainly estimate the frequencies of the harmonic components arranged in descending order of magnitude. Having established the consistency of the $\hat{\omega}_{r,n}$, we then deduce that

$$p \lim_{n\to\infty} \hat{A}_{r,n} = A_{r,0}, \quad p \lim_{n\to\infty} \hat{B}_{r,n} = B_{r,0}, \quad p \lim_{n\to\infty} \hat{v}_n = v,$$

by arguments of the type used in § 3 following equation (3·12).

Finally, if we denote (5·2) by $U_n(\mathbf{A}, \mathbf{B}, \boldsymbol{\omega})$, where $\mathbf{A} = (A_1, \ldots, A_q)$, $\mathbf{B} = (B_1, \ldots, B_q)$, we see that $U_n(\mathbf{A}, \mathbf{B}, \boldsymbol{\omega})$ is of the form

$$\sum_{t=1}^{n} X_t^2 + \sum_{t=1}^{n} f_{n,r}\{A_r, B_r, \omega_r, \mathbf{X}^{(n)}\}.$$

Therefore applying the mean value theorem as in § 4 gives us q sets of equations each of the form (4·1)–(4·3), whence

$$\begin{aligned}\{n^{\frac{1}{2}}(\hat{A}_{r,n} - A_{r,0}),\ n^{\frac{1}{2}}(\hat{B}_{r,n} - B_{r,0}),\ n^{\frac{3}{2}}(\hat{\omega}_{r,n} - \omega_{r,0})\}\\ = -\{n^{-\frac{1}{2}}(U_n)_{A_{r,0}},\ n^{-\frac{1}{2}}(U_n)_{B_{r,0}},\ n^{-\frac{3}{2}}(U_n)_{\omega_{r,0}}\}\{\mathbf{W}_{r,n}^*\}^{-1},\end{aligned} \tag{5·10}$$

in an obvious notation. Then from results of the type (4·5)–(4·7), for example,

$$(U_n)_{A_{r,0}} = -2 \sum_{t=1}^{n} \epsilon_t \cos(\omega_{r,0} t) + O_p(1),$$

and an application of the central limit theorem as in § 4, it follows that the row vectors on the right hand sides of (5·10) are asymptotically distributed independently as $N(\mathbf{0}, 2v\mathbf{W}_{r,0})$, where the matrix $\mathbf{W}_{r,0}$ is obtained by replacing A_0 and B_0 in the expression (2·9) for $\mathbf{W}_0$ by $A_{r,0}$ and $B_{r,0}$, respectively. Again as in § 4, we can show that

$$p \lim_{n\to\infty} \mathbf{W}_{r,n}^* = \mathbf{W}_{r,0}.$$

Therefore, we reach the conclusion that the row vectors on the left hand sides of (5·10) are asymptotically distributed independently as $N(\mathbf{0}, 2v\mathbf{W}_{r,0}^{-1})$. This is the required generalization of Theorem 2.

In practice the determination of the estimators $\hat{\omega}_{r,n}$ by maximizing $\phi_n(\boldsymbol{\omega})$ subject to a restriction such as (5·6) would be very laborious, and the awkward problem of the appropriate choice of the minimum separation to be used for a particular set of data would also arise. However, it is not hard to see that an asymptotically equivalent procedure, which is much simpler, is obtained as follows. We first determine $\hat{\omega}_{1,n}$ by maximizing $I_n(\omega)$ unconditionally, then $\hat{\omega}_{2\,n}$ by maximizing unconditionally

$$\left|\sum_{t=1}^{n} \{X_t - \hat{A}_{1,n}\cos(\hat{\omega}_{1,n}t) - \hat{B}_{1,n}\sin(\hat{\omega}_{1.n}t)\}\, e^{i\omega t}\right|^2, \tag{5·11}$$

where $\hat{A}_{1,n}$ and $\hat{B}_{1,n}$ are defined by (5·3), and so on, $\hat{\omega}_{q,n}$ being finally determined by maximizing unconditionally

$$\left|\sum_{t=1}^{n}\left[X_t - \sum_{r=1}^{q-1}\{\hat{A}_{r,n}\cos(\hat{\omega}_{r,n}t) + \hat{B}_{r,n}\sin(\hat{\omega}_{r,n}t)\}\right] e^{i\omega t}\right|^2. \tag{5·12}$$

Here one must strictly introduce the additional assumption that the $A^2_{r,0}+B^2_{r,0}$ $(1 \leqslant r \leqslant q)$ are all unequal, but this is clearly a very mild restriction.

