STAT 613 Midterm (1 hour 15 minutes) April 13th, 2012
Marks will be given for clarity of the solution.
Good Luck!

(1) The object of this question is to use the log-likelihood ratio test to derive the χ-squared test for independence (in the case of two by two tables). In other words, derive the distribution of the test statistic

$$T = \frac{(O_1 - E_1)^2}{E_1} + \frac{(O_2 - E_2)^2}{E_2} + \frac{(O_3 - E_3)^2}{E_3} + \frac{(O_4 - E_4)^2}{E_4},$$

under the null that there is no association between the categorical variables C and R, where and $E_1 = n_3 \times n_1/N$, $E_2 = n_4 \times n_1/N$, $E_3 = n_3 \times n_2/N$ and $E_2 = n_4 \times n_2/N$.

	C_1	C_2	Subtotal
R_1	O_1	O_2	n_1
R_2	O_3	O_4	n_2
Subtotal	n_3	n_4	N

State all results you use. (hint: It may be useful to use the Taylor approximation $x\log(x/y) \approx (x-y) + \frac{1}{2}(x-y)^2/y$). [10]

(2) Consider the following shifted exponential mixture distribution

$$f(x;\lambda_1,\lambda_2,p,a) = p\frac{1}{\lambda_1}\exp(-x/\lambda_1)I(x \geq 0) + (1-p)\frac{1}{\lambda_2}\exp(-(x-a)/\lambda_2)I(x \geq a),$$

where p, λ_1, λ_2 and a are unknown.

[15]

(i) Make a plot of the above mixture density.

Considering the cases $x \geq a$ and $x < a$ separately, calculate the probability of belonging to each of the mixtures, given the observation X_i (ie. Define the variable δ_i, where $P(\delta_i = 0) = p$, $f(x|\delta_i = 0) = \frac{1}{\lambda_1}\exp(-x/\lambda_1)$ etc. and calculate $P(\delta_i = 0|X_i = x)$ and $P(\delta_i = 0|X_i = x)$).

(ii) Show how the EM-algorithm can be used to estimate $a, p, \lambda_1, \lambda_2$. At each iteration you should be able to obtain explicit solutions for *most* of the parameters, give as many details as you can.

Hint: It may be beneficial for you to use profiling too.

(iii) From your knowledge of estimation of these parameters, what do you conjecture the rates of convergence to be? Will they all be the same, or possibly different?

(iv) Not part of the exams: code the estimator. Through simulations try to verify your conjecture in (iii).