Verification of the asymptotic equivalence is somewhat tedious although straightforward, so we omit the details. Essentially what one does is as follows. One first shows that $\hat{\omega}_{1,n} - \hat{\omega}_{1,0} = o_p(n^{-1})$, using

$$\max_{0 \leqslant \omega \leqslant \pi}\left\{\left|I_n(\omega) - \tfrac{1}{2}n^{-1}\sum_{r=1}^{q}(A^2_{r,0}+B^2_{r,0})|M_n(\omega-\omega_{r,0})|^2\right|\right\} = O_p(n^{\frac{3}{4}}), \tag{5·13}$$

and hence that $p\lim_{n\to\infty}(\hat{A}_{1,n}) = A_{1,0}$ and $p\lim_{n\to\infty}(\hat{B}_{1,n}) = B_{1,0}$, arguing as in § 3; then that the terms $A_{r,0}\cos(\omega_{r,0}t) + B_{r,0}\sin(\omega_{r,0}t)$ in $E(X_t)$ for $r \geqslant 2$ have no effect on the limiting behaviour of the first order partial derivatives of $U_n(\mathbf{A}, \mathbf{B}, \boldsymbol{\omega})$ with respect to A_1, B_1 and ω_1 at $(A_1, B_1, \omega_1) = (A_{1,0}, B_{1,0}, \omega_{1,0})$ and the second order partial derivatives at

$$(A_1, B_1, \omega_1) = (A^*_{1,0}, B^*_{1,0}, \omega^*_{1,0}).$$

Further an argument analogous to that used in § 4 yields the same asymptotic distribution for $(\hat{A}_{1,n}, \hat{B}_{1\,n}, \hat{\omega}_{1,n})$ as was found above. Next, using the fact that what may be called the periodogram function corrected for the effect of the first harmonic component, namely (5·11) multiplied by $2/n$, differs from the function

$$\tfrac{1}{2}n^{-1}\sum_{r=2}^{q}(A^2_{r,0}+B^2_{r,0})\,|M_n(\omega-\omega_{r,0})|^2$$

by a quantity which is at most $O_p(n^{\frac{3}{4}})$, which may be proved in the same way as (5·13), one obtains $\hat{\omega}_{2,n} - \omega_{2,0} = o_p(n^{-1})$, $p\lim_{n\to\infty}\hat{A}_{2,n} = A_{2,0}$ and $p\lim_{n\to\infty}\hat{B}_{2,n} = B_{2,0}$. Then, if the function $U_n(\mathbf{A}, \mathbf{B}, \boldsymbol{\omega})$ is modified by replacing X_t in (5·2) by

$$X_t - \hat{A}_{1,n}\cos(\hat{\omega}_{1,n}t) - \hat{B}_{1,n}\sin(\hat{\omega}_{1,n}t), \tag{5·14}$$

one shows that the limiting behaviour of the first order partial derivatives with respect to A_2, B_2 and ω_2 at $(A_2, B_2, \omega_2) = (A_{2,0}, B_{2,0}, \omega_{2,0})$ and the second order partial derivatives at $(A_2, B_2, \omega_2) = (A^*_{2,0}, B^*_{2,0}, \omega^*_{2,0})$ is unaffected by the replacement of the last two terms in (5·14) by $-A_{1,0}\cos(\omega_{1,0}t)$ and $-B_{1,0}\sin(\omega_{1,0}t)$; the preceding argument then yields the asymptotic distribution for $(\hat{A}_{2,n}, \hat{B}_{2,n}, \hat{\omega}_{2,n})$. One continues similarly, finally dealing with the expression (5·12), proportional to the periodogram function corrected for the effect of the first $q-1$ harmonic components.

6. Effect of correcting for the series mean

It will suffice to illustrate this for the case of a single harmonic component. Suppose that

$$X_t = C_0 + A_0 \cos(\omega_0 t) + B_0 \sin(\omega_0 t) + \epsilon_t, \tag{6·1}$$

and let $\tilde{I}_n(\omega)$, $\tilde{U}_n(A, B, \omega)$ be the expressions obtained from $I_n(\omega)$, $U_n(A, B, \omega)$ defined by (2·7) and (2·4), by replacing X_t by $X_t - \bar{X}$, where $\bar{X} = n^{-1} \sum_{t=1}^{n} X_t$. Then

$$\tfrac{1}{2}n\{\tilde{I}_n(\omega) - I_n(\omega)\} = -2\bar{X}\mathscr{R}\left\{M_n(\omega) \sum_{t=1}^{n} X_t e^{-i\omega t}\right\} + \{\bar{X}|M_n(\omega)|\}^2, \tag{6·2}$$

$$\tilde{U}_n(A, B, \omega) - U_n(A, B, \omega) = -n\bar{X}^2 + 2\bar{X} \sum_{t=1}^{n} \{A \cos(\omega t) + B \sin(\omega t)\}. \tag{6·3}$$

We may take $C_0 = 0$ without loss of generality, since $I_n(.)$ and $U_n(.)$ are invariant under a change of origin. Clearly

$$\bar{X} = n^{-1}\{D_0 M_n(\omega_0) + D_0^* M_n(-\omega_0)\} + n^{-1} \sum_{t=1}^{n} \epsilon_t = O_p(n^{-\frac{1}{2}}). \tag{6·4}$$

Also from (3·9), we have

$$\max_{0 \leqslant \omega \leqslant \pi} \left(\left| \sum_{t=1}^{n} X_t e^{-i\omega t} \right|\right) = O_p(n).$$

Hence, from (6·2), we see that

$$\tilde{I}_n(\omega) - I_n(\omega) = O_p(n^{\frac{1}{2}}). \tag{6·5}$$

Thus (3·9) will still hold if $I_n(\omega)$ is replaced by $\tilde{I}_n(\omega)$. The argument of § 3 then gives us (3·1), that is, $\hat{\omega}_n - \omega_0 = o_p(n^{-1})$. Also we have now

$$\hat{A}_n + i\hat{B}_n = \frac{2}{n} \sum_{t=1}^{n} (D_0 e^{i\omega_0 t} + D_0^* e^{-i\omega_0 t} + \epsilon_t) e^{i\hat{\omega}_n t} - \frac{2}{n} \bar{X} M_n(\hat{\omega}_n). \tag{6·6}$$

The last term on the right hand side of (6·6) is clearly at most $O_p(n^{-\frac{1}{2}})$ and so, as in § 3, we find that $\hat{A}_n$ and $\hat{B}_n$ are consistent. The estimator of v is

$$\hat{v}_n = n^{-1} \sum_{t=1}^{n} X_t^2 - \bar{X}^2 - \tfrac{1}{2}(\hat{A}_n^2 + \hat{B}_n^2),$$

and it is easy to see that this also is consistent.

Again from (6·3) we have

$$(\tilde{U}_n)_{A_0} - (U_n)_{A_0} + i\{(\tilde{U}_n)_{B_0} - (U_n)_{B_0}\} = 2\bar{X} M_n(\omega_0) = O_p(n^{-\frac{1}{2}}), \tag{6·7}$$

$$(\tilde{U}_n)_{\omega_0} - (U_n)_{\omega_0} = 2\bar{X}\{D_0 M_n'(\omega_0) + D_0^* M_n'(-\omega_0)\} = O_p(n^{\frac{1}{2}}). \tag{6·8}$$

It follows that $\{n^{-\frac{1}{2}}(\tilde{U}_n)_{A_0}, n^{-\frac{1}{2}}(\tilde{U}_n)_{B_0}, n^{-\frac{3}{2}}(\tilde{U}_n)_{\omega_0}\}$ has the same limiting distribution as $\{n^{-\frac{1}{2}}(U_n)_{A_0}, n^{-\frac{1}{2}}(U_n)_{B_0}, n^{-\frac{3}{2}}(U_n)_{\omega_0}\}$. Finally, on differentiating (6·7) and (6·8) with respect to ω_0, we obtain expressions which are respectively $O_p(n^{\frac{1}{2}})$ and $O_p(n^{\frac{3}{2}})$. This result still holds if A_0, B_0 and ω_0 are replaced respectively by A_n^*, B_n^* and ω_n^*, and consequently we find that the matrix $\mathbf{W}_n^*$ obtained by replacing U_n by $\tilde{U}_n$ in (4·23) converges in probability to $\mathbf{W}_0$. The argument in § 4 can thus be applied to show that Theorem 2 still holds if X_t is replaced by $X_t - \bar{X}$.

This paper is a revised version of a Stanford University report prepared during the tenure of a Visiting Professorship in 1968. This work constitutes part of a programme of research in time series analysis under the direction of Professor E. Parzen, supported by the United States Office of Naval Research and the United States Army Research Office.

REFERENCES

HANNAN, E. J. (1960). *Time Series Analysis*. London: Methuen.

LOÈVE, M. (1963). *Probability Theory*, 3rd edition. New York: Van Nostrand.

RAO, C. R. (1952). *Advanced Statistical Methods in Biometric Research*. New York: Wiley.

RAO, C. R. (1965). *Linear Statistical Inference*. New York: Wiley.

WALKER, A. M. (1964). Asymptotic properties of least-squares estimates of parameters of the spectrum of a stationary non-deterministic time series. *J. Aust. Math. Soc.* **4**, 363–84.

WALKER, A. M. (1965). Some asymptotic results for the periodogram of a stationary time series. *J. Aust. Math. Soc.* **5**, 107–28.

WHITTLE, P. (1952). The simultaneous estimation of a time series' harmonic components and covariance structure. *Trab. Estad.* **3**, 43–57.

WHITTLE, P. (1953). Estimation and information in stationary time series. *Ark. Mat. Astr. Fys.* **2**, 423–34.

WHITTLE, P. (1954). Appendix 2 in *A Study in the Analysis of Stationary Time Series*, by H. Wold. Uppsala: Almqvist and Wiksell.

WHITTLE, P. (1959). Sur la distribution du maximum d'un polynome trigonométrique à coefficients aleátoires. *Colloques Internationaux du Centre National de la Recherche Scientifique*, LXXXVII. Paris.

[*Received November* 1969. *Revised September* 1970]

Some key words: Harmonic components model with independent residuals; Asymptotic distribution of least squares regression estimates in time series